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A dimension reduction result in the framework of structured deformations

José Matias and Pedro Miguel Santos

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Abstract. Structured deformations provide a model to non-classical deformations of continua suitable for the description of deformations of materials whose kinematics requires analysis at both the macroscopic and microscopic levels. In this work we apply dimension reduction techniques in order to derive models for thin structures in the framework of structured deformations of continua.

Keywords. Structured Deformations, Dimension Reduction, Functions of Bounded Variation.

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1 Introduction

Structured deformations were first introduced by Del Piero and Owen [15] and later generalized by Owen and Paroni [19]. The model introduced in [15] (first order structured deformations) provides a class of deformations which is appropriate to describe complicated processes of fracture at the macroscopic level and also permits to identify processes of microfracture that describe a continuum with structure. Choksi and Fonseca [8] extended the notion of first order structured deformation to the setting of special functions of bounded variation. Precisely, the authors defined a first order structured deformation as a pair (g, G) where the macroscopic deformation g is an element of $SBV(\Omega;\mathbb{R}^d)$ and G is an integrable tensor field in Ω, and have proved that given such a pair there exist deformations u_n in $SBV(\Omega;\mathbb{R}^d)$ such that

$$u_n \xrightarrow{L^1} g \quad \text{and} \quad \nabla u_n \overset{\mathcal{M}(\Omega)}{\rightharpoonup} G.$$

Then the energy of (g, G) was defined as

$$\mathcal{I}(g, G) := \inf_{\{u_n\}\subset SBV(\Omega;\mathbb{R}^d)} \left\{ \liminf_{n\to\infty} E(u_n), \;\; u_n \xrightarrow{L^1} g, \;\; \nabla u_n \overset{\mathcal{M}(\Omega)}{\rightharpoonup} G \right\},$$

where

$$E(u) = \int_\Omega W(\nabla u)\, dx + \int_{S(u)} \psi([u], \nu(u))\, d\mathcal{H}^{N-1}$$

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for any $u \in SBV(\Omega;\mathbb{R}^d)$, and an integral representation of $\mathcal{I}(g,G)$ was derived. Note that the energy of (g,G) corresponds to the most economical way to build up deformations using SBV- approximations.

In this work we consider a model for first order structured deformations departing from a different initial energy $E(u)$ which includes second order derivatives (see (1.1) below; see Carriero Leaci and Tomarelli [9] and [10] for other second order variational problems). Our goal is to derive a model for thin structures (in the context of first order structured deformations) through dimensional reduction techniques. The need for second derivatives relies on the fact that, in order to avoid the formation of holes in the target lower dimensional domain, all the jumps in the approximating sequences must be properly aligned (see Remark 1.3 below).

Let $\Omega \subset \mathbb{R}^N$ be an open bounded set. For $u \in SBV^2(\Omega;\mathbb{R}^d)$ we define

$$\begin{aligned} E(u) \;=\; & \int_\Omega W(\nabla u, \nabla^2 u)\, dx + \int_{S_u\cap\Omega} \Psi_1([u],\nu(u))\, d\mathcal{H}^{N-1} \\ & + \int_{S_{\nabla u}\cap\Omega} \Psi_2([\nabla u],\nu(\nabla u))\, d\mathcal{H}^{N-1}. \end{aligned} \tag{1.1}$$

In [7] we studied the relaxed energy

$$I(g,G) = \inf_{\{u_n\in SBV^2(\Omega;\mathbb{R}^3)\}} \left\{ \liminf_{n\to\infty} E(u_n)\colon\; u_n \in SBV^2(\Omega;\mathbb{R}^3),\; u_n \xrightarrow{L^1} g,\; \nabla u_n \xrightarrow{L^1} G \right\}, \tag{1.2}$$

under the following hypotheses for the bulk and interfacial densities

(H_1) : there exists $C>0$ such that

$$\frac{1}{C}|B| - C \le W(A,B) \le C\big(1+|B|\big)$$

for all $A \in \mathbb{R}^{d\times N}$ and $B \in \mathbb{R}^{d\times N\times N}$;

(H_2) : there exists $C>0$ such that

$$|W(A_1,B_1) - W(A_2,B_2)| \le C\big(|A_1-A_2| + |B_1-B_2|\big)$$

for all $A_i \in \mathbb{R}^{d\times N}$ and $B_i \in \mathbb{R}^{d\times N\times N}$, $i=1,2$;

(H_3) : there exists $0<\alpha<1$ and $L>0$ such that

$$\Big| W^\infty(A,B) - \frac{W(A,tB)}{t} \Big| \le \frac{C}{t^\alpha}$$

for all $t>L$, $A\in\mathbb{R}^{d\times N}$, $B\in\mathbb{R}^{d\times N\times N}$ with $|B|=1$, where W^∞ denotes, as usual, the *recession function* of W in the variable B, i.e.,

$$W^{\infty}(A,B)=\limsup_{t\to+\infty}\frac{W(A,tB)}{t};$$

(H_4) : there exist $c_1>0,\ C_1>0$ such that

$$c_1|\lambda|\le\Psi_1(\lambda,\nu)\le C_1|\lambda|,$$

for all $\lambda\in\mathbb{R}^d$ and $\nu\in S^{N-1}$;

(H_5): there exist $c_2>0,\ C_2>0$ such that

$$c_2|\Lambda|\le\Psi_2(\Lambda,\nu)\le C_2|\Lambda|,$$

for all $\nu\in S^{N-1}$ and $\Lambda\in\mathbb{R}^{d\times N}$;

(H_6): (homogeneity of degree one)

$$\Psi_1(t\lambda,\nu)=t\Psi_1(\lambda,\nu),\qquad \Psi_2(t\Lambda,\nu)=t\Psi_2(\Lambda,\nu)$$

for all $\nu\in S^{N-1}$, $\lambda\in\mathbb{R}^d$, $\Lambda\in\mathbb{R}^{d\times N}$ and $t>0$;

(H_7): (sub-additivity)

$$\Psi_1(\lambda_1+\lambda_2,\nu)\le\Psi_1(\lambda_1,\nu)+\Psi_1(\lambda_2,\nu),$$
$$\Psi_2(\Lambda_1+\Lambda_2,\nu)\le\Psi_2(\Lambda_1,\nu)+\Psi_2(\Lambda_2,\nu)$$

for all $\nu\in S^{N-1}$, $\lambda_i\in\mathbb{R}^d$, $\Lambda_i\in\mathbb{R}^{d\times N}$, $i=1,2$.

Remark 1.1. We extend $\Psi_i, i=1,2$ as homogeneous functions of degree one in the second variable to all of $\mathbb{R}^N$ (respectively $\mathbb{R}^{d\times N}$).

In [7], under the hypotheses $(H_1)-(H_7)$, an integral representation of the energy $I(g,G)$ was derived for $g\in BV^2(\Omega;\mathbb{R}^d)$ and $G\in BV(\Omega;\mathbb{R}^{d\times N})$, with the infimum taken over all sequences $u_n\in SBV^2(\Omega;\mathbb{R}^d),\ u_n\ \xrightarrow{L^1} g, \nabla u_n\ \xrightarrow{L^1} G$. Namely, given $A,B\in\mathbb{R}^{d\times N}$ and $C\in\mathbb{R}^{d\times N\times N}$ and defining

$$W_1(A)=\inf_{u\in SBV(Q;\mathbb{R}^d)}\left\{\int_{Su\cap Q}\Psi_1([u],\nu(u))\,d\mathcal{H}^{N-1}, u|_{\partial Q}=0, \nabla u=A \text{ a.e. in } Q\right\},$$

$$\begin{aligned}W_2(B,C)\;=\;&\inf_{v\in SBV(Q;\mathbb{R}^{d\times N})}\left\{\int_Q W(B,\nabla v(x))\,dx\right.\\&\left.+\int_{Q\cap S_v}\Psi_2([v],\nu(v))\,d\mathcal{H}^{N-1}, v|_{\partial Q}=C\,x\right\},\end{aligned}$$

$$\gamma_1(\lambda,\nu)=\inf_{u\in SBV(Q_\nu;\mathbb{R}^d)}\left\{\int_{Q_\nu\cap S_u}\Psi_1([u],\nu(u))\,d\mathcal{H}^{N-1},u|_{\partial Q_\nu}=\gamma_{(\lambda,\nu)},\nabla u=0\right\},$$

where

$$\gamma_{(\lambda,\nu)}(x):=\begin{cases}\lambda & \text{if } x.\nu>0\\ \\ 0 & \text{if } x.\nu<0,\end{cases}$$

$$\gamma_2(\Lambda,\Gamma,\nu)=\inf_{v\in SBV(Q_\nu;\mathbb{R}^{d\times N})}\left\{\int_{Q_\nu}W^\infty(v,\nabla v)\,dx+\int_{Q_\nu\cap S_v}\Psi_2([v],\nu(v))\,d\mathcal{H}^{N-1},\right.$$

$$\left. v|_{\partial Q_\nu}=\gamma_{(\Lambda,\Gamma,\nu)}\right\}$$

where

$$\gamma_{(\Lambda,\Gamma,\nu)}(x):=\begin{cases}\Lambda & \text{if } x.\nu>0\\ \\ \Gamma & \text{if } x.\nu<0,\end{cases}$$

and

$$\begin{aligned}W_2^\infty(A,B) &= \inf_{v\in SBV(Q;\mathbb{R}^{d\times N})}\left\{\int_Q W^\infty(A,\nabla v(x))\,dx\right.\\ &\left.+\int_{Q\cap S_v}\Psi_2([v],\nu(v))\,d\mathcal{H}^{N-1},v|_{\partial Q}=Bx\right\},\end{aligned}$$

the following result was proved:

Theorem 1.2. *Under hypotheses $(H_1)-(H_7)$ we have that for all $(g,G)\in BV^2(\Omega;\mathbb{R}^d)\times BV(\Omega;\mathbb{R}^{d\times N})$, the functional*

$$I(g,G)=\inf_{\{u_n\in SBV^2(\Omega;\mathbb{R}^3)\}}\left\{\liminf_{n\to\infty}E(u_n):\ u_n\in SBV^2(\Omega;\mathbb{R}^3),\ u_n\ \overset{L^1}{\longrightarrow} g,\ \nabla u_n\ \overset{L^1}{\longrightarrow} G\right\},$$

is given by

$$\begin{aligned}I(g,G) &= \int_\Omega\Big(W_1(G-\nabla g)+W_2(G,\nabla G)\Big)\,dx+\int_{S_g}\gamma_1([g],\nu(g))d\mathcal{H}^{N-1}(x)\\ &+\int_{S_G}\gamma_2(G^+,G^-,\nu(G))\,d\mathcal{H}^{N-1}(x)+\int_\Omega W_1(-\frac{dD^cg}{d|D^cg|})\,d|D^cg|\\ &+\int_\Omega W_2^\infty(G,\frac{dD^cG}{d|D^cG|})\,d|D^cG|.\end{aligned}$$

In what follows let $N = 3$, $\epsilon > 0$, $\omega \subset \mathbb{R}^2$ an open bounded set and $\Omega_\epsilon = w \times (0, \epsilon)$ We denote simply by Ω the subset of $\mathbb{R}^3$ corresponding to $\Omega_1 = w \times (0,1) = w \times I$. If $x \in \mathbb{R}^3$ then $x_\alpha := (x_1, x_2) \in \mathbb{R}^2$ is the vector of the first two components of x.

In this work we consider three dimensional structures with vanishing thickness ϵ. The energy associated to these structures is of the form:

$$E_\epsilon(v) := \int_{\Omega_\epsilon} W(\nabla v, \nabla^2 v)\, dy + \int_{S_v} \Psi_1([v], \nu(v))\, d\mathcal{H}^2 + \int_{S_{\nabla v}} \Psi_2([\nabla v], \nu(\nabla v))\, d\mathcal{H}^2,$$

where $v \in SBV^2(\Omega_\epsilon; \mathbb{R}^3)$. Let $y = (y_\alpha, y_3) \in \Omega_\epsilon$ and changing variables define $x = (x_\alpha, x_3) \in \Omega$ through $x_\alpha = y_\alpha$ and $x_3 = \frac{y_3}{\epsilon}$. Then

$$u(x_\alpha, x_3) := v(x_\alpha, \epsilon x_3)$$

is clearly a function in $SBV^2(\Omega; \mathbb{R}^3)$ and the integral above becomes

$$\begin{aligned} E_\epsilon(u) \;=\; \epsilon \Bigg[& \int_\Omega W(\nabla_\alpha u | \frac{1}{\epsilon} \nabla_3 u, \nabla^2_{\alpha,\beta} u | \frac{1}{\epsilon} \nabla^2_{\alpha 3} u | \frac{1}{\epsilon} \nabla^2_{3\beta} u | \frac{1}{\epsilon^2} \nabla^2_{33} u)\, dx \\ & + \int_{S_u} \Psi_1([u], \nu_\alpha(u) | \frac{1}{\epsilon} \nu_3(u))\, d\mathcal{H}^2 \\ & + \int_{S_{\nabla u}} \Psi_2([\nabla_\alpha u] | \frac{1}{\epsilon} [\nabla_3 u], \nu_\alpha(\nabla u) | \frac{1}{\epsilon} \nu_3(\nabla u))\, d\mathcal{H}^2 \Bigg] \end{aligned}$$

where $\alpha, \beta \in \{1, 2\}$.

We introduce now the ϵ-scaled $3D$ energies $J_\epsilon := \frac{E_\epsilon}{\epsilon}$, and our aim is to derive the asymptotic behaviour as $\epsilon \to 0^+$, in the sense of Γ-convergence (see [12], [13], [3] and [14]). More precisely we consider

$$I(g, b, G) := \inf_{u_n \in SBV^2(\Omega; \mathbb{R}^3)} \left\{ \liminf_{\epsilon_n \to 0} J_{\epsilon_n}(u_n),\ u_n \xrightarrow{L^1} g,\ \frac{1}{\epsilon_n} \nabla_3 u_n \xrightarrow{L^1} b,\ \nabla u_n \xrightarrow{L^1} G \right\},$$

where $(g, G) \in BV^2(\omega; \mathbb{R}^3) \times BV(\omega; \mathbb{R}^{d \times N})$ and $b \in BV(w; \mathbb{R}^3)$. The remark below gives the motivation for the definition of I, in particular the convergence and the appearance of fields independent of the transversal variable x_3 in the limit space.

Remark 1.3. Suppose we have a sequence of deformations clamped in the boundary and with finite total energy. Thus, for a given sequence $\{\epsilon_n\}$, we have a sequence $\{v_n\} \subset SBV^2(\Omega_{\epsilon_n}; \mathbb{R}^3)$ such that $v_n = x$ in a neighborhood of $\partial\omega \times (0, \epsilon_n)$. After rescaling we obtain a new sequence $\{u_n\} \subset SBV^2(\Omega;\mathbb{R}^3)$ such that $u_n = (x_\alpha, \epsilon_n x_3)$ and

$$\sup_n J_{\epsilon_n}(u_n) < \infty.$$

From the growth conditions $(H1)$, $(H4)$ and $(H5)$, we obtain

$$\sup_n \left(|D(\nabla u_n)|(\Omega) + \int_{S_{u_n}} |[u_n]| \right) < \infty,$$

which, together with the boundary condition, implies the boundedness of u_n and ∇u_n in the BV-norm. Thus, up to a subsequence, we have $u_n \to g$ and $\nabla u_n \to G$ in L^1. Now, defining $b_n := \frac{\nabla_3 u_n}{\epsilon_n}$, and using $(H1)$ and $(H5)$ we get that

$$\sup_n |D(b_n)|(\Omega) < \infty,$$

which, together with the boundary condition $b_n = (0,0,1)$ implies the boundedness of b_n in BV-norm and consequently the existence of a subsequence such that $b_n \to b$ in L^1. The field g represents the deformation of the mid-surface and the field b represents the rotation and compression of the normal sections.

On the other hand, using the same growth conditions, we have

$$\sup_n \left(|D_3(\nabla u_n)|(\Omega) + \int_{S_{u_n}} |[u_n]\nu_3| + |D_3(b_n)|(\Omega) \right) < C\epsilon_n,$$

which, together with boundary conditions, implies that the limit fields g, G and b do not depend on x_3.

Remark 1.4. We will prove in the sequel that the functional I above does not depend on the sequence $\{\epsilon_n\}$.

The overall plan of this work in the ensuing sections will be as follows: Section 2 collects the main notations and results used throughout. In Section 3 we prove that $I(g,b,G)$ can be decomposed into two first order functionals $I_1(g,b,G)$ and $I_2(b,G)$. Precisely, we show that

$$I(g,b,G) = I_1(g,b,G) + I_2(b,G),$$

with

$$I_1(g,b,G) = \inf_{u_n \in SBV^2(\Omega;\mathbb{R}^3)} \left\{ \liminf_{\epsilon_n \to 0} \int_{S_{u_n}} \Psi_1([u_n], \nu_\alpha(u_n)|\frac{1}{\epsilon_n}\nu_3(u_n))\, d\mathcal{H}^2 \right.$$
$$\left. u_n \xrightarrow{L^1} g,\ \frac{1}{\epsilon_n}\nabla_3 u_n \xrightarrow{L^1} b,\ \nabla_\alpha u_n \xrightarrow[n\to\infty]{L^1} G \right\},$$

and

$$I_2(b,G) = \inf_{h_n\in SBV(\Omega;\mathbb{R}^{3\times 3})}\left\{\liminf_{\epsilon_n\to 0}\left[\int_\Omega W(h_n,\nabla_\alpha h_n|\frac{1}{\epsilon_n}\nabla_3 h_n)\,dx\right.\right.$$
$$\left.+\int_{S_{h_n}}\Psi_2([h_n],\nu_\alpha(h_n)|\frac{1}{\epsilon_n}\nu_3(h_n))\,d\mathcal{H}^2\right]$$
$$\left. h_n \xrightarrow[n\to\infty]{L^1}(b,G)\right\}.$$

Section 4 is devoted to determine the integral representation of I_1 in BV, which turns out to depend only on (g,G). We prove the following Theorem (see Section 2 for notations).

Theorem 1.5. *Let* $(g,G)\in BV^2(\omega;\mathbb{R}^3)\times BV(\omega;\mathbb{R}^{3\times 2})$. *Then, for* Ψ_1 *satisfying* $(H_4),(H_6),(H_7)$*,*

$$I_1(g,G) := \inf_{u_n}\left\{\liminf_{\epsilon_n\to 0}\int_{S_{u_n}}\Psi_1([u_n],\nu_\alpha(u_n)|\frac{1}{\epsilon_n}\nu_3(u_n))\,d\mathcal{H}^2(x)\right.$$
$$\left. u_n\in SBV^2(w;\mathbb{R}^3)\ u_n \xrightarrow{L^1} g,\ \frac{1}{\epsilon_n}\nabla_3 u_n \xrightarrow{L^1} b,\ \nabla_\alpha u_n \xrightarrow{L^1} G\right\}$$

is given by

$$I_1(g,G)=\int_w \hat{W}_1^{(2)}(G-\nabla g)\,dx_\alpha+\int_{S_g}\hat{\gamma}_1^{(2)}([g],\nu(g))\,d\mathcal{H}^1+\int_\omega \hat{W}_1^2(-\frac{dD^cg}{d|D^cg|})\,d|D^cg|,$$

where

$$\hat{W}_1^{(2)}(A)=\inf_{u\in SBV(Q';\mathbb{R}^3)}\left\{\int_{Su\cap Q'}\bar{\Psi}_1([u],\nu(u))\,d\mathcal{H}^1, u|_{\partial Q}=0, \nabla u = A \text{ a.e. in } Q'\right\},$$

and

$$\hat{\gamma}_1^{(2)}(\lambda,\nu)=\inf_{u\in SBV(Q'_\nu;\mathbb{R}^3)}\left\{\int_{Q'_\nu\cap S_u}\bar{\Psi}_1([u],\nu(u))\,d\mathcal{H}^1, u|_{\partial Q'_\nu}=\gamma_{(\lambda,\nu)}, \nabla u=0\right\},$$

with

$$\gamma_{(\lambda,\nu)}(x_\alpha) := \begin{cases}\lambda & \text{if } x_\alpha.\nu>0\\ 0 & \text{if } x_\alpha.\nu<0\end{cases}$$

and $\bar{\Psi}_1$ defined as follows

$$\bar{\Psi}_1(\lambda, \nu_\alpha) := \inf\left\{ \frac{1}{\sqrt{|\nu_\alpha|^2 + t^2}} \Psi_1(\lambda, \nu_\alpha, t) : t \in \mathbb{R} \right\}.$$

In Section 5 we derive the integral representation of $I_2(b, G)$ in BV. This representation relies on the Global Method for Relaxation, introduced by Bouchitté Fonseca Mascarenhas in [5]. Precisely we prove the following result:

Theorem 1.6. *Let $(b, G) \in BV(w; \mathbb{R}^3) \times BV(w; \mathbb{R}^{3\times 2})$. Then for W and Ψ_2 satisfying (H_1), (H_2) , (H_3), (H_5), (H_6) and (H_7), the functional*

$$\begin{aligned} I_2(b, G) \quad = \quad & \inf_{h_n \in SBV(\Omega;\mathbb{R}^{3\times 3})} \left\{ \liminf_{\epsilon_n \to 0} \left[\int_\Omega W(h_n, \nabla_\alpha h_n | \frac{1}{\epsilon_n} \nabla_3 h_n)\, dx \right.\right. \\ & \left. + \int_{S_{h_n}} \Psi_2([h_n], \nu_\alpha(h_n) | \frac{1}{\epsilon_n} \nu_3(h_n))\, d\mathcal{H}^2 \right] \\ & \left. h_n \xrightarrow[n\to\infty]{L^1} (G, b) \right\} \end{aligned}$$

is given by

$$\begin{aligned} I_2(b, G) \;=\; & \int_\omega \hat{W}_2(b, G, \nabla b, \nabla G)\, dx + \int_{\omega \cap S((b,G))} \hat{\Psi}_2((b, G)^+, (b, G)^-, \nu((b, G)))\, d\mathcal{H}^1 \\ & + \int_\omega \hat{W_2}^\infty \left(b, G, \frac{dC(b, G)}{d|C(b, g)|} \right) d|C(b, g)|, \end{aligned}$$

where

$$\begin{aligned} \hat{W}_2(A, B_\alpha) := \inf \Bigg\{ & \int_{Q'} \bar{W}_2(A, \nabla u)\, dy + \int_{Q' \cap S_u} \bar{\Psi}_2([u], \nu(u))\, d\mathcal{H}^1 : \\ & u \in SBV(Q'; \mathbb{R}^3), u|_{\partial Q'} = B_\alpha y \Bigg\}, \end{aligned} \tag{1.3}$$

$$\begin{aligned} \hat{\Psi}_2(\lambda, \theta, \nu) := \inf \Bigg\{ & \int_{Q'_\nu} \bar{W}_2^\infty(u, \nabla u)\, dy + \int_{Q'_\nu \cap S(u)} \bar{\Psi}_2([u], \nu(u)) d\mathcal{H}^1 : \\ & u \in SBV(Q'_\nu; \mathbb{R}^3), u|_{\partial Q'_\nu} = u_{\lambda,\theta,\nu} \Bigg\}, \end{aligned} \tag{1.4}$$

where

$$u_{\lambda,\theta,\nu}(y) := \begin{cases} \lambda & \text{if } y\cdot\nu > 0, \\ \\ \theta & \text{otherwise.} \end{cases}$$

The functions $\bar{W}_2$ *and* $\bar{\Psi}_2$ *are defined as follows. Decomposing the pair* $(A,B) \in \mathbb{R}^{3\times 3}\times\mathbb{R}^{3\times 3\times 3}$ *into* $(A,B_\alpha,B_3)\in\mathbb{R}^{3\times 3}\times\mathbb{R}^{3\times 3\times 2}\times\mathbb{R}^{3\times 3\times 1}$ *we define*

$$\bar{W}_2(A,B_\alpha) := \inf_{B_3\in\mathbb{R}^{3\times 3\times 1}} W(A,B_\alpha,B_3),$$

and for $\lambda\in\mathbb{R}^{3\times 3},\ \nu_\alpha\in S^1$, *let*

$$\bar{\Psi}_2(\lambda,\nu_\alpha) := \inf\left\{\frac{1}{\sqrt{|\nu_\alpha|^2+t^2}}\Psi_2(\lambda,\nu_\alpha,t) : t\in\mathbb{R}\right\}.$$

2 Preliminaries

The purpose of this section is to give a brief overview of the concepts and results that are used in the sequel. Almost all these results are stated without proofs as they can be readily found in the references given below.

2.1 Notation

Throughout the text $w\subset\mathbb{R}^2$ will denote an open bonded set and for $\epsilon>0, \Omega_\epsilon = w\times(0,\epsilon)$. We denote simply by Ω the subset of $\mathbb{R}^3$ corresponding to $\Omega_1 = w\times(0,1) = w\times I$. If $x\in\mathbb{R}^3$ then $x_\alpha := (x_1,x_2)\in\mathbb{R}^2$ is the vector of the first two components of x.

We wil use the following notations:

- $\mathcal{A}(\Omega)$ (resp. $\mathcal{A}(w)$) is the family of all open subsets of Ω (resp. w),
- $\mathcal{M}(\Omega)$ (resp. $\mathcal{M}(w)$) is the set of finite Radon measures on Ω (resp. w),
- $||\mu||$ stands for the total variation of a measure $\mu\in\mathcal{M}(\Omega)$ (resp. $\mathcal{M}(w)$),
- S^{N-1} stands for the unit sphere in $\mathbb{R}^N$,
- Q denotes the unit cube of $\mathbb{R}^3$ centered at the origin with one side orthogonal to e_3,
- $Q(x,\delta)$ denotes a cube in $\mathbb{R}^3$ centered at $x\in\Omega$ with side length δ and with one side orthogonal to e_3,
- $Q_\nu(x,\delta)$ is the cube centered at $x\in\Omega$ with side length δ and with one side orthogonal to $\nu\in S^2$,

- when related to $\mathbb{R}^2$ and w we use the previous notations with the obvious adaptations with Q' in place of Q,
- C represents a generic constant,
- $\lim\limits_{n,m\to\infty} := \lim\limits_{n\to\infty}\lim\limits_{m\to\infty}$ while $\lim\limits_{m,n\to\infty} := \lim\limits_{m\to\infty}\lim\limits_{n\to\infty}$.

2.2 BV-functions

We start by recalling some facts on functions of bounded variation which will be used afterwards. We refer to Ambrosio, Fusco and Pallara [1], Evans and Gariepy [16], Federer [17], Giusti [18] and Ziemer [20] for a detailed theory on this subject.

A function $u \in L^1(\Omega;\mathbb{R}^d)$ is said to be of *bounded variation*, and we write $u \in BV(\Omega;\mathbb{R}^d)$, if all its first distributional derivatives $D_j u_i \in \mathcal{M}(\Omega)$ for $i = 1, ..., d$ and $j = 1, ..., N$. The matrix-valued measure whose entries are $D_j u_i$ is denoted by Du. The space $BV(\Omega;\mathbb{R}^d)$ is a Banach space when endowed with the norm

$$\|u\|_{BV} = \|u\|_{L^1} + ||Du||(\Omega).$$

By the Lebesgue Decomposition theorem Du can be split into the sum of two mutually singular measures $D^a u$ and $D^s u$ (the absolutely continous part and singular part, respectively, of Du with respect to the Lebesgue measure $\mathcal{L}^N$). By ∇u we denote the Radon-Nikodým derivative of $D^a u$ with respect to $\mathcal{L}^N$, so that we can write

$$Du = \nabla u \mathcal{L}^N \lfloor \Omega + D^s u.$$

Let Ω_u be the set of points where the approximate limits of u exists and S_u the *jump set* of this function, i.e., the set of points $x \in \Omega \setminus \Omega_u$ for which there exists a, $b \in \mathbb{R}^N$ and a unit vector $\nu \in S^{N-1}$, normal to S_u at x, such that $a \neq b$ and

$$\lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^N} \int_{\{y\in Q_\nu(x,\varepsilon):(y-x)\cdot\nu>0\}} |u(y) - a|\, dy = 0 \tag{2.1}$$

and

$$\lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^N} \int_{\{y\in Q_\nu(x,\varepsilon):(y-x)\cdot\nu<0\}} |u(y) - b|\, dy = 0. \tag{2.2}$$

The triple (a, b, ν) uniquely determined by (2.1) and (2.2) up to permutation of (a, b), and a change of sign of ν and is denoted by $(u^+(x), u^-(x), \nu_u(x))$.

If $u \in BV(\Omega)$ it is well known that S_u is countably $N - 1$ rectifiable, i.e.

$$S_u = \bigcup_{n=1}^{\infty} K_n \cup E,$$

where $\mathcal{H}^{N-1}(E) = 0$ and K_n are compact subsets of C^1 hypersurfaces. Furthermore, $\mathcal{H}^{N-1}((\Omega \setminus \Omega_u) \setminus S_u) = 0$ and the following decomposition holds

$$Du = \nabla u \mathcal{L}^N \lfloor \Omega + [u] \otimes \nu_u \mathcal{H}^{N-1} \lfloor S_u + C_u,$$

where $[u] := u^+ - u^-$ and C_u is the Cantor part of the measure Du, i.e., $C_u = D^s u \lfloor (\Omega_u)$.

We next recall some properties of BV functions used in the sequel. We start with the following Lemma whose proof can be found in [8]:

Lemma 2.1. *Let $u \in BV(\Omega; \mathbb{R}^d)$. There exist piecewise constant functions u_n such that $u_n \to u$ in $L^1(\Omega; \mathbb{R}^d)$ and*

$$||Du||(\Omega) = \lim_{n \to \infty} ||Du_n||(\Omega) = \lim_{n \to \infty} \int_{S_{u_n}} |[u_n](x)| \, d\mathcal{H}^{N-1}.$$

The space of *special functions of bounded variation*, $SBV(\Omega; \mathbb{R}^d)$, introduced by De Giorgi and Ambrosio in [11] to study free discontinuity problems, is the space of functions $u \in BV(\Omega; \mathbb{R}^d)$ such that $C_u = 0$, i.e. for which

$$Du = \nabla u \mathcal{L}^N + [u] \otimes \nu_u \mathcal{H}^{N-1} \lfloor S_u.$$

The next result is a Lusin type theorem for gradients due to Alberti [2] and is essential to our arguments.

Theorem 2.2. *Let $f \in L^1(\Omega; \mathbb{R}^{d \times N})$. There exists $u \in SBV(\Omega; \mathbb{R}^d)$ and a Borel function $g : \Omega \to \mathbb{R}^{d \times N}$ such that*

$$Du = f \mathcal{L}^N + g \mathcal{H}^{N-1} \lfloor S_u,$$

$$\int_{S_u} |g| \, d\mathcal{H}^{N-1} \leq C ||f||_{L^1(\Omega; \mathbb{R}^{d \times N})}.$$

Remark 2.3. From the proof of Theorem 2.2 it also follows that

$$||u||_{L^1(\Omega)} \leq 2C ||f||_{L^1(\Omega; \mathbb{R}^{d \times N})}.$$

Following Carriero, Leaci and Tomarelli (see [9] and [10]) we define

$$SBV^2(\Omega; \mathbb{R}^d) = \{ v \in SBV(\Omega; \mathbb{R}^d), \ \nabla v \in SBV(\Omega; \mathbb{R}^{d \times N}) \}.$$

If $u \in SBV^2(\Omega; \mathbb{R}^d)$ we use the notation $\nabla^2 u = \nabla(\nabla u)$, that is, $\nabla^2 u$ is the absolutely continuous part of $D(\nabla u)$ with respect to Lebesgue measure.

We will also denote by

$$BV^2(\Omega; \mathbb{R}^d) = \{ v \in BV(\Omega; \mathbb{R}^d), \ \nabla v \in BV(\Omega; \mathbb{R}^{d \times N}) \}.$$

2.3 Integral representation results

Here we recall Theorem 3.12 in [5]. Let:

$$\mathcal{F} : BV(\Omega;\mathbb{R}^d) \times \mathcal{A}(\Omega) \to [0,+\infty)$$

satisfying:

i) $\mathcal{F}(u,:)$ is the restriction to $\mathcal{A}(\Omega)$ of a Radon measure,

ii) $\mathcal{F}(u,:)$ is $L^1(A;\mathbb{R}^d)$- lower semicontinuous,

iii) $\frac{1}{C}|Du|(A) \leq \mathcal{F}(u,A) \leq C\left(\mathcal{L}^N(A) + |Du|(A)\right)$ for some $C > 0$,

iv) There exists a modulus of continuity $\phi(t)$ satisfying

$$|\mathcal{F}(u(.-z)+b;z+A) - \mathcal{F}(u,A)| \leq \phi(|b|+|z|)\left(\mathcal{L}^N(A) + |Du|(A)\right).$$

Define the set function:

$$m(u;A) := \inf\left\{\mathcal{F}(v;A), v|_{\partial A} = u|_{\partial A},\, v \in BV(\Omega;\mathbb{R}^d)\right\},$$

and let

$$f(x_0,a,\zeta) := \limsup_{\epsilon\to 0^+} \frac{m(a+\zeta(.-x_0);Q(x_0,\epsilon))}{\epsilon^N}, \tag{2.3}$$

$$g(x_0,\lambda,\theta,\nu) := \limsup_{\epsilon\to 0^+} \frac{m(u_{\lambda,\theta,\nu}(.-x_0);Q_\nu(x_0,\epsilon))}{\epsilon^{N-1}} \tag{2.4}$$

for all $x_0 \in \Omega, a, \theta, \lambda \in \mathbb{R}^d, \zeta \in \mathbb{R}^{d\times N}$, where

$$u_{\lambda,\theta,\nu}(y) := \begin{cases} \lambda & \text{if } y\cdot\nu > 0, \\ \\ \theta & \text{otherwise.} \end{cases}$$

Then the following full representation result of $\mathcal{F}$ on $BV(\Omega;\mathbb{R}^d)$ holds:

Theorem 2.4. *Under hypotheses $i), ii), iii)$ and $iv)$,*

$$\begin{aligned} \mathcal{F}(u;A) &= \int_\Omega f(x,u,\nabla u)\,dx + \int_{S_u\cap A} g(x,u^+,u^-,\nu_u)\,d\mathcal{H}^{N-1} \\ &\quad + \int_A f^\infty\left(x,u,\frac{dC(u)}{d|C(u)|}\right)\,d|C(u)|, \end{aligned}$$

where f and g are defined by (2.3) *and* (2.4) *respectively and f^∞ denotes the recession function of f in the last variable.*

3 Decomposition

We start by noting that for any $(g,b,G) \in BV^2(\omega : \mathbb{R}^3) \times BV(\omega\ \mathbb{R}^3) \times BV(\omega;\mathbb{R}^{3\times 2})$ and for any fixed sequence $\epsilon_n \to 0$ there exists $u_n \in SBV^2(\Omega;\mathbb{R}^3)$ such that $u_n \to g, \frac{1}{\epsilon_n}\nabla_3 u_n \to b, \nabla u_n \to (G,0)$ in L^1. In fact, given $(g,G) \in BV(\omega\ \mathbb{R}^3) \times BV(\omega;\mathbb{R}^{3\times 2})$ and $b \in BV(\omega;\mathbb{R}^3)$, by Theorem 2.2 there exists $h \in SBV(\omega;\mathbb{R}^3)$ such that $\nabla h(x_\alpha) = G(x_\alpha)$ a.e. $x_\alpha \in w$ and

$$||D^s h||(\omega) \le C_1 ||G||_{L^1(\omega;\mathbb{R}^3)} \tag{3.1}$$

for some $C_1 \equiv C_1(N) > 0$. By Lemma 2.1, there exist $\{v_n\}$ piecewise constant such that

$$v_n \underset{L^1(\omega;\mathbb{R}^3)}{\longrightarrow} g - h \quad \text{and} \quad ||Dv_n||(\omega) = ||D^s v_n||(\omega) \to ||Dg - Dh||(\omega).$$

Define $u_n \in SBV(\Omega;\mathbb{R}^3)$ by $u_n(x_\alpha, x_3) := v_n(x_\alpha) + h(x_\alpha) + \epsilon_n b(x_\alpha) x_3$. Clearly $\nabla_\alpha u_n(x) \xrightarrow{L^1} G(x_\alpha),\ u_n \to g$ in $L^1(\Omega;\mathbb{R}^3)$ and $\frac{\nabla_3 u_n}{\epsilon_n} \to b$ in $L^1(\Omega;\mathbb{R}^3)$.

We show next that our second order functional $I(g,b,G)$ can in fact be decomposed into two first order functionals. This is mainly due to Theorem 2.2.

For $(g,b,G) \in BV^2(\omega;\mathbb{R}^3) \times BV(\omega;\mathbb{R}^3) \times BV(\omega;\mathbb{R}^{3\times 2})$ we write:

$$\begin{aligned} I(g,b,G) = \inf_{u_n \in SBV^2(\Omega;\mathbb{R}^3)} \Bigg\{ \liminf_{\epsilon_n \to 0} \Bigg[& \int_\Omega W(\nabla_\alpha u_n | b_n, \nabla^2_{\alpha,\beta} u_n | \nabla_\alpha b_n | \frac{1}{\epsilon_n} \nabla^2_{3\beta} u_n | \frac{1}{\epsilon_n} \nabla_3 b_n) dx \\ & + \int_{S_{u_n}} \Psi_1([u_n], \nu_\alpha(u_n) | \frac{1}{\epsilon_n} \nu_3(u_n))\, d\mathcal{H}^2 \\ & + \int_{S_{\nabla u_n}} \Psi_2([\nabla_\alpha u_n] | [b_n], \nu_\alpha(\nabla u_n) | \frac{1}{\epsilon_n} \nu_3(\nabla u_n))\, d\mathcal{H}^2 \Bigg] \\ & u_n \xrightarrow{L^1} g,\ \ b_n \xrightarrow{L^1} b,\ \ \nabla_\alpha u_n \xrightarrow{L^1} G \Bigg\}, \end{aligned}$$

where,

$$b_n := \frac{\nabla_3 u_n}{\epsilon_n}$$

We will show that we can decompose $I(g,b,G)$ into

$$I(g,b,G) = I_1(g,b,G) + I_2(b,G)$$

with

$$I_1(g,b,G) = \inf_{u_n \in SBV^2(\Omega;\mathbb{R}^3)} \left\{ \liminf_{\epsilon_n \to 0} \int_{S_{u_n}} \Psi_1([u_n], \nu_\alpha(u_n)|\frac{1}{\epsilon_n}\nu_3(u_n))\, d\mathcal{H}^2 \right.$$

$$\left. u_n \xrightarrow{L^1} g,\ \frac{1}{\epsilon_n}\nabla_3 u_n \xrightarrow{L^1} b,\ \nabla_\alpha u_n \xrightarrow[n\to\infty]{L^1} G \right\},$$

and

$$I_2(b,G) = \inf_{h_n \in SBV(\Omega;\mathbb{R}^{3\times 3})} \left\{ \liminf_{\epsilon_n \to 0} \left[\int_\Omega W(h_n, \nabla_\alpha h_n|\frac{1}{\epsilon_n}\nabla_3 h_n)\, dx \right.\right.$$

$$\left. + \int_{S_{h_n}} \Psi_2([h_n], \nu_\alpha(h_n)|\frac{1}{\epsilon_n}\nu_3(h_n))\, d\mathcal{H}^2 \right]$$

$$\left. h_n \xrightarrow[n\to\infty]{L^1} (b,G) \right\}.$$

If we put $v_n = \nabla_\alpha u_n$ and $h_n = (v_n, b_n)$ it is immediate to see that

$$I(g,b,G) \geq I_1(g,b,G) + I_2(b,G).$$

Next we prove the opposite inequality. Let $u_n \in SBV^2(\Omega;\mathbb{R}^d)$ and $\epsilon_n \to 0$ with

$$u_n \xrightarrow[n\to\infty]{L^1} g,\ \frac{1}{\epsilon_n}\nabla_3 u_n \xrightarrow[n\to\infty]{L^1} b,\ \nabla_\alpha u_n \xrightarrow[n\to\infty]{L^1} G$$

be such that

$$I_1(g,b,G) = \lim_{n\to\infty} \int_{S_{u_n}} \Psi_1([u_n], \nu_\alpha(u_n)|\frac{1}{\epsilon_n}\nu_3(u_n))\, d\mathcal{H}^2(x).$$

Now let $v_n \in SBV(\Omega;\mathbb{R}^{3\times 2}),\ b_n \in SBV(\Omega;\mathbb{R}^3)$ with

$$v_n \xrightarrow[n\to\infty]{L^1} G,\ b_n \xrightarrow[n\to\infty]{L^1} b$$

be such that, setting $h_n = (v_n, b_n)$,

$$I_2(b,G) = \lim_{n\to\infty} \left[\int_\Omega W(h_n, \nabla_\alpha h_n|\frac{1}{\epsilon_n}\nabla_3 h_n)\, dx \right.$$

$$\left. + \int_{S_{h_n}} \Psi_2([h_n], \nu_\alpha(h_n)|\frac{1}{\epsilon_n}\nu_3(h_n))\, d\mathcal{H}^2 \right].$$

Remark 3.1. The equalities above for I_1 and I_2 can be attained if we pass to a subsequence of $\{\epsilon_n\}$ (for which we still use the same notation). In the next sections we will prove that I_1 and I_2 are independent of the sequence ϵ_n.

We now proceed in two steps
Step 1: Define

$$w_n^1 := u_n + \epsilon_n \phi_n^1,$$

where, by Theorem 2.2, ϕ_n^1 satisfies:

$$\nabla \phi_n^1 = (v_n - \nabla_\alpha u_n, 0), \tag{3.2}$$

$$|D\phi_n^1|(\Omega) \leq \int_\Omega |v_n - \nabla_\alpha u_n| \, dx, \tag{3.3}$$

and

$$\phi_n^1 \to 0 \text{ in } L^1. \tag{3.4}$$

Notice that clearly w_n^1 is still an admissible sequence for $I_1(g, b, G)$. In fact

$$I_1(g,b,G) = \lim_{n\to\infty} \int_{S_{w_n^1}} \Psi_1([w_n^1], \nu_\alpha(w_n^1)|\frac{1}{\epsilon_n}\nu_3(w_n^1)) \, d\mathcal{H}^2,$$

since, by (H_4), by Remark (1.1), (3.2) and (3.3) the extra term in the energy is controlled by

$$\Big(\sqrt{|\nu_\alpha(\phi_n^1)|^2 + \frac{1}{\epsilon_n^2}|\nu_3(\phi_n^1)|^2}\Big)\epsilon_n \int_\Omega |v_n - \nabla_\alpha u_n| \, dx,$$

which goes to zero as $\epsilon_n \to 0$ since $(v_n - \nabla_\alpha u_n) \underset{n\to\infty}{\overset{L^1}{\longrightarrow}} 0$.
Notice that

$$\nabla_\alpha w_n^1 - v_n = (1 - \epsilon_n)\left(\nabla_\alpha u_n - v_n\right).$$

Iterating the process, we construct functions ϕ_n^k, using Theorem (2.2) by:

$$\nabla \phi_n^k = (v_n - \nabla_\alpha u_n - \epsilon_n \sum_{j=1}^{k-1} \phi_n^j, 0), \tag{3.5}$$

$$|D\phi_n^k|(\Omega) \leq (1-\epsilon_n)^{k-1} \int_\Omega |\nabla_\alpha u_n - v_n| \, dx \tag{3.6}$$

and

$$\phi_n^k \underset{n\to\infty}{\overset{L^1}{\longrightarrow}} 0. \tag{3.7}$$

Now define a sequence

$$w_n^k := u_n + \epsilon_n \phi_n^k$$

where k $(= k_n)$ will be chosen in the next step.
Note that

$$\nabla_\alpha w_n^k - v_n = (1-\epsilon_n)^k \left(\nabla_\alpha u_n - v_n\right).$$

Notice that $w_n^{k_n}$ is still a sequence realizing the value for $I_1(g, b, G)$.

Step 2: For $\bar{w}_n := w_n^{k_n}$ (k_n to be defined latter), let ϕ_n^α, ϕ_n^3 once again obtained by Theorem 2.2 be such that

$$\phi_n^\alpha \xrightarrow[n\to\infty]{L^1} 0, \tag{3.8}$$

$$\nabla \phi_n^\alpha = (v_n - \nabla_\alpha \bar{w}_n, 0), \quad |D\phi_n^\alpha|(\Omega) \leq \int_\Omega |\nabla_\alpha \bar{w}_n - v_n| \, dx, \tag{3.9}$$

$$\phi_n^3 \xrightarrow[n\to\infty]{L^1} 0, \tag{3.10}$$

and

$$\nabla \phi_n^3 = (0, b_n - \frac{\nabla_3 \bar{w}_n}{\epsilon_n}) = (0, b_n - \frac{\nabla_3 u_n}{\epsilon_n}), \quad |D\phi_n^3|(\Omega) \leq \int_\Omega |b_n - \frac{\nabla_3 u_n}{\epsilon_n}| \, dx. \tag{3.11}$$

Set:

$$\tilde{w}_n = \bar{w}_n + \phi\alpha_n + \epsilon_n \phi_n^3,$$

which is an admissible sequence for $I(g, b, G)$. Hence,

$$
\begin{aligned}
I(g,b,G) \;\le\; & \liminf_{n\to\infty}\left\{\int_\Omega W(\nabla_\alpha \tilde{w}_n|b_n, \nabla^2_{\alpha,\beta}\tilde{w}_n|\nabla_\alpha b_n|\frac{1}{\epsilon_n}\nabla^2_{3\beta}\tilde{w}_n|\frac{1}{\epsilon_n}\nabla_3 b_n)\,dx\right.\\
& + \int_{S_{w_n}} \Psi_1([\tilde{w}_n],\nu_\alpha(\tilde{w}_n)|\frac{1}{\epsilon_n}\nu_3(\tilde{w}_n))\,d\mathcal{H}^2\\
& \left. + \int_{S_{\nabla\tilde{w}_n}} \Psi_2([\nabla_\alpha \tilde{w}_n]|[b_n],\nu_\alpha(\nabla\tilde{w}_n)|\frac{1}{\epsilon_n}\nu_3(\nabla\tilde{w}_n))\,d\mathcal{H}^2\right\}\\
& \le \liminf_{n\to\infty}\left\{\int_{S_{u_n}} \Psi_1([\bar{w}_n],\nu_\alpha(\bar{w}_n)|\frac{1}{\epsilon_n}\nu_3(\bar{w}_n))\,d\mathcal{H}^2\right.\\
& + C\Big(\sqrt{|\nu_\alpha(\phi_n^\alpha)|^2 + \frac{1}{\epsilon_n^2}|\nu_3(\phi_n^\alpha)|^2}\Big)(1-\epsilon_n)^{k_n}\int_\Omega |v_n - \nabla_\alpha u_n|\,dx\\
& + C\Big(\sqrt{|\nu_\alpha(\phi_n^3)|^2 + \frac{1}{\epsilon_n^2}|\nu_3(\phi_n^3)|^2}\Big)\epsilon_n\int_\Omega |b_n - \frac{\nabla_3 u_n}{\epsilon_n}|\,dx\\
& + \int_\Omega W(h_n,\nabla_\alpha h_n\,\frac{1}{\epsilon_n}\nabla_3 h_n)\,dx\\
& \left. + \int_{S_{h_n}} \Psi_2([h_n],\nu_\alpha(h_n)|\frac{1}{\epsilon_n}\nu_3(h_n))\,d\mathcal{H}^2\right\}\\
& \le I_1(g,b,G) + I_2(b,G)\\
& + \limsup_{n\to\infty}\left\{C\Big(\sqrt{|\nu_\alpha(\phi_n^\alpha)|^2 + \frac{1}{\epsilon_n^2}|\nu_3(\phi_n^\alpha)|^2}\Big)\epsilon_n\frac{(1-\epsilon_n)^{k_n}}{\epsilon_n}\int_\Omega |v_n - \nabla_\alpha u_n|\,dx\right.\\
& \left. + C\Big(\sqrt{|\nu_\alpha(\phi_n^3)|^2 + \frac{1}{\epsilon_n^2}|\nu_3(\phi_n^3)|^2}\Big)\epsilon_n\int_\Omega |b_n - \frac{\nabla_3 u_n}{\epsilon_n}|\,dx\right\}.
\end{aligned}
$$

The desired inequality follows if we assume that k_n satisfies the condition that

$$\frac{(1-\epsilon_n)^{k_n}}{\epsilon_n} \text{ is a bounded sequence.} \tag{3.12}$$

In what follows we will therefore treat separately $I_1(g,b,G)$ and $I_2(b,G)$.

4 Integral representation of $I_1(g,b,G)$

4.0.1 Lower bound

Fix $\{\epsilon_n\}$ and denote by

$$I_1{}^{\epsilon_n}(g,b,G) = \inf_{u_n\in SBV^2(\Omega;\mathbb{R}^3)}\left\{\liminf_{\epsilon_n\to 0}\int_{S_{u_n}}\Psi_1([u_n],\nu_\alpha(u_n)|\frac{1}{\epsilon_n}\nu_3(u_n))\,d\mathcal{H}^2\right.$$

$$\left. u_n \xrightarrow{L^1} g,\ \frac{1}{\epsilon_n}\nabla_3 u_n \xrightarrow{L^1} b,\ \nabla_\alpha u_n \xrightarrow[n\to\infty]{L^1} G\right\},$$

Given an arbitrary sequence $u_n \xrightarrow[n\to\infty]{L^1} g$ with $\nabla_\alpha u_n \xrightarrow[n\to\infty]{L^1} G$ and $\frac{1}{\epsilon_n}\nabla_3 u_n \xrightarrow[n\to\infty]{L^1} b$, define

$$L_1 := \liminf_{n\to\infty}\int_{S_{u_n}\cap\Omega}\Psi_1([u_n],\nu_\alpha(u_n)|\frac{1}{\epsilon_n}\nu_3(u_n))\,d\mathcal{H}^2.$$

Then clearly

$$L_1 \geq \liminf_{n\to\infty}\int_{S_{u_n}\cap\Omega}\bar{\Psi}_1([u_n],\nu_\alpha(u_n))\,d\mathcal{H}^2.$$

For x_3 fixed set now $u_n^{x_3}(x_\alpha) := u_n(x_\alpha,x_3)$. Then, by Theorem 3.1.1 in [6],

$$\begin{aligned} L_1 &\geq \liminf_{n\to\infty}\int_{S_{u_n}\cap\Omega}\bar{\Psi}_1([u_n],\nu_\alpha(u_n))\,d\mathcal{H}^2 \\ &\geq \liminf_{n\to\infty}\int_0^1\int_{S_{u_n^{x_3}}\cap\omega}\bar{\Psi}_1([u_n^{x_3}],\nu_\alpha(u_n^{x_3}))\,d\mathcal{H}^1(x_\alpha)\,dx_3. \end{aligned}$$

Note that

$$u_n^{x_3} \xrightarrow[n\to\infty]{L^1} g, \qquad \nabla u_n^{x_3} \xrightarrow[n\to\infty]{L^1} G.$$

Hence, by Fatou's Lemma and the integral representation in Theorem 1.2 (see Remark 4.1 and 4.2), and using the arbitrariness of the sequence u_n we arrive at the inequality below

$$I_1{}^{\epsilon_n}(g,b,G)(w) \geq \int_w \hat{W}_1^{(2)}(G-\nabla g)\,dx_\alpha + \int_{S_g}\hat{\gamma}_1^{(2)}([g],\nu(g))\,d\mathcal{H}^1 + \int_\omega \hat{W}_1^{(2)}(-dD^c g),$$

where

$$\hat{W}_1^{(2)}(A) = \inf_{u\in SBV(Q';\mathbb{R}^3)}\left\{\int_{Su\cap Q'}\bar{\Psi}_1([u],\nu(u))\,d\mathcal{H}^1, u|_{\partial Q} = 0, \nabla u = A \text{ a.e. in } Q'\right\},$$

and

$$\hat{\gamma}_1^{(2)}(\lambda,\nu) = \inf_{u\in SBV(Q'_\nu;\mathbb{R}^3)} \left\{ \int_{Q'_\nu \cap S_u} \bar{\Psi}_1([u],\nu(u))\, d\mathcal{H}^1, u|_{\partial Q'_\nu} = \gamma_{(\lambda,\nu)}, \nabla u = 0 \right\},$$

with

$$\gamma_{(\lambda,\nu)}(x_\alpha) := \begin{cases} \lambda & \text{if } x_\alpha.\nu > 0 \\ \\ 0 & \text{if } x_\alpha.\nu < 0. \end{cases}$$

Remark 4.1. We note that $\bar{\Psi}_1$ is continuous. In fact, if $(\lambda_n,\nu_n) \to (\lambda,\nu)$ in $\mathbb{R}^3 \times S^1$ and if we assume that $\lim_{n\to\infty} \bar{\Psi}_1(\lambda_n,\nu_n) = \liminf_{n\to\infty} \bar{\Psi}_1(\lambda_n,\nu_n)$ we get that

$$\lim_{n\to\infty} \bar{\Psi}_1(\lambda_n,\nu_n) = \lim_{n\to\infty} \Psi_1(\lambda_n,\tau_n) = \Psi_1(\lambda,\tau)$$

where $\tau_n = \frac{(\nu_n,t_n)}{\sqrt{(|\nu_n|^2+t_n^2)}}$ is a sequence in S^2 which we assume to be convergent to a point $\tau \in S^2$. On the other hand if we write $\tau = \lim_{n\to\infty} \frac{(\nu,t_n)}{\sqrt{(|\nu|^2+t_n^2)}}$ and use the definition of $\bar{\Psi}_1$ we get that

$$\Psi_1(\lambda,\tau) = \lim_{n\to\infty} \frac{\Psi_1(\lambda,\nu,t_n)}{\sqrt{(|\nu|^2+t_n^2)}} \geq \bar{\Psi}_1(\lambda,\nu),$$

which together with the previous equality gives the lower semicontinuity of $\bar{\Psi}_1$. For the upper semicontinuity of $\bar{\Psi}_1$ we consider again a sequence $(\lambda_n,\nu_n) \to (\lambda,\nu)$ in $\mathbb{R}^3 \times S^1$ and t_n such that $\bar{\Psi}_1(\lambda,\nu) = \lim_{n\to\infty} \frac{\Psi_1(\lambda,\nu,t_n)}{\sqrt{(|\nu|^2+t_n^2)}}$ and thus

$$\limsup_{n\to\infty} \bar{\Psi}_1(\lambda_n,\nu_n) \leq \lim_{n\to\infty} \frac{\Psi_1(\lambda_n,\nu_n,t_n)}{\sqrt{(|\nu_n|^2+t_n^2)}} = \lim_{n\to\infty} \frac{\Psi_1(\lambda,\nu,t_n)}{\sqrt{(|\nu|^2+t_n^2)}} = \bar{\Psi}_1(\lambda,\nu).$$

Remark 4.2. We do not know if $\bar{\Psi}_1$ inherits the subadditivity from Ψ_1 but it keeps the Lipschitz continuity and the growth conditions from Ψ_1 which are also sufficient to apply Theorem 1.2.

4.0.2 Upper bound

In order to derive the upper bound for ${I_1}^{\epsilon_n}(g,b,G)$ we rely on Theorem 11 of [6] which is an application of the Global method in SBV_p to Brittle Thin Films. More precisely, defining for $\delta > 0$

$$\begin{aligned}\mathcal{F}_\delta(g,A) &= \inf_{\{g_n\}}\Bigg\{\liminf_{\epsilon_n\to 0}\delta\int_{A\times I}\left|\left(\nabla_\alpha g_n|\frac{\nabla_3 g_n}{\epsilon_n}\right)\right|^2\,dx\\ &\quad+\int_{S_{g_n}\cap(A\times I)}\delta+\Psi_1\left([g_n],\nu_\alpha(g_n),\frac{1}{\epsilon_n}\nu_3(g_n)\right)\,d\mathcal{H}^2\\ &\quad g_n\overset{L^1}{\longrightarrow} g\Bigg\},\end{aligned}$$

Theorem 11 in [6] implies that for $A \in \mathcal{A}(w)$

$$\mathcal{F}_\delta(g;A) = \delta\int_A|\nabla g|^2\,dx_\alpha + \int_{S_g\cap A}(\delta+\mathcal{R}_2\bar{\Psi}_1([g],\nu(g))\,d\mathcal{H}^1,$$

where $\mathcal{R}_2$ denotes the BV-elliptic envelope.
Therefore since

$$I_1{}^{\epsilon_n}(g,\nabla g;\omega)\le\mathcal{F}_\delta(g;\omega),$$

by letting $\delta\to 0^+$ and using the inequality $\mathcal{R}_2\bar{\Psi}_1\le\bar{\Psi}_1$, we conclude that

$$I_1{}^{\epsilon}(g,\nabla g)\le\int_{S_g}\bar{\Psi}_1([g],\nu(g))\,d\mathcal{H}^1.$$

Consider now $g \in BV^2(w;\mathbb{R}^3)$, $G \in BV(w;\mathbb{R}^{3\times 2})$ and let $g_n \in SBV^2(w;\mathbb{R}^3)$ be such that

$$g_n\overset{L^1}{\longrightarrow} g,\ \nabla g_n\overset{L^1}{\longrightarrow} G.$$

Then

$$\begin{aligned}I_1{}^{\epsilon_n}(g,G;A) &\le \liminf_{n\to\infty} I_1{}^{\epsilon_n}(g_n,\nabla g_n;A)\\ &\le \liminf_{n\to\infty}\int_{S_{g_n}\cap A}\bar{\Psi}_1([g_n],\nu(g_n))\,d\mathcal{H}^1.\end{aligned}$$

Therefore, taking into account the integral representation obtained in [7],

$$I_1{}^{\epsilon_n}(g,G)(w)\le\int_w\hat{W}_1^{(2)}(G-\nabla g)\,dx_\alpha+\int_{S_g}\hat{\gamma}_1^{(2)}([g],\nu(g))\,d\mathcal{H}^1+\int_\omega\hat{W}_1^{(2)}(-dD^c g),$$

for $\hat{W}_1^{(2)}$ and $\hat{\gamma}_1^{(2)}$ defined previously. This together with the lower bound gives the desired characterization. In fact I_1 only depends on (g,G) and does not depend on ϵ_n (thus we drop the superscrit) and obtain the result stated in Theorem 1.5.

5 Integral representation of $I_2(b, G)$

Again, for a given sequence $\{\epsilon_n\}$ and $(b, G) \in BV(w; \mathbb{R}^3) \times BV(w; \mathbb{R}^{3\times 2})$, we define

$$
\begin{aligned}
I_2^{\epsilon_n}(b, G) \quad = \quad & \inf_{h_n \in SBV(\Omega;\mathbb{R}^{3\times 3})} \Bigg\{ \liminf_{\epsilon_n \to 0} \Bigg[\int_\Omega W(h_n, \nabla_\alpha h_n | \frac{1}{\epsilon_n} \nabla_3 h_n)\, dx \\
& + \int_{S_{h_n}} \Psi_2([h_n], \nu_\alpha(h_n) | \frac{1}{\epsilon_n} \nu_3(h_n))\, d\mathcal{H}^2(x) \Bigg] \\
& h_n \xrightarrow[n\to\infty]{L^1} (G, b) \Bigg\}.
\end{aligned}
$$

5.1 Lower bound

Let $h_n \to (G, b)$ be an arbitrary sequence and for fixed $\{\epsilon_n\}$ define

$$
\begin{aligned}
L_2 : \quad = \quad & \liminf_{n\to\infty} \Bigg[\int_\Omega W(h_n, \nabla_\alpha h_n | \frac{1}{\epsilon_n} \nabla_3 h_n)\, dx \\
& + \int_{S_{h_n}} \Psi_2([h_n], \nu_\alpha(h_n) | \frac{1}{\epsilon_n} \nu_3(h_n))\, d\mathcal{H}^2(x) \Bigg].
\end{aligned}
$$

Clearly,

$$
\begin{aligned}
L_2 \quad \geq \quad & \liminf_{n\to\infty} \Bigg[\int_\Omega \bar{W}_2(h_n, \nabla_\alpha h_n)\, dx \\
& + \int_{S_{h_n}} \bar{\Psi}_2([h_n], \nu_\alpha(h_n))\, d\mathcal{H}^2(x) \Bigg].
\end{aligned}
$$

Let now x_3 be fixed and set $h_n^{x_3}(x_\alpha) = h_n(x_\alpha, x_3)$. Then, by Theorem 3.1.1 of [6],

$$
\begin{aligned}
L_2 \geq & \liminf_{n\to\infty}\left[\int_\Omega \bar{W}_2(h_n,\nabla_\alpha h_n)\,dx\right.\\
&\left.+\int_{S_{h_n}}\bar{\Psi}_2([h_n],\nu_\alpha(h_n))\,d\mathcal{H}^2(x)\right]\\
&\geq \liminf_{n\to\infty}\left[\int_0^1\int_w \bar{W}_2(h_n,\nabla_\alpha h_n^{x_3})\,dx_\alpha\,dx_3\right.\\
&\left.+\int_0^1\int_{S_{h_n^{x_3}}\cap w}\bar{\Psi}_2([h_n^{x_3}],\nu_\alpha(h_n^{x_3}))\,d\mathcal{H}^1(x_\alpha)\,dx_3\right].
\end{aligned}
$$

Noting that $h_n^{x_3} \xrightarrow[n\to\infty]{L^1} (b,G)$, by Fatou's Lemma a lower bound for $I_2{}^{\epsilon_n}(b,G)$ will be given by Theorem 4.2.2 in [5].

5.2 Upper bound

Let $A\in\mathcal{A}(w)$. We write $h=(b,G)$ and consequently

$$
\begin{aligned}
I_2(h;A) = & \inf_{h_n\in SBV(\Omega;\mathbb{R}^{3\times 3})}\left\{\liminf_{\epsilon_n\to 0}\int_{A\times I} W(h_n,\nabla_\alpha h_n|\frac{1}{\epsilon_n}\nabla_3 h_n)\,dx\right.\\
&\left.+\int_{S_{h_n}\cap(A\times I)}\Psi_2([h_n],\nu_\alpha(h_n)|\frac{1}{\epsilon_n}\nu_3(h_n))\,d\mathcal{H}^2,\ h_n \xrightarrow[n\to\infty]{L^1} h\right\}.
\end{aligned}
$$

For a fixed sequence $\{\epsilon_n\}$ we write $I_2^{\epsilon_n}(h;A)$ as before. Now observe that $I_2{}^{\epsilon_n}$ satisfies the conditions $i)$ to $iv)$ of Theorem 2.4. In fact, following the arguments in [5] it is easy to check that $I_2{}^{\epsilon_n}(h;.)$ is the restriction of a bounded Radon measure to $\mathcal{A}(\omega)$. We just sketch the proof of subadditivity property.

Lemma 5.1. *Let $A,B,C\in\mathcal{A}(\omega)$ with $A\subset\subset B\subset C$. Then*

$$
I_2{}^{\epsilon_n}(h;C)\leq I_2{}^{\epsilon_n}(h;B)+I_2{}^{\epsilon_n}(h;C\backslash\bar{A}). \tag{5.1}
$$

Proof. We note that for fixed h and $A\subset\omega$ we can find a subsequence $\{\epsilon_n\}$ (not relabelled) such that

$$
\begin{aligned}
I_2{}^{\epsilon_n}(h;A) = & \lim_{n\to\infty}\int_{A\times I} W(h_n,\nabla_\alpha h_n|\frac{1}{\epsilon_n}\nabla_3 h_n)\,dx\\
&+\int_{S_{h_n}\cap(A\times I)}\Psi_2([h_n],\nu_\alpha(h_n)|\frac{1}{\epsilon_n}\nu_3(h_n))\,d\mathcal{H}^2.
\end{aligned}
$$

By repeating this argument, and considering a diagonal subsequence, we arrive at countable class of open subsets (A_i) of ω and a subsequence $\{\varepsilon_n\}$ (not relabelled) such that there exists $\{h_n^{(i)}\}$ verifying

$$h_n^{(i)} \xrightarrow[n\to\infty]{L^1} h$$

and

$$\begin{aligned} I_2{}^{\epsilon_n}(h;A_i) &= \lim_{n\to\infty} \int_{A_i\times I} W(h_n^{(i)}, \nabla_\alpha h_n^{(i)} | \frac{1}{\epsilon_n}\nabla_3 h_n^{(i)})\, dx \\ &\quad + \int_{S_{h_n^{(i)}}\cap(A_i\times I)} \Psi_2([h_n^{(i)}], \nu_\alpha(h_n^{(i)})| \frac{1}{\epsilon_n}\nu_3(h_n^{(i)}))\, d\mathcal{H}^2. \end{aligned}$$

For the sequence $\{A_i\}$ we can choose all the squares contained in ω with centers at points with rational coordinates and rational side length, and their unions. Now using a density argument together with bounds we can extend the previous equality to all open subsets of ω. To finish the proof we can use the same argument as in [7], together with the argument above to choose an appropriate subsequence of $\{\varepsilon_n\}$ (such that the equality holds for every open set).

□

By a diagonalization argument it is easy to check that $I_2{}^{\epsilon_n}(.;A)$ is lower semicontinuous with respect to $L^1(A;\mathbb{R}^d)$ convergence.

By using the constant sequence $h_n = h$ and the lower semicontinuity of the total variation, we get the upper and the lower bounds, respectively, as follows

$$\frac{1}{C}|Dh|(A) \le I_2{}^{\epsilon_n}(h;A) \le C(\mathcal{L}^2(A) + |Dh|(A)).$$

We also have, due to (H_2), the following property of continuity by translation

$$|I_2{}^{\epsilon_n}(h;A) - I_2{}^{\epsilon_n}(h(.-a)+b; b+A)| \le C|b|\mathcal{L}^2(A)$$

for $A \subset \omega$ such that $a + A \subset \omega$.

Properties i)-iv) allow us to apply Theorem 2.4 and obtain that $I_2{}^{\epsilon_n}$ has an integral representation of the form

$$I_2{}^{\epsilon_n}(h;A) = \int_A W_0(h,\nabla h)\, dx + \int_{A\cap S(h)} \Psi_0([h],\nu(h))\, d\mathcal{H}^1 + \int_A W_0^\infty(h, dC(h)).$$

We next prove that

$$W_0(a,F_\alpha) \le \bar{W}_2(a,F_\alpha),\ \forall a \in \mathbb{R}^{3\times 3},\ \forall F_\alpha \in \mathbb{R}^{3\times 3\times 2}, \tag{5.2}$$

and that

$$\Psi_0(\lambda,\nu_\alpha) \le \bar{\Psi}_2(\lambda,\nu_\alpha), \forall \lambda \in \mathbb{R}^{3\times 3},\ \forall \nu_\alpha \in S^1. \tag{5.3}$$

In order to prove (5.2) consider

$$h_n = a + F_\alpha(x_\alpha - x_0) + \epsilon_n F_3 x_n,$$

where $F_3 \in \mathbb{R}^3$ is such that

$$\bar{W}_2(a, F_\alpha) = W(a, F_\alpha, F_3).$$

Notice that, by (H_1) the infimum in the definition fo $\bar{W}_2$ is attained. Since

$$h_n \xrightarrow[n\to\infty]{L^1} h = a + F_\alpha(x_\alpha - x_0),$$

by the definition of I_2, we have that

$$\begin{aligned} I_2(a + F_\alpha(x_\alpha - x_0); Q(x_0,\delta)) &\le \liminf_{\epsilon_n \to 0} \int_{Q(x_0,\delta)} W(a + F_\alpha(x_\alpha - x_0) + \epsilon_n F_3 x_3, F_\alpha | F_3)\, dx \\ &\le \int_{Q(x_0,\delta)} W(a, F_\alpha, F_3)\, dx + C \int_{Q(x_0,\delta)} F_\alpha(x_\alpha - x_0)\, dx, \end{aligned}$$

where we used (H_2) in the last inequality. Dividing both terms in the previous inequality by δ^N and letting $\delta \to 0^+$, we end up with (5.2).

We proceed similarly to prove (5.3), now with

$$h_n = \begin{cases} \lambda & \text{if } \nu_\alpha \cdot (x_\alpha - x_0) + \epsilon_n \nu_3 > 0 \\ \\ 0 & \text{if } \nu_\alpha \cdot (x_\alpha - x_0) + \epsilon_n \nu_3 \le 0 \end{cases}$$

as test functions. Since $h_n \xrightarrow[n\to\infty]{L^1} u_{x_0}^{\lambda,\nu_\alpha}(x)$ given by

$$u_{x_0}^{\lambda,\nu_\alpha}(x) = \begin{cases} \lambda & \text{if } \nu_\alpha \cdot (x_\alpha - x_0) > 0 \\ \\ 0 & \text{if } \nu_\alpha \cdot (x_\alpha - x_0) \le 0 \end{cases}$$

we conclude that

$$\begin{aligned} I_2(u_{x_0}^{\lambda,\nu_\alpha}(x); Q(x_0,\delta)) \quad &\le \quad \liminf_{n\to\infty} \int_{Q(x_0,\delta)} W(h_n, 0)\, dx \\ &\quad + \Psi_2(\lambda, \nu_\alpha, \mu) |\{\nu_\alpha \cdot (x_\alpha - x_0) = 0 \cap (Q(x_0,\delta)\}| \end{aligned}$$

where $\mu \in \mathbb{R}$ such that $\bar{\Psi}_2(\lambda, \nu_\alpha) := \frac{1}{\sqrt{|\nu_\alpha|^2+\mu^2}}\Psi_2(\lambda, \nu_\alpha, \mu)$. The existence of such μ results from the coercivity condition (H_5). Dividing both terms by δ^{N-1} letting $\delta \to 0^+$ (and for x_0 a point in S_h)

The upper bound results now of relaxing from SBV to BV, together with an application of Theorem 4.2.2 in [5]. Combining the upper and the lower bound we have proved Theorem 1.6.

Remark 5.2. Notice that the integral representation of $I_2(b, G)$ is a generalization of the result derived by Braides and Fonseca in [4] for Brittle Thin Films.

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References

[1] Ambrosio, L., N. Fusco and D. Pallara, *Functions of Bounded Variation and Free Discontinuity Problems*, Oxford University Press, 2000.

[2] Alberti, G., A Lusin type Theorem for gradients, *J. Funct. Anal.* **100** (1991), pp. 110-118.

[3] A. Braides, *Γ-convergence for beginners*, Oxford Lecture Series in Mathematics and its Applications, 22, Oxford University Press, Oxford, 2002.

[4] Braides, A. and I. Fonseca, Brittle Thin Films, *Appl. Math. Optm.* **44** (2001) pp. 299-323.

[5] Bouchitté, G. I. Fonseca and L. Mascarenhas, A Global Method for Relaxation, *Arch. Rational Mech. Anal.* **145** (1998), pp. 51-98.

[6] Bouchitté, G. I. Fonseca, G. Leoni and L. Mascarenhas, A Global Method for Relaxation in $W^{1,p}$ and in SBV_p, *Arch. Rational Mech. Anal.* **165** (2002), pp. 187-242.

[7] Baía, M. P. M. Santos and J. Matias, A relaxation result in the framework of structured deformations, to appear.

[8] Choksi, R. and I.Fonseca, Bulk and Interfacial Energies for Structured Deformations of Continua, *Arch. Rational Mech. Anal.* **138** (1997), pp. 37-103.

[9] Carriero, M., A. Leaci and F. Tomarelli, A second order model in image segmentation: Blake and Zisserman Functional, *Progress in Nonlinear Diff. Equations* **25** (1996), pp. 57-72.

[10] Carriero, M., A. Leaci and F. Tomarelli, Second Order Variational Problems with Free Discontinuity and Free Gradient Discontinuity, *Calculus of Variations: Topics from the Mathematical Heritage of E. De Giorgi, Quad. Mat.* **14** (2004), pp. 135-186.

[11] De Giorgi, E. and L. Ambrosio, Un nuovo tipo di funzionale del calcolo delle variazioni, *Atti Accad. Naz. Lincei* **82** (1988), pp. 199-210.

[12] E. De Giorgi and G. Dal Maso, Γ-convergence and calculus of variations, *Mathematical theories of optimization* (1981), pp. 121-143, Lecture Notes in Math. **979**, Springer, Berlin, 1983.

[13] E. De Giorgi and T. Franzoni, Su un tipo di convergenza variazionale, *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur.* (8) **58** (1975), No. 6, pp. 842-850.

[14] Dal Maso, G. *An Introduction to* Γ*-convergence*, Birkhäuser, 1993.

[15] Del Piero, G., D. Owen, Structured Deformations of Continua, *Arch. Rational Mech. Anal.* **124** (1993), pp. 99-155.

[16] Evans, L. C. and R. F. Gariepy. *Measure Theory and Fine Properties of Functions*, Studies in Advanced Mathematics, CRC Press, 1992.

[17] Federer, H, *Geometric Measure Theory*, Springer, Berlin, 1969.

[18] Giusti, E. *Minimal Surfaces and Functions of Bounded Variation*, Birkhäuser, 1984.

[19] Owen, D. and R. Paroni, Second-order structured deformations, *Arch. Rational Mech. Anal.* **155** (2000), pp. 215-235.

[20] Ziemer, W. *Weakly Differentiable Functions*, Springer-Verlag, 1989.

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Author information

José Matias, Centro de Análise Matemática, Geometria e Sistemas Dinâmicos, Departamento de Matemática, Instituto Superior Técnico Av. Rovisco Pais 1, 1049-001 Lisboa, Portugal.
E-mail: `jmatias@math.ist.utl.pt`

Pedro Miguel Santos, Centro de Análise Matemática, Geometria e Sistemas Dinâmicos, Departamento de Matemática, Instituto Superior Técnico Av. Rovisco Pais 1, 1049-001 Lisboa, Portugal.
E-mail: `pmsantos@math.ist.utl.pt`