A relaxation result in the framework of structured deformations

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Abstract

Structured deformations are non-classical deformations of continua suitable for the description of materials whose kinematics requires analysis at both the macroscopic and microscopic levels. In this work we obtain an integral representation of an energy for structured deformations of continua as a first step to the study of thin defective cristalline structures.

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1 Introduction

The model established by Del Piero and Owen [18] for first order structured deformations set a basis to address problems in continuum mechanics where an analysis at macroscopic and microscopic levels is required, dividing the study of the deformations of continua in two parts: the part arising from smooth changes and the part due to slips and separations at smaller length scales (disarrangements). The first part is known to be associated with limits of gradients of approximating deformations, and the second one corresponds to their jump effects.

Choksi and Fonseca [11] extended the notion of structured deformation to the setting of special functions of bounded variation. Namely, the authors defined a structured deformation as a pair (g, G) where the macroscopic deformation g is an element of $SBV(\Omega;\mathbb{R}^d)$ and G is an integrable tensor field in Ω, and proved that given such a pair there exist deformations u_n in $SBV(\Omega;\mathbb{R}^d)$ with

$$u_n \xrightarrow{L^1} g \quad \text{and} \quad \nabla u_n \overset{\mathcal{M}(\Omega)}{\rightharpoonup} G,$$

and that in addition an integral representation of

$$I(g,G) := \inf_{\{u_n\}\subset SBV(\Omega;\mathbb{R}^d)} \left\{ \liminf_{n\to\infty} E(u_n),\ \ u_n \xrightarrow{L^1} g, \quad \nabla u_n \overset{\mathcal{M}(\Omega)}{\rightharpoonup} G \right\},$$

$$E(u) = \int_\Omega W(\nabla u)\, dx + \int_{S_u} \psi([u], \nu(u))\, d\mathcal{H}^{N-1}, \quad u \in SBV(\Omega;\mathbb{R}^d),$$

can be obtained in terms of some appropriate bulk and interfacial energy densities.

Here we consider a model for structured deformations in the full BV setting departing from a different initial energy which includes second order derivatives (see Carriero, Leaci and Tomarelli [12] and [13] for other second order variational problems). The reason for this approach relies on the subject of a forthcoming paper, where a model for thin defective cristalline structures is obtained by dimensional reduction techniques, and where to avoid the formation of holes in the target lower dimensional domain all the jumps in the approximating sequences must be properly aligned.

Precisely, we define the space of generalized structured deformations as

$$GSD(\Omega;\mathbb{R}^d) := BV^2(\Omega;\mathbb{R}^d) \times BV(\Omega;\mathbb{R}^{d\times N}),$$

and for any $(g, G) \in GSD(\Omega;\mathbb{R}^d)$ consider the relaxed energy

$$I(g,G) = \inf_{\{u_n\}\subset SBV^2(\Omega;\mathbb{R}^d)} \left\{ \liminf_{n\to\infty} E(u_n),\ u_n \xrightarrow{L^1} g,\ \nabla u_n \xrightarrow{L^1} G \right\} \tag{1.1}$$

where, for $u \in SBV^2(\Omega;\mathbb{R}^d)$,

$$E(u) = \int_\Omega W(\nabla u, \nabla^2 u)\, dx + \int_{S_u} \Psi_1([u], \nu(u))\, d\mathcal{H}^{N-1} + \int_{S_{\nabla u}} \Psi_2([\nabla u], \nu(\nabla u))\, d\mathcal{H}^{N-1}.$$

We assume the bulk and interfacial energy density functions to satisfy the following hypotheses

(H_1) : there exists $C > 0$ such that

$$\frac{1}{C}(|B|) - C \le W(A, B) \le C\big(1 + |B|\big)$$

for all $A \in \mathbb{R}^{d\times N}$ and $B \in \mathbb{R}^{d\times N\times N}$;

(H_2) : there exists $C > 0$ such that

$$|W(A_1, B_1) - W(A_2, B_2)| \le C\big(|A_1 - A_2| + |B_1 - B_2|\big)$$

for all $A_i \in \mathbb{R}^{d\times N}$ and $B_i \in \mathbb{R}^{d\times N\times N}$, $i = 1, 2$;

$(H_3):$ there exists $0<\alpha<1$ and $L>0$ such that

$$\Big|W^\infty(A,B)-\frac{W(A,tB)}{t}\Big|\le\frac{C}{t^\alpha}$$

for all $t>L$, $A\in\mathbb{R}^{d\times N}$, $B\in\mathbb{R}^{d\times N\times N}$ with $|B|=1$, where W^∞ denotes, as usual, the *recession function* of W in the variable B i.e.,

$$W^\infty(A,B)=\limsup_{t\to\infty}\frac{W(A,tB)}{t};$$

$(H_4):$ there exist $c_1>0$ and $C_1>0$ such that

$$c_1|\lambda|\le\Psi_1(\lambda,\nu)\le C_1|\lambda|$$

for all $\lambda\in\mathbb{R}^d$ and $\nu\in S^{N-1}$;

(H_5): there exist $c_2>0$ and $C_2>0$ such that

$$c_2|\Lambda|\le\Psi_2(\Lambda,\nu)\le C_2|\Lambda|$$

for all $\nu\in S^{N-1}$ and $\Lambda\in\mathbb{R}^{d\times N}$;

(H_6): (homogeneity of degree one)

$$\Psi_1(t\lambda,\nu)=t\Psi_1(\lambda,\nu),\qquad\Psi_2(t\Lambda,\nu)=t\Psi_2(\Lambda,\nu)$$

for all $\nu\in S^{N-1}$, $\lambda\in\mathbb{R}^d$, $\Lambda\in\mathbb{R}^{d\times N}$ and $t>0$;

(H_7): (sub-additivity)

$$\Psi_1(\lambda_1+\lambda_2,\nu)\le\Psi_1(\lambda_1,\nu)+\Psi_1(\lambda_2,\nu),$$
$$\Psi_2(\Lambda_1+\Lambda_2,\nu)\le\Psi_2(\Lambda_1,\nu)+\Psi_2(\Lambda_2,\nu)$$

for all $\nu\in S^{N-1}$, $\lambda_i\in\mathbb{R}^d$, $\Lambda_i\in\mathbb{R}^{d\times N}$, $i=1,2$.

Remark 1.1 We extend Ψ_i, $i=1,2$, as homogeneous functions of degree one in the second variable to all of $\mathbb{R}^N$.

Given $A,B,\Lambda,\Gamma\in\mathbb{R}^{d\times N}$, $C\in\mathbb{R}^{d\times N\times N}$, $\lambda\in\mathbb{R}^d$ and $\nu\in\mathcal{S}^{N-1}$ we define

- $$W_1(A)=\inf_{u\in SBV^2(Q;\mathbb{R}^d)}\left\{\int_{S_u}\Psi_1([u],\nu(u))\,d\mathcal{H}^{N-1},\,u|_{\partial Q}=0,\nabla u=A\text{ a.e. in }Q\right\}\tag{1.2}$$

- $$W_2(B,C)=\inf_{v\in SBV(Q;\mathbb{R}^{d\times N})}\left\{\int_Q W(B,\nabla v(x))\,dx+\int_{S_v}\Psi_2([v],\nu(v))\,d\mathcal{H}^{N-1},\,v|_{\partial Q}(x)=C\,x\right\}$$

- $$\gamma_1(\lambda,\nu)=\inf_{u\in SBV^2(Q_\nu;\mathbb{R}^d)}\left\{\int_{S_u}\Psi_1([u],\nu(u))\,d\mathcal{H}^{N-1},u|_{\partial Q_\nu}=\gamma_{(\lambda,\nu)},\,\nabla u=0\text{ a.e. in }Q_\nu\right\}\tag{1.3}$$

where

$$\gamma_{(\lambda,\nu)}(x) := \begin{cases} \lambda & \text{if } x\cdot\nu > 0 \\ 0 & \text{if } x\cdot\nu < 0 \end{cases}$$

- $\gamma_2(\Lambda,\Gamma,\nu) = \inf_{v\in SBV(Q_\nu;\mathbb{R}^{d\times N})} \left\{ \int_{Q_\nu} W^\infty(v,\nabla v)\,dx + \int_{S_v} \Psi_2([v],\nu(v))\,d\mathcal{H}^{N-1}, v|_{\partial Q_\nu} = \gamma_{(\Lambda,\Gamma,\nu)} \right\}$

where

$$\gamma_{(\Lambda,\Gamma,\nu)}(x) := \begin{cases} \Lambda & \text{if } x\cdot\nu > 0 \\ \Gamma & \text{if } x\cdot\nu < 0. \end{cases}$$

Our main result reads as follows.

Theorem 1.2 *Under hypotheses* $(H_1)-(H_7)$

$$\begin{aligned} I(g,G) \;&=\; \int_\Omega \Big(W_1(G-\nabla g) + W_2(G,\nabla G) \Big)\,dx + \int_{S_g} \gamma_1([g],\nu(g))d\mathcal{H}^{N-1} \\ &\quad + \int_{S_G} \gamma_2(G^+,G^-,\nu(G))\,d\mathcal{H}^{N-1} + \int_\Omega W_1\left(-\frac{dD^c g}{d|D^c g|}\right)\,d|D^c g| \\ &\quad + \int_\Omega W_2^\infty\left(G,\frac{dD^c G}{d|D^c G|}\right)\,d|D^c G| \end{aligned}$$

for all $(g,G)\in GSD(\Omega;\mathbb{R}^d)$.

Remark 1.3 It is easy to check that the recession function of W_2 in the second variable is given by

$$W_2^\infty(B,C) = \inf_{v\in SBV(Q;\mathbb{R}^{d\times N})} \left\{ \int_Q W^\infty(B,\nabla v(x))\,dx + \int_{S_v} \Psi_2([v],\nu(v))\,d\mathcal{H}^{N-1}, v|_{\partial Q}(x) = C\,x \right\}$$

for all $(B,C)\in\mathbb{R}^{d\times N}\times\mathbb{R}^{d\times N\times N}$.

To prove Theorem 1.2 we start by deriving a similar relaxation result in the SBV setting (see Theorem 3.1). Theorem 1.2 follows mainly from this representation and from Reshetnyak's Theorem (see Theorem 2.2). The analysis is based on an appropriate decomposition of $I(g,G)$ into two first order functionals. One of them captures the effect of the structured deformation through the energy density W_1. Its dependence on $G-\nabla g$ (deformation due to disarrangements at the microscopic level) is realized by the diffusion of jumps in the approximating sequence (see Del Piero and Owen [18], Choksi and Fonseca [11]). Our contribution is precisely the derivation of this relaxed energy since the other one comes from the results of Bouchitté, Fonseca and Mascarenhas [10].

The overall plan of this work in the ensuing sections will be as follows: Section 2 collects the main notations and results used throughout. Section 3 is devoted to the SBV case. The proof of Theorem 1.2 is obtained in Section 4.

2 Preliminaries

The purpose of this section is to give a brief overview of the concepts and results that are used in the sequel. All these results are stated without proofs as they can be readily found in the references given below.

2.1 Notation

Throughout the text $\Omega \subset \mathbb{R}^N$, $N \geq 1$, will denote an open bounded set and we will use the following notations:

- $\mathcal{A}(\Omega)$ is the family of all open subsets of Ω,
- $\mathcal{M}(\Omega)$ is the set of finite Radon measures on Ω,
- $||\mu||$ stands for the total variation of a measure $\mu \in \mathcal{M}(\Omega)$,
- S^{N-1} stands for the unit sphere in $\mathbb{R}^N$,
- e_i denotes the i^{th} element of the canonical basis of $\mathbb{R}^N$ for $i = 1, \dots N$,
- Q denotes the unit cube centered at the origin with one side orthogonal to e_N,
- $Q(x, \delta)$ denotes a cube centered at $x \in \Omega$ with side length δ and with one side orthogonal to e_N,
- $Q_\nu(x, \delta)$ is the cube centered at $x \in \Omega$ with side length δ and with one side orthogonal to $\nu \in S^{N-1}$,
- $Q_\nu := Q_\nu(0, 1)$,
- $\mathbb{R}^{d\times N\times N}$ is the set of real tensors of order $d \times N \times N$, $d \geq 1$,
- C represents a generic constant,
- $\lim\limits_{n,m} := \lim\limits_{n} \lim\limits_{m}$ while $\lim\limits_{m,n} := \lim\limits_{m} \lim\limits_{n}$.

2.2 Measure Theory

We start by recalling a generalization of the Besicovitch Differentiation Theorem due to Ambrosio and Dal Maso [4].

Theorem 2.1 *If λ and μ are Radon measures in Ω, $\mu \geq 0$, then there exists a Borel set $E \subset \Omega$ such that $\mu(E) = 0$, and for every $x \in \operatorname{supp} \mu \backslash E$*

$$\frac{d\lambda}{d\mu}(x) := \lim_{\epsilon \to 0} \frac{\lambda(x + \epsilon C)}{\mu(x + \epsilon C)}$$

exists and is finite whenever C is a bounded, convex, open set containing the origin.

We also recall Reshetnyak's Theorem on weak convergence of vector measures (see Reshetnyak [28]; see also Ambrosio, Fusco and Pallara [5]).

Theorem 2.2 *Let μ, μ_n be $\mathbb{R}^d-$valued finite Radon measures in Ω such that $\mu_n \overset{*}{\rightharpoonup} \mu$ in Ω and such that $||\mu_n||(\Omega) \to ||\mu||(\Omega)$. Then*

$$\lim_{n\to\infty} \int_\Omega f\left(x, \frac{\mu_n}{||\mu_n||}(x)\right) d||\mu_n||(x) = \int_\Omega f\left(x, \frac{\mu}{||\mu||}(x)\right) d||\mu||(x)$$

for every continuous and bounded function $f : \Omega \times S^{d-1} \to \mathbb{R}$.

2.3 BV-functions

In this part we briefly summarize some facts on functions of bounded variation that will be used afterwards. We refer to Ambrosio, Fusco and Pallara [5], Evans and Gariepy [19], Federer [20], Giusti [23] and Ziemer [29] for a detailed description of this subject.

A function $u \in L^1(\Omega;\mathbb{R}^d)$ is said to be of *bounded variation*, and we write $u \in BV(\Omega;\mathbb{R}^d)$, if all its first distributional derivatives $D_j u_i \in \mathcal{M}(\Omega)$ for $i = 1, ..., d$ and $j = 1, ..., N$. The matrix-valued measure whose entries are $D_j u_i$ is denoted by Du. The space $BV(\Omega;\mathbb{R}^d)$ is a Banach space when endowed with the norm

$$\|u\|_{BV} = \|u\|_{L^1} + ||Du||(\Omega).$$

By the Lebesgue Decomposition Theorem Du can be split into the sum of two mutually singular measures $D^a u$ and $D^s u$ (the absolutely continous part and singular part, respectively, of Du with respect to the Lebesgue measure $\mathcal{L}^N$). By ∇u we denote the Radon-Nikodým derivative of $D^a u$ with respect to $\mathcal{L}^N$, so that we can write

$$Du = \nabla u \mathcal{L}^N \lfloor \Omega + D^s u.$$

Let Ω_u be the set of points where the approximate limit of u exists, i.e., $x \in \Omega$ such that there exist $z \in \mathbb{R}^N$ with

$$\lim_{\varepsilon \to 0} \fint_{Q(x,\varepsilon)} |u(y) - z| \, dy = 0.$$

If $x \in \Omega_u$ and $z = u(x)$ we say that u is *approximately continuous* at x (or that x is a Lebesgue point of u). The function u is approximately continuous $\mathcal{L}^N$-a.e. $x \in \Omega_u$ and

$$\mathcal{L}^N(\Omega \setminus \Omega_u) = 0. \tag{2.4}$$

Let S_u be the *jump set* of this function, i.e., the set of points $x \in \Omega \setminus \Omega_u$ for which there exists a, $b \in \mathbb{R}^N$ and a unit vector $\nu \in S^{N-1}$, normal to S_u at x, such that $a \neq b$ and

$$\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon^N} \int_{\{y \in Q_\nu(x,\varepsilon) : \, (y-x)\cdot\nu > 0\}} |u(y) - a| \, dy = 0 \tag{2.5}$$

and

$$\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon^N} \int_{\{y \in Q_\nu(x,\varepsilon) : \, (y-x)\cdot\nu < 0\}} |u(y) - b| \, dy = 0. \tag{2.6}$$

The triple (a, b, ν) uniquely determined by (2.5) and (2.6) up to permutation of (a, b), and a change of sign of ν and is denoted by $(u^+(x), u^-(x), \nu_u(x))$.

If $u \in BV(\Omega)$ it is well known that S_u is countably $N-1$ rectifiable, i.e.

$$S_u = \bigcup_{n=1}^{\infty} K_n \cup E,$$

where $\mathcal{H}^{N-1}(E) = 0$ and K_n are compact subsets of C^1-hypersurfaces. Furthermore, $\mathcal{H}^{N-1}((\Omega \setminus \Omega_u) \setminus S_u) = 0$ and the following decomposition holds

$$Du = \nabla u \mathcal{L}^N \lfloor \Omega + [u] \otimes \nu_u \mathcal{H}^{N-1} \lfloor S_u + D^c u,$$

where $[u] := u^+ - u^-$ and $D^c u$ is the Cantor part of the measure Du, i.e., $D^c u = D^s u \lfloor (\Omega_u)$.

If Ω is an open and bounded set with Lipschitz boundary then the outer unit normal to $\partial\Omega$ (denoted by ν) exists $\mathcal{H}^{N-1}$ a.e. and the trace for functions in $BV(\Omega;\mathbb{R}^d)$ is defined.

We next recall some useful results on BV functions used in the sequel.

Theorem 2.3 *(Approximate Differentiability) If $u \in BV(\Omega;\mathbb{R}^d)$, then for $\mathcal{L}^N$-a.e. $x \in \Omega$*

$$\lim_{\epsilon\to 0} \frac{1}{\epsilon^N} \left\{ \int_{Q(x,\epsilon)} |u(y) - u(x) - \nabla u(x).(y-x)|^{\frac{N}{N-1}}\, dy \right\}^{\frac{N-1}{N}} = 0.$$

Lemma 2.4 *Let $u \in BV(\Omega;\mathbb{R}^d)$. There exist piecewise constant functions u_n such that $u_n \to u$ in $L^1(\Omega;\mathbb{R}^d)$ and*

$$||Du||(\Omega) = \lim_{n\to\infty} ||Du_n||(\Omega) = \lim_{n\to\infty} \int_{S_{u_n}} |[u_n](x)|\, dH^{N-1}(x).$$

The space of *special functions of bounded variation*, $SBV(\Omega;\mathbb{R}^d)$, introduced by De Giorgi and Ambrosio in [14] to study free discontinuity problems, is the space of functions $u \in BV(\Omega;\mathbb{R}^d)$ such that $D^c u = 0$, i.e. for which

$$Du = \nabla u \mathcal{L}^N + [u] \otimes \nu_u \mathcal{H}^{N-1} \lfloor S_u.$$

The next result is a Lusin type theorem for gradients due to Alberti [2] and is essential for our arguments.

Theorem 2.5 *Let $f \in L^1(\Omega;\mathbb{R}^{d\times N})$. There exists $u \in SBV(\Omega;\mathbb{R}^d)$ and a Borel function $g : \Omega \to \mathbb{R}^{d\times N}$ such that*

$$Du = f\mathcal{L}^N + g\mathcal{H}^{N-1}\lfloor S_u,$$

$$\int_{S_u} |g|\, d\mathcal{H}^{N-1} \leq C||f||_{L^1(\Omega;\mathbb{R}^{d\times N})}.$$

Remark 2.6 From the proof of Theorem 2.5 it follows also that

$$||u||_{L^1(\Omega)} \leq C||f||_{L^1(\Omega;\mathbb{R}^{d\times N})}.$$

The following technical result is a simplified version of Lemma 4.3 in Matias [26].

Lemma 2.7 *Let $\Omega \subset \mathbb{R}^N$ be open and bounded and let $A \in \mathbb{R}^{d\times N}$. Then there exists $u \in SBV(\Omega;\mathbb{R}^d)$ such that $u_{\lfloor \partial Q} = 0$ and $\nabla u = A$ a.e in Ω. In addition*

$$|D^s u|(\Omega) \leq C(N)|A|\,|\Omega|.$$

Following Carriero, Leaci and Tomarelli (see [12] and [13]) we define

$$SBV^2(\Omega;\mathbb{R}^d) = \{v \in SBV(\Omega;\mathbb{R}^d),\ \nabla v \in SBV(\Omega;\mathbb{R}^{d\times N})\}.$$

If $u \in SBV^2(\Omega;\mathbb{R}^d)$ we use the notation $\nabla^2 u = \nabla(\nabla u)$, that is, $\nabla^2 u$ is the absolutely continuous part of $D(\nabla u)$ with respect to Lebesgue measure. We will also denote by

$$BV^2(\Omega;\mathbb{R}^d) = \{v \in BV(\Omega;\mathbb{R}^d),\ \nabla v \in BV(\Omega;\mathbb{R}^{d\times N})\}.$$

3 Integral representation in SBV

In this section we derive an integral representation of I (see (1.1)) in the SBV setting. Precisely, we define the space of structured deformations as

$$SD((\Omega;\mathbb{R}^d) := SBV^2(\Omega;\mathbb{R}^d) \times SBV(\Omega;\mathbb{R}^{d\times N})$$

and our objective is to prove the following result.

Theorem 3.1 *Under hypotheses $(H_1)-(H_7)$ we have that for all $(g,G) \in SD((\Omega;\mathbb{R}^d)$*

$$\begin{aligned} I(g,G) &= \int_\Omega \Big(W_1(G-\nabla g) + W_2(G,\nabla G)\Big)\,dx + \int_{S_g} \gamma_1([g],\nu(g))d\mathcal{H}^{N-1} \\ &\quad + \int_{S_G} \gamma_2(G^+,G^-,\nu(G))\,d\mathcal{H}^{N-1}. \end{aligned} \tag{3.7}$$

Given $(g,G) \in SD((\Omega;\mathbb{R}^d)$ to prove (3.7) we start by defining the energy

$$I_1(g,G) = \inf_{\{u_n\}\subset SBV^2(\Omega;\mathbb{R}^d)} \left\{ \liminf_{n\to\infty} \int_{S_{u_n}} \Psi_1([u_n],\nu(u_n))\,d\mathcal{H}^{N-1},\ u_n \xrightarrow{L^1} g,\ \nabla u_n \xrightarrow{L^1} G \right\} \tag{3.8}$$

We note that under hypothesis (H_4) $I_1(g,G) < \infty$. In fact, by Theorem 2.5 there exists $h \in SBV(\Omega;\mathbb{R}^d)$ such that $\nabla h(x) = G(x)$ a.e. $x \in \Omega$ and

$$||D^s h||(\Omega) \le C||G||_{L^1(\Omega;\mathbb{R}^d)} \tag{3.9}$$

for some $C \equiv C(N) > 0$. By Lemma 2.4, there exist a sequence $\{v_n\}$ of piecewise constant functions such that

$$v_n \underset{n\to\infty}{\overset{L^1(\Omega;\mathbb{R}^d)}{\longrightarrow}} g-h \quad \text{and} \quad ||Dv_n||(\Omega) = ||D^s v_n||(\Omega) \underset{n\to\infty}{\longrightarrow} ||Dg - Dh||(\Omega).$$

Define $u_n \in SBV(\Omega;\mathbb{R}^d)$ by $u_n := v_n + h$. Clearly $\nabla u_n(x) = G(x)$ for a.e $x \in \Omega$ and $u_n \to g$ in $L^1(\Omega;\mathbb{R}^d)$. Thus by hypothesis (H_4) and by (3.9)

$$\begin{aligned} I_1(g,G) &\le \liminf_{n\to\infty} \int_{S_{u_n}} \Psi_1([u_n],\nu(u_n))\,d\mathcal{H}^{N-1} \\ &\le C \liminf_{n\to\infty} ||D^s u_n||(\Omega) \\ &\le C\left\{ \int_\Omega (|\nabla g| + |G|)\,dx + ||D^s g||(\Omega) \right\}. \end{aligned} \tag{3.10}$$

We now observe that $I(g,G)$ can be decomposed as

$$I(g,G) = I_1(g,G) + I_2(G) \tag{3.11}$$

where

$$I_2(G) = \inf_{\{v_n\}\subset SBV(\Omega;\mathbb{R}^{d\times N})} \left\{ \liminf_{n\to\infty} \int_\Omega W(v_n,\nabla v_n)\,dx + \int_{S_{v_n}} \Psi_2([v_n],\nu(v_n))\,d\mathcal{H}^{N-1},\ v_n \xrightarrow{L^1} G \right\}. \tag{3.12}$$

In fact, it is immediate to see that

$$I_1(g,G) + I_2(G) \leq I(g,G).$$

On the other hand, let $u_n \in SBV^2(\Omega;\mathbb{R}^d)$ with $u_n \to g$ in L^1 and $\nabla u_n \to G$ in L^1 such that

$$I_1(g,G) = \lim_{n\to\infty} \int_{S_{u_n}} \Psi_1([u_n],\nu(u_n))\, d\mathcal{H}^{N-1}$$

and let $v_n \in SBV(\Omega;\mathbb{R}^{d\times N})$ with $v_n \to G$ in L^1 such that

$$I_2(G) = \lim_{n\to\infty} \int_\Omega W(v_n,\nabla v_n)\, dx + \int_{S_{v_n}} \Psi_2([v_n],\nu(v_n))\, d\mathcal{H}^{N-1}.$$

By Theorem 2.5 let $h_n \in SBV(\Omega;\mathbb{R}^d)$ be such that $\nabla h_n = v_n - \nabla u_n$ and by Lemma 2.4 let $\tilde{h}_n$ be a piecewise constant function with $||\tilde{h}_n - h_n||_{L^1} < \frac{1}{n}$ and $|\,||D\tilde{h}_n||(\Omega) - ||Dh_n||(\Omega)| < \frac{1}{n}$. Then the sequence

$$w_n = u_n + h_n - \tilde{h}_n$$

is admissible for $I(g,G)$ and

$$\begin{aligned}
I(g,G) \quad &\leq \quad \liminf_{n\to\infty} \left[\int_\Omega W(\nabla w_n,\nabla^2 w_n)\, dx + \int_{S_{w_n}} \Psi_1([w_n],\nu(w_n))\, d\mathcal{H}^{N-1}\right.\\
&+ \quad \left.\int_{S_{\nabla w_n}} \Psi_2([\nabla w_n],\nu(\nabla w_n))\, d\mathcal{H}^{N-1}\right]\\
&\leq \quad \lim_{n\to\infty} \int_{S_{u_n}} \Psi_1([u_n],\nu(u_n))\, d\mathcal{H}^{N-1}\\
&+ \quad \limsup_{n\to\infty} \int_{S_{h_n}} \Psi_1([h_n],\nu(h_n))\, d\mathcal{H}^{N-1} + \limsup_{n\to\infty} \int_{S_{\tilde{h}_n}} \Psi_1([\tilde{h}_n],\nu(\tilde{h}_n))\, d\mathcal{H}^{N-1}\\
&+ \quad \lim_{n\to\infty} \left[\int_\Omega W(v_n,\nabla v_n)\, dx + \int_{S_{v_n}} \Psi_2([v_n],\nu(v_n))\, d\mathcal{H}^{N-1}\right]\\
&\leq \quad I_1(g,G) + I_2(G) + C\int_\Omega |v_n - \nabla u_n|\, dx
\end{aligned}$$

where we have used (H_4), (H_7), Theorem 2.5 and Lemma 2.4 in the last step. Inequality $I_1(g,G) + I_2(G) \geq I(g,G)$ follows by letting $n\to\infty$ since $v_n \to G$ and $\nabla u_n \to G$ in L^1.

From Theorem 4.1.4 in Bouchitté, Fonseca and Mascarenhas [10] the hypotheses on W and Ψ_2 lead to the following integral representation for I_2

$$I_2(G) \quad = \quad \int_\Omega W_2(G,\nabla G)\, dx + \int_{S_G} \gamma_2(G^+,G^-,\nu(G))\, d\mathcal{H}^{N-1}.$$

Therefore to prove (3.7) it is enough to show that

$$I_1(g,G) = \int_\Omega W_1(G-\nabla g)\, dx + \int_{S_g} \gamma_1([g],\nu(g)) d\mathcal{H}^{N-1}. \tag{3.13}$$

We divide the argument as follows. First we start by introducing a local version of $I_1(g,G)$ defined on $\mathcal{A}(\Omega)$ and show that $I_1(g,G,\cdot)$ is the restriction to $\mathcal{A}(\Omega)$ of a Radon measure absolutely continuous with respect to $\mathcal{L}^N + \mathcal{H}^{N-1}\lfloor S_g$ (Subsection 3.1). In Subsections 3.2 and 3.3 we prove that for all $A \in \mathcal{A}(\Omega)$

$$I_1(g,G,A) = \int_A W_1\left(G - \nabla g\right)\, dx + \int_{A\cap S(g)} \gamma_1\left([g], \nu(g)\right)\, d\mathcal{H}^{N-1}$$

from where (3.13) follows taking $A = \Omega$.

3.1 Localization

We start by localizing $I_1(g,G)$, i.e., we define for $A \in \mathcal{A}(\Omega)$

$$I_1(g,G,A) \;\; := \inf_{\{u_n\}\subset SBV^2(A;\mathbb{R}^d)} \left\{ \liminf_{n\to\infty} \int_{S_{u_n}} \Psi_1([u_n],\nu(u_n))\, d\mathcal{H}^{N-1},\ u_n \overset{L^1(A;\mathbb{R}^d)}{\longrightarrow} g,\ \nabla u_n \overset{L^1(A;\mathbb{R}^{d\times N})}{\longrightarrow} G \right\}.$$

Remark 3.2 We note that the localized version of (3.10) still holds. Namely there exists $C > 0$ such that for all $A \in \mathcal{A}(\Omega)$

$$I_1(g,G,A) \le C\left[\int_A (|\nabla g| + |G|)\, dx + ||D^s g||(A)\right]. \tag{3.14}$$

We next prove that $I_1(g,G,\cdot)\lfloor\mathcal{A}(\Omega)$ is a Radon measure. For this purpose we first show that $I_1(g,G,\cdot)$ is nested-subadditive.

Lemma 3.3 *Let $A, B, C \in \mathcal{A}(\Omega)$ with $A \subset\subset B \subset C$. Then*

$$I_1(g,G,C) \le I_1(g,G,B) + I_1(g,G,C\backslash\bar{A}). \tag{3.15}$$

Proof. Let $u_n \in SBV^2(B;\mathbb{R}^d)$ and $v_n \in SBV^2(C\backslash\bar{A};\mathbb{R}^d)$ be two sequences such that $u_n \to g$ in $L^1(B;\mathbb{R}^d)$, $\nabla u_n \to G$ in $L^1(B;\mathbb{R}^{d\times N})$, $v_n \to g$ in $L^1(C\backslash\bar{A};\mathbb{R}^d)$, $\nabla v_n \to G$ in $L^1(C\backslash\bar{A};\mathbb{R}^{d\times N})$, and that in addition

$$I_1(g,G,B) \;\; = \;\; \lim_{n\to\infty}\int_{S_{u_n}} \Psi_1([u_n],\nu(u_n))\, d\mathcal{H}^{N-1}$$

and

$$I_1(g,G,C\backslash\bar{A}) \;\; = \;\; \lim_{n\to\infty}\int_{S_{v_n}} \Psi_1([v_n],\nu(v_n))\, d\mathcal{H}^{N-1}.$$

Note that

$$u_n - v_n \to 0 \text{ in } L^1(B\cap(C\backslash\bar{A});\mathbb{R}^d) \tag{3.16}$$

and

$$\nabla u_n - \nabla v_n \to 0 \text{ in } L^1(B\cap(C\backslash\bar{A});\mathbb{R}^{d\times N}).$$

For $\delta > 0$ define

$$A_\delta := \{x \in B,\ \operatorname{dist}(x, A) < \delta\}.$$

Let $d(x) := \operatorname{dist}(x, A),\ x \in C$. Since the distance function to a fixed set is Lipschitz continuous (see Exercise 1.1 in Ziemer [29]), we can apply the change of variables formula (see Theorem 2, Section 3.4.3, in Evans and Gariepy [19]), to get that

$$\int_{A_\delta \setminus \bar{A}} |u_n - v_n| Jd(x)\, dx = \int_0^\delta \left[\int_{d^{-1}(y)} |u_n - v_n| d\mathcal{H}^{N-1}(x) \right] dy$$

and, as $Jd(\cdot)$ is bounded and (3.16) holds, then for almost every $\rho \in [0, \delta]$ it follows that

$$\lim_{n\to\infty} \int_{d^{-1}(\rho)} |u_n - v_n|\, d\mathcal{H}^{N-1} = \lim_{n\to\infty} \int_{\partial A_\rho} |u_n - v_n|\, d\mathcal{H}^{N-1} = 0. \tag{3.17}$$

Fix ρ_0 such that (3.17) holds. We observe that A_{ρ_0} is a set with locally Lipschitz boundary since it is a level set of a Lipschitz function (see e.g. Evans and Gariepy [24]). Hence we can consider $u_n, v_n, \nabla u_n, \nabla v_n$ on ∂A_{ρ_0} in the sense of traces and define

$$w_n = \begin{cases} u_n & \text{in } \bar{A}_{\rho_0} \\ v_n & \text{in } C \setminus \bar{A}_{\rho_0}. \end{cases}$$

Then

$$I_1(g, G, C) \quad \leq \quad \liminf_{n\to\infty} \int_{S_{w_n}} \Psi_1([w_n], \nu(w_n))\, dH^{N-1}$$

and using (H4) and (3.16) we get (3.15).

■

Theorem 3.4 *Assume that hypothesis (H_4) holds. Then $I_1(g, G, \cdot)\lfloor\mathcal{A}(\Omega)$ is a Radon measure absolutely continuous with respect to $\mathcal{L}^N + \mathcal{H}^{N-1}\lfloor S_g$.*

Proof. Let $u_n \in SBV^2(\Omega; \mathbb{R}^d)$ be such that $u_n \to g$ in $L^1(\Omega; \mathbb{R}^d)$, $\nabla u_n \to G$ in $L^1(\Omega; \mathbb{R}^{d\times N})$ and

$$I_1(g, G, \Omega) = \lim_{n\to\infty} \int_{S_{u_n}} \Psi_1([u_n], \nu(u_n))\, d\mathcal{H}^{N-1},$$

and define for all Borel set $B \subset \mathbb{R}^N$

$$\mu_n(B) \quad := \quad \int_{S_{u_n} \cap B} \Psi_1([u_n], \nu(u_n))\, d\mathcal{H}^{N-1}.$$

By (H4) the sequence of non-negative Radon measures $\{\mu_n\}$ is uniformly bounded in $\mathcal{M}(\mathbb{R}^N)$ and thus, upon passing if necessary to a subsequence, we conclude that

$$\mu_n \overset{*}{\rightharpoonup} \mu \text{ in } \mathcal{M}(\mathbb{R}^N).$$

Let us show that for all $V \in \mathcal{A}(\Omega)$

$$\mu(V) = I_1(g, G, V). \tag{3.18}$$

Given $V \in \mathcal{A}(\Omega)$, let $\epsilon > 0$ and take $W \subset\subset V$ such that $\mu(V \backslash W) < \epsilon$. It follows that

$$\begin{aligned}
\mu(V) &\leq \mu(W) + \epsilon \\
&= \mu(\Omega) - \mu(\Omega \backslash W) + \epsilon \\
&\leq I_1(g, G, \Omega) - I_1(g, G, \Omega \backslash \bar{W}) + \epsilon \\
&\leq I_1(g, G, V) + \epsilon
\end{aligned}$$

where we have used the equality $\mu(\Omega) = \mu(\overline{\Omega})$ and Lemma 3.3. Thus, letting $\epsilon \to 0$, we get

$$\mu(V) \leq I_1(g, G, V). \tag{3.19}$$

Let us see now the reverse inequality. Define for $A \in \mathcal{A}(\Omega)$

$$\lambda(A) := \int_A (|\nabla g| + |G|)\, dx + ||D^s g||(A). \tag{3.20}$$

Let $K \subset\subset V$ be a compact set such that $\lambda(V \backslash K) < \epsilon$, and choose an open set W such that $K \subset\subset W \subset\subset V$. Using again Lemma 3.3, (3.20), and (3.14)

$$\begin{aligned}
I_1(g, G, V) &\leq I_1(g, G, W) + I_1(g, G, V \backslash K) \\
&\leq \mu(\bar{W}) + C\lambda(V \backslash K) \\
&\leq \mu(V) + C\epsilon,
\end{aligned}$$

which, together with (3.19), yields to (3.18) by letting $\epsilon \to 0$. ∎

3.2 Lower Bound

The objective of this part is to show that for all $A \in \mathcal{A}(\Omega)$

$$I_1(g, G, A) \geq \int_A W_1\left(G(x) - \nabla g(x)\right)\, dx + \int_{A \cap S(g)} \gamma_1\left([g], \nu(g)\right)\, d\mathcal{H}^{N-1}. \tag{3.21}$$

To prove inequality (3.21) let $u_n \in SBV^2(\Omega; \mathbb{R}^N)$ be such that $u_n \underset{n\to\infty}{\overset{L^1}{\longrightarrow}} g$, $\nabla u_n \underset{n\to\infty}{\overset{L^1}{\longrightarrow}} G$ and

$$\lim_{n\to\infty} \int_{S_{u_n}} \Psi_1([u_n], \nu(u_n))\, d\mathcal{H}^{N-1} < \infty.$$

Define μ_n by

$$\mu_n(B) = \int_{B \cap S(u_n)} \Psi_1([u_n], \nu(u_n)) d\mathcal{H}^{N-1},$$

for all Borel sets $B \subset \Omega$. Since, by the hypotheses on Ψ_1, the sequence of Radon measures $\{\mu_n\}$ is bounded then there exists (up to a subsequence) $\mu \in \mathcal{M}(\Omega)$ with $\mu_n \overset{*}{\rightharpoonup} \mu$ in $\mathcal{M}(\Omega)$. We now show that

$$\frac{d\mu}{d\mathcal{L}^N}(x_0) \geq W_1(G(x_0) - \nabla g(x_0)) \tag{3.22}$$

for $\mathcal{L}^N$-almost every $x_0 \in \Omega$ and that

$$\frac{d\mu}{d\mathcal{H}^{N-1}\lfloor S_g}(x_0) \geq \gamma_1([g](x_0), \nu(g)(x_0)) \tag{3.23}$$

for $\mathcal{H}^{N-1}\lfloor S_g$-almost every $x_0 \in \Omega$.

Proof of (3.22)**:** Let $x_0 \in \Omega$ be a point of approximate differentiability of g and of approximately continuity of G (see Theorem 2.3 and (2.4)), and such that $\frac{d\mu}{d\mathcal{L}^N}(x_0)$ exists. Let $\{\delta_k\} \to 0$ be such that $\mu(\partial Q(x_0, \delta_k)) = 0$. Then $\lim_{n\to\infty} \mu_n(Q(x_0, \delta_k)) = \mu(Q(x_0, \delta_k)$ and

$$\begin{aligned}
\frac{d\mu}{d\mathcal{L}^N}(x_0) &= \lim_{k\to\infty} \frac{\mu(Q(x_0, \delta_k))}{\mathcal{L}^N(Q(x_0, \delta_k))} \\
&= \lim_{k,\,n\to\infty} \frac{1}{\delta_k^N} \int_{S_{u_n} \cap Q(x_0,\delta_k)} \Psi_1([u_n], \nu(u_n))\, d\mathcal{H}^{N-1} \\
&= \lim_{k,\,n\to\infty} \frac{1}{\delta_k} \int_{Q\cap\{y:\ x_0+\delta_k y\ \in S_{u_n}\}} \Psi_1([u_n](x_0 + \delta_k y), \nu(u_n)(x_0 + \delta_k y)) d\mathcal{H}^{N-1}(y).
\end{aligned}$$

Defining

$$v_{n,k}(y) := \frac{u_n(x_0 + \delta_k y) - g(x_0)}{\delta_k},\ y \in Q$$

we have that

$$\frac{d\mu}{d\mathcal{L}^N}(x_0) = \lim_{k,n\to\infty} \int_{Q\cap S_{v_{n,k}}} \Psi_1([v_{n,k}], \nu(v_{n,k}))\, d\mathcal{H}^{N-1}. \tag{3.24}$$

As x_0 is a point of approximate differentiability of g then

$$v_{n,k} \xrightarrow[k,\,n\to\infty]{L^1} \nabla g(x_0)(\cdot) \tag{3.25}$$

since

$$\begin{aligned}
\lim_{n\to\infty} \int_Q |v_{n,k}(y) - \nabla g(x_0)\, y|\, dy &= \lim_{n\to\infty} \int_Q \left| \frac{u_n(x_0 + \delta_k\, y) - g(x_0)}{\delta_k} - \nabla g(x_0)\, y \right| dy \\
&= \int_Q \left| \frac{g(x_0 + \delta_k\, y) - g(x_0)}{\delta_k} - \nabla g(x_0)\, y \right| dy \\
&= \frac{1}{\delta_k^N} \int_{Q(x_0,\delta_k)} \left| \frac{g(z) - g(x_0) - \nabla g(x_0)(z - x_0)}{\delta_k} \right| dy
\end{aligned}$$

by a change of variables. Similarly, as x_0 is also an approximately continuity point of G,

$$\nabla v_{k,\,n} \xrightarrow[k,n\to\infty]{L^1} G(x_0). \tag{3.26}$$

Next we change the sequence $\{v_{n,k}\}$ to comply in (3.24) with the definition of W_1 (see (1.2)). We start by setting

$$w_{n,k}(y) = v_{n,k}(y) - \nabla g(x_0) y,\ y \in Q,$$

and following the argument used in 3.17 we choose $r_{n,k} \in]0,1[$ such that

$$r_{n,k} \underset{k,n\to\infty}{\longrightarrow} 1$$

and

$$\int_{\partial Q(0,r_{n,k})} |w_{n,k}| d\mathcal{H}^{N-1} \underset{k,n\to\infty}{\longrightarrow} 0. \tag{3.27}$$

By Theorem 2.5 let $\rho_{n,k}$ be such that

$$\nabla \rho_{n,k}(y) = G(x_0) - \nabla v_{n,k}(y), \;\; y \in Q,$$

and define

$$z_{n,k} := w_{n,k} + \rho_{n,k} \;\text{ in }\; Q(0, r_{n,k}).$$

Notice that

$$\nabla z_{n,k} = G(x_0) - \nabla g(x_0) \;\text{ in }\; Q(0, r_{n,k}).$$

In addition, by (3.26) $\nabla \rho_{n,k} \underset{k,n\to\infty}{\longrightarrow} 0$, and then by Theorem 2.5

$$|D^s \rho_{n,k}|(Q(0, r_{n,k})) \underset{k,n\to\infty}{\longrightarrow} 0. \tag{3.28}$$

Thus by the continuity of the trace with respect to the intermediate topology (see Proposition 3.88 in Ambrosio, Fusco and Pallara [5]) it follows that

$$\int_{\partial Q(0,r_{n,k})} |\rho_{n,k}| \, d\mathcal{H}^{N-1} \underset{k,n\to\infty}{\longrightarrow} 0. \tag{3.29}$$

Applying Lemma 2.7 in $Q\backslash(Q(0, r_{n,k}))$ let $\{\eta_{n,k}\}$ be a sequence of functions such that

$$\nabla \eta_{n,k}(y) = G(x_0) - \nabla g(x_0) \quad \text{in } Q\backslash(Q(0, r_{n,k})),$$

$$\eta_{n,k} = 0 \quad \text{on } \partial(Q\backslash(Q(0, r_{n,k})) \tag{3.30}$$

and

$$|D^s \eta_{n,k}|(Q\backslash(Q(0, r_{n,k})) \leq C(N)|Q\backslash(Q(0, r_{n,k})|. \tag{3.31}$$

Then the sequence

$$\tilde{z}_{n,k}(y) := \begin{cases} z_{n,k}(y), & y \in Q(0, r_{n,k}) \\ \eta_{n,k}(y), & y \in Q\backslash(Q(0, r_{n,k})) \end{cases}$$

is admissible fot $W^1(G(x_0) - \nabla g(x_0))$, and in addition by (H_4), (H_7) and (3.30) we have, for any n

and k, that

$$
\begin{aligned}
\int_{Q\cap S_{\tilde z_{n,k}}} \Psi_1([\tilde z_{n,k}],\nu(\tilde z_{n,k}))\,d\mathcal{H}^{N-1} \;\le\; & \int_{Q(0,r_{n,k})\cap S_{z_{n,k}}} \Psi_1([z_{n,k}],\nu(z_{n,k}))\,d\mathcal{H}^{N-1} \\
& + C\left[\int_{\partial Q(0,r_{n,k})} |z_{n,k}|\,d\mathcal{H}^{N-1} + \int_{[Q\setminus Q(0,r_{n,k})]\cap S_{\eta_{n,k}}} |[\eta_{n,k}]|\,d\mathcal{H}^{N-1}\right] \\
\le\; & \int_{Q(0,r_{n,k})\cap S_{v_{n,k}}} \Psi_1([v_{n,k}],\nu(v_{n,k}))\,d\mathcal{H}^{N-1} \\
& + C\left[\int_{\partial Q(0,r_{n,k})} |w_{n,k}|\,d\mathcal{H}^{N-1} + \int_{\partial Q(0,r_{n,k})} |\rho_{n,k}|\,d\mathcal{H}^{N-1}\right. \\
& \left. + \int_{Q(0,r_{n,k})\cap S_{\rho_{n,k}}} |[\rho_{n,k}]|\,d\mathcal{H}^{N-1} + \int_{[Q\setminus Q(0,r_{n,k})]\cap S_{\eta_{n,k}}} |[\eta_{n,k}]|\,d\mathcal{H}^{N-1}\right]
\end{aligned}
$$

Therefore by (3.27), (3.28),(3.29) and (3.31) we get that

$$
\liminf_{k,\,n\to\infty} \int_{Q\cap S_{\tilde z_{n,k}}} \Psi_1([\tilde z_{n,k}],\nu(\tilde z_{n,k}))\,d\mathcal{H}^{N-1} \le \lim_{k,\,n\to\infty} \int_{Q\cap S_{v_{n,k}}} \Psi_1([v_{n,k}],\nu(v_{n,k}))\,d\mathcal{H}^{N-1}
$$

which together with (3.24) implies (3.22).

Proof of (3.23)**:** Let $x_0 \in S_g$ such that

$$
\lim_{\delta\to 0} \frac{\mathcal{H}^{N-1}(S_g\cap Q_\nu(x_0,\delta))}{\delta^{N-1}} = 1 \tag{3.32}
$$

and

$$
\lim_{\delta\to 0} \frac{1}{\delta^{N-1}} \int_{Q_\nu(x_0,\delta)} |G(x)|\,dx = 0, \tag{3.33}
$$

where $\nu \equiv \nu(g)(x_0)$. We note that for $\mathcal{H}^{N-1}$-a.e $x_0 \in S_g$ equalities (3.32) and (3.33) hold. In fact see Ambrosio, Fusco and Pallara [5] for (3.32). Equality (3.33) holds by (3.32) and the fact that

$$
\frac{d|G|\mathcal{L}^N}{d\mathcal{H}^{N-1}\lfloor S_g} = 0.
$$

Let $\{\delta_k\}\to 0$ be such that $\mu(\partial Q_\nu(x_0,\delta_k)) = 0$. Then $\lim_{n\to\infty} \mu_n(Q_\nu(x_0,\delta_k)) = \mu(Q_\nu(x_0,\delta_k)$ and

$$
\begin{aligned}
\frac{d\mu}{d\mathcal{H}^{N-1}\lfloor S_g}(x_0) &= \lim_{k\to\infty}\lim_{n\to\infty} \frac{1}{\mathcal{H}^{N-1}(S_g\cap Q_\nu(x_0,\delta_k))}\mu_n(Q_\nu(x_0,\delta_k)) \\
&= \lim_{k\to\infty}\lim_{n\to\infty} \frac{1}{\mathcal{H}^{N-1}(S_g\cap Q_\nu(x_0,\delta_k))} \int_{S_{u_n}\cap Q_\nu(x_0,\delta)} \Psi_1([u_n],\nu(u_n))\,d\mathcal{H}^{N-1} \\
&= \lim_{k\to\infty}\lim_{n\to\infty} \frac{\delta_k^{N-1}}{\mathcal{H}^{N-1}(S_g\cap Q_\nu(x_0,\delta_k))} \int_{Q_\nu\cap\{y:\, x_0+\delta_k y\in S_{u_n}\}} \Psi_1\big([u_n](x_0+\delta_k\, y),\nu(u_n)(x_0+\delta_k y)\big)\,d\mathcal{H}^{N-1} \\
&= \lim_{k\to\infty}\lim_{n\to\infty} \int_{Q_\nu\cap\{y:\, x_0+\delta_k y\in S_{u_n}\}} \Psi_1\big([u_n](x_0+\delta_k\, y),\nu(u_n)(x_0+\delta_k y)\big)\,d\mathcal{H}^{N-1}
\end{aligned}
$$

by (3.32). Defining

$$w^1_{n,k}(y) = u_n(x_0 + \delta_k y) - g^-(x_0), \quad y \in Q_\nu,$$

it follows that

$$\frac{d\mu}{d\mathcal{H}^{N-1}\lfloor S_g}(x_0) \quad = \quad \lim_{k\to\infty}\lim_{n\to\infty}\int_{Q_\nu\cap S_{w^1_{n,k}}}\Psi_1\big([w^1_{n,k}](y), \nu(w^1_{n,k})(y)\big)\, d\mathcal{H}^{N-1}.$$

By (2.5) and (2.6) we have

$$w^1_{n,k} \underset{n,k\to\infty}{\overset{L^1}{\longrightarrow}} \gamma_{([g](x_0),\nu)},$$

and in addition by (3.33)

$$\nabla w^1_{n,k} \underset{n,k\to\infty}{\overset{L^1}{\longrightarrow}} 0$$

since, for all k, $\nabla u_n(x_0 + \delta_k \cdot) \underset{n\to\infty}{\longrightarrow} G(x_0 + \delta_k \cdot)$.

Using Theorem 2.5 and following the arguments of Lemma 3.3 we note that it is possible to modify $w^1_{n,k}$ so that $\nabla w^1_{n,k} = 0$ and $w^1_{n,k}|_{\partial Q_\nu} = \gamma_{([g](x_0),\nu)}$. Thus by definition of γ_1 (see (1.3)) inequality (3.23) holds.

Proof of (3.21)**:** Denote by μ_a the absolutely continuous part of μ with respect to Lebesgue measure and by μ^s_g the absolutely continuous part of μ with respect to $H^{N-1}\lfloor S_g$. Since μ is a positive measure we conclude that

$$\begin{aligned}
\liminf_{n\to\infty}\mu_n(A) &\geq \mu(A) \\
&\geq \int_A \mu_a(x)\, dx + \int_{A\cap S_g}\mu^s_g(x)\, d\mathcal{H}^{N-1}(x) \\
&\geq \int_A W_1\left(G(x) - \nabla g(x)\right)\, dx + \int_{A\cap S_g}\gamma_1\left([g], \nu(g)\right)\, d\mathcal{H}^{N-1}.
\end{aligned}$$

Taking the infimum over all sequences $u_n \in SBV^2(\Omega;\mathbb{R}^d)$, $u_n \underset{n\to\infty}{\overset{L^1}{\longrightarrow}} g$, $\nabla u_n \underset{n\to\infty}{\overset{L^1}{\longrightarrow}} G$ inequality (3.21) holds.

3.3 Upper bound

The objective of this part is to show that for all $A \in \mathcal{A}(\Omega)$

$$I_1(g, G, A) \leq \int_A W_1\left(G(x) - \nabla g(x)\right)\, dx + \int_{A\cap S(g)}\gamma_1\left([g], \nu(g)\right)\, d\mathcal{H}^{N-1}.$$

For this purpose, it is enough to prove that

$$\frac{dI_1(g, G, \cdot)}{d\mathcal{L}^N}(x_0) \leq W_1(G(x_0) - \nabla g(x_0)), \quad \mathcal{L}^N \text{ -a.e } x_0 \in \Omega \tag{3.34}$$

and

$$\frac{dI_1(g,G,\cdot)}{d\mathcal{H}^{N-1}\lfloor S_g}(x_0) \le \gamma_1([g](x_0),\nu(g)(x_0)), \quad \mathcal{H}^{N-1}\text{ -a.e } x_0 \in S_g. \tag{3.35}$$

Proof of (3.34)**:** Let x_0 be a point of approximate continuity for G and ∇g, that is, such that

$$\frac{1}{\delta^N}\left\{\int_{Q(x_0,\delta)} |G(x)-G(x_0)| + |\nabla g(x) - \nabla g(x_0)|\, dx\right\} \underset{\delta\to 0}{\longrightarrow} 0. \tag{3.36}$$

Let $\epsilon > 0$ and consider $u \in SBV^2(\Omega;\mathbb{R}^N)$ such that

$$W_1(G(x_0)-\nabla g(x_0)) + \epsilon \ge \int_{Q\cap Su} \Psi_1([u],\nu(u))\, d\mathcal{H}^{N-1}, \tag{3.37}$$

$u|_{\partial Q} = 0$ and $\nabla u(x) = G(x_0) - \nabla g(x_0)$ for a.e. $x \in Q$. Extend u by periodicity to all of $\mathbb{R}^N$ and define for $n \in \mathbb{N}$ and $\delta > 0$

$$u_{n,\delta}(x) = \frac{\delta}{n}\, u\left(\frac{n(x-x_0)}{\delta}\right).$$

Given $\delta > 0$ apply Theorem 2.5 and let $\rho_\delta \in SBV^2(Q(x_0,\delta);\mathbb{R}^N)$ be a function such that

$$\nabla\rho_\delta(x) = G(x) - G(x_0) + \nabla g(x_0) - \nabla g(x) \tag{3.38}$$

$\mathcal{L}^N$-a.e. $x \in Q(x_0,\delta)$ and satisfying that

$$||D\rho_\delta||(Q(x_0,\delta)) \le C(N)\int_{Q(x_0,\delta)} |G(x)-G(x_0)| + |\nabla g(x_0) - \nabla g(x)|\, dx.$$

Note that by (3.36)

$$\frac{||D\rho_\delta||(Q(x_0,\delta))}{\delta^N} \underset{\delta\to 0}{\longrightarrow} 0. \tag{3.39}$$

In addition, using Lemma 2.4 define a sequence of piecewise constant functions $\rho_{n,\delta}$ such that for all $\delta > 0$

$$\rho_{n,\delta} \underset{n\to\infty}{\overset{L^1}{\longrightarrow}} -\rho_\delta \quad \text{and} \quad ||D\rho_{n,\delta}||(Q(x_0,\delta)) \underset{n\to\infty}{\longrightarrow} ||D\rho_\delta||(Q(x_0,\delta)). \tag{3.40}$$

Let now define

$$w_{n,\delta}(x) := g(x) + u_{n,\delta}(x) + \rho_\delta(x) + \rho_{n,\delta}(x), \quad x \in Q(x_0,\delta).$$

Clearly $w_{n,\delta} \in SBV^2(Q(x_0,\delta);\mathbb{R}^d)$, $w_{n,\delta} \underset{n\to\infty}{\overset{L^1}{\longrightarrow}} g$ and $\nabla w_{n,\delta} \underset{n\to\infty}{\overset{L^1}{\longrightarrow}} G$.
As for each $\delta > 0$ the sequence $w_{n,\delta}$ is admissible for I_1 and

$$\frac{dI_1(g,G,.)}{d\mathcal{L}^N}(x_0) = \lim_{\delta\to 0}\frac{I_1(g,G,Q(x_0,\delta))}{\delta^N}.$$

Then

$$\frac{dI_1(g,G,.)}{d\mathcal{L}^N}(x_0) \le \liminf_{\delta\to 0}\liminf_{n\to\infty}\left\{\frac{1}{\delta^N}\int_{S_{w_{n,\delta}}\cap Q(x_0,\delta)} \psi_1([w_{n,\delta}],\nu(w_{n,\delta}))\, d\mathcal{H}^{N-1}\right\}$$

and by (H_7)

$$\begin{aligned}\frac{dI_1(g,G,.)}{d\mathcal{L}^N}(x_0) \quad &\leq \quad \liminf_{\delta\to 0}\liminf_{n\to\infty}\left\{\frac{1}{\delta^N}\int_{S_g\cap Q(x_0,\delta)}\Psi_1([g],\nu(g))\,d\mathcal{H}^{N-1}\right.\\ &+ \quad \frac{1}{\delta^N}\int_{\left\{x_0+\frac{\delta}{n}S_u\right\}\cap Q(x_0,\delta)}\Psi_1\left(\frac{\delta}{n}[u]\left(\frac{n(x-x_0)}{\delta}\right),\nu(u)\left(\frac{n(x-x_0)}{\delta}\right)\right)\,d\mathcal{H}^{N-1}\\ &+ \quad \frac{1}{\delta^N}\int_{S_{\rho_\delta}\cap Q(x_0,\delta)}\Psi_1([\rho_\delta],\nu(\rho_\delta))\,d\mathcal{H}^{N-1}\\ &+ \quad \left.\frac{1}{\delta^N}\int_{S_{\rho_{n,\delta}}\cap Q(x_0,\delta)}\Psi_1([\rho_{n,\delta}],\nu(\rho_{n,\delta}))\,d\mathcal{H}^{N-1}\right\}.\end{aligned}$$

We observe that by (H_4)

$$\frac{1}{\delta^N}\int_{S_g\cap Q(x_0,\delta)}\Psi_1([g],\nu(g))\,d\mathcal{H}^{N-1}\underset{\delta\to 0}{\longrightarrow}0$$

since

$$\frac{d|D^s g|}{d\mathcal{L}^N}(x_0)=0.$$

Moreover

$$\frac{1}{\delta^N}\int_{S_{\rho_\delta}\cap Q(x_0,\delta)}\Psi_1([\rho_\delta],\nu(\rho_\delta))\,d\mathcal{H}^{N-1}\underset{\delta\to 0}{\longrightarrow}0$$

and

$$\lim_{\delta\to 0}\lim_{n\to\infty}\frac{1}{\delta^N}\int_{S_{\rho_{n,\delta}}\cap Q(x_0,\delta)}\Psi_1([\rho_{n,\delta}],\nu(\rho_{n,\delta}))\,d\mathcal{H}^{N-1}=0$$

by (H_4), (3.39) and (3.40). Finally, changing variables we obtain that

$$\frac{1}{\delta^N}\int_{\left\{x_0+\frac{\delta}{n}S_u\right\}\cap Q(x_0,\delta)}\Psi_1\left(\frac{\delta}{n}[u]\left(\frac{n(x-x_0)}{\delta}\right),\nu(u)\left(\frac{n(x-x_0)}{\delta}\right)\right)\,d\mathcal{H}^{N-1}=\int_{Q\cap S_u}\Psi_1([u],\nu(u))\,d\mathcal{H}^{N-1},$$

from where

$$\frac{dI_1(g,G,.)}{d\mathcal{L}^N}(x_0)\leq\int_{Q\cap S_u}\Psi_1([u],\nu(u))\,d\mathcal{H}^{N-1}.$$

As a consequence, letting $\varepsilon\to 0$ in (3.37) inequality (3.34) follows.

Proof of (3.35)**:** Following an argument of Ambrosio, Mortolla, Tortorelli [6] it suffices to prove (3.35) for $g=\lambda\chi_E$ with $\lambda\in\mathbb{R}$ and where χ_E is the characteristic function of a set of finite perimeter E.

We will start by addressing the case where E *is a polyhedron.* Let $x_0\in S_g$ be such that

$$\lim_{\delta\to 0}\frac{1}{\delta^{N-1}}\int_{Q_\nu(x_0,\delta)}|G(x)|\,dx=0 \tag{3.41}$$

where $\nu\equiv\nu(g)(x_0)$. Given $\epsilon>0$ let now $u\in SBV(Q_\nu;\mathbb{R}^d)$ be such that $\nabla u=0, u|_{\partial Q_\nu}=\gamma_{(\lambda,\nu)}$ and

$$\gamma_1(\lambda,\nu)+\epsilon\geq\int_{Q_\nu}\Psi_1([u],\nu(u))\,d\mathcal{H}^{N-1} \tag{3.42}$$

(see (1.3)). For $\delta > 0$ small enough define

$$D^n_\nu(x_0,\delta) := Q_\nu(x_0,\delta) \cap \left\{ x : \ \frac{|(x-x_0)\cdot\nu|}{\delta} < \frac{1}{2n} \right\},$$

$$Q^+_\nu(x_0,\delta) = Q_\nu(x_0,\delta) \cap \left\{ x : \ \frac{(x-x_0)\cdot\nu}{\delta} > 0 \right\}$$

and $Q^-_\nu(x_0,\delta)$ in an analogous way. Set now

$$u_{n,\delta}(x) = \begin{cases} \lambda & x \in Q^+_\nu(x_0,\delta)\backslash D^n_\nu(x_0,\delta) \\ u\left(\frac{n(x-x_0)}{\delta}\right) & x \in D^n_\nu(x_0,\delta) \\ 0 & x \in Q^-_\nu(x_0,\delta)\backslash D^n_\nu(x_0,\delta) \end{cases}$$

where u has been extended by periodicity to all of $\mathbb{R}^N$. Note that for $x \in D^n_\nu(x_0,\delta)$ we have that $\frac{n}{\delta}|(x-x^0)\cdot\nu| < \frac{1}{2}$. Clearly $u_{n,\delta} \underset{n\to\infty,\delta\to 0}{\overset{L^1}{\longrightarrow}} \tilde{\gamma}_{(\lambda,\nu)}$ where, for $x \in Q_\nu(x_0,\delta)$,

$$\tilde{\gamma}_{(\lambda,\nu)}(x) := \begin{cases} \lambda & \text{if } \ x\cdot\nu > 0 \\ 0 & \text{if } \ x\cdot\nu < 0. \end{cases}$$

By Theorem 2.5 there exists $\zeta_\delta \in SBV(Q_\nu(x_0,\delta);\mathbb{R}^d)$ such that $\nabla\zeta_\delta = G$ and

$$|D^s\zeta_\delta|(Q_\nu(x_0,\delta)) \le C||G||_{L^1(Q_\nu(x_0,\delta);\mathbb{R}^{d\times N})}. \tag{3.43}$$

Moreover by Lemma 2.4 there exists a sequence $\zeta_{n,\delta}$ of piecewise constant functions defined on $Q_\nu(x_0,\delta)$ such that $\zeta_{n,\delta} \underset{n\to\infty}{\overset{L^1}{\longrightarrow}} \zeta_\delta$ and

$$|D\zeta_{n,\delta}|(Q_\nu(x_0,\delta)) \underset{n\to\infty}{\longrightarrow} |D\zeta_\delta|(Q_\nu(x_0,\delta)). \tag{3.44}$$

Set

$$w_{n,\delta} = u_{n,\delta} + \zeta_\delta - \zeta_{n,\delta}.$$

Clearly $w_{n,\delta}$ is admissible for $I_1(g,G,Q_\nu(x_0,\delta))$. Therefore

$$\begin{aligned} \frac{dI_1(g,G,.)}{d\mathcal{H}^{N-1}\lfloor S_g}(x_0) &= \lim_{\delta\to 0}\frac{1}{\delta^{N-1}} I_1(g,G,Q_\nu(x_0,\delta)) \\ &\le \lim_{\delta\to 0}\liminf_{n\to\infty}\frac{1}{\delta^{N-1}}\left\{ \int_{Q_\nu(x_0,\delta)\cap S_{w_{n,\delta}}} \Psi_1([w_{n,\delta}](x),\nu(w_{n,\delta})(x))\,d\mathcal{H}^{N-1}\right\} \\ &= \lim_{\delta\to 0}\liminf_{n\to\infty}\frac{1}{\delta^{N-1}}\left\{ \int_{Q_\nu(x_0,\delta)\cap S_{u_{n,\delta}}} \Psi_1([u_{n,\delta}](x),\nu(u_{n,\delta})(x))\,d\mathcal{H}^{N-1} \right. \\ &+ \int_{Q_\nu(x_0,\delta)\cap S_{\zeta_\delta}} \Psi_1([\zeta_\delta](x),\nu(\zeta_\delta)(x))\,d\mathcal{H}^{N-1} \\ &+ \left. \int_{Q_\nu(x_0,\delta)\cap S_{\zeta_{n,\delta}}} \Psi_1([\zeta_{n,\delta}](x),\nu(\zeta_{n,\delta})(x))\,d\mathcal{H}^{N-1}\right\} \\ &=: J_1 + J_2 + J_3 \end{aligned}$$

The terms J_2 and J_3 go to zero due to (H_4), (3.43), (3.44) and (3.41). Moreover

$$
\begin{aligned}
J_1 &= \lim_{\delta\to 0}\liminf_{n\to\infty}\frac{1}{\delta^{N-1}}\int_{Q_\nu(x_0,\delta)\cap S_{u_{n,\delta}}} \Psi_1([u_{n,\delta}](x),\nu(u_{n,\delta})(x))\,d\mathcal{H}^{N-1}\\
&= \lim_{\delta\to 0}\liminf_{n\to\infty}\frac{1}{\delta^{N-1}}\int_{D^n_\nu(x_0,\delta)\cap\left\{x:\ \frac{n(x-x_0)}{\delta}\in S_u\right\}} \Psi_1\left([u]\left(\frac{n(x-x_0)}{\delta}\right),\nu(u)\left(\frac{n(x-x_0)}{\delta}\right)\right)\,d\mathcal{H}^{N-1}\\
&= \lim_{\delta\to 0}\liminf_{n\to\infty}\frac{1}{n^{N-1}}\int_{nQ_\nu\cap\left\{y:\ |y\cdot\nu|<\frac{1}{2}\right\}\cap S_u} \Psi_1\big([u](y),\nu(u)(y)\big)\,d\mathcal{H}^{N-1}(y)\\
&= \int_{Q_\nu\cap S_u}\Psi_1\big([u](y),\nu(u)(y)\big)\,d\mathcal{H}^{N-1}(y).
\end{aligned}
$$

Thus

$$
\frac{dI_1(g,G,.)}{d\mathcal{H}^{N-1}\lfloor S_g}(x_0)\leq\int_{Q_\nu\cap S_u}\Psi_1\big([u](y),\nu(u)(y)\big)\,d\mathcal{H}^{N-1}(y)
$$

and consequently (3.35) follows by letting $\epsilon\to 0$ in (3.42).

To treat the case of sets *of finite perimeter* we need the following results.

Proposition 3.5 *Let Ψ_1 satisfy (H_4) and (H_7). Then there exist constants $C_1, C_2 > 0$ such that*

$$
|\gamma_1(\lambda,\nu)-\gamma_1(\lambda',\nu)|\leq C_1|\lambda-\lambda'|,\quad\forall\lambda,\lambda'\in\mathbb{R}^d, \tag{3.45}
$$

$$
|W_1(A)-W_1(B)|\leq C_2|A-B|,\quad\forall A,B\in\mathbb{R}^{d\times N}. \tag{3.46}
$$

Moreover γ_1 is upper semicontinuous with respect to ν.

Proof. We proof (3.46) being the proof of (3.45) similar to that of Proposition 4.3 in [11]. We start by showing that

$$
W_1(A)\leq W_1(B)+C_1|B-A|,\quad\forall A,B\in\mathbb{R}^{d\times N}.
$$

Fixed $\epsilon>0$ let $u\in SBV(Q;\mathbb{R}^d)$ be such that $u|_{\partial Q}=0, \nabla u=A$ and

$$
\epsilon+W_1(A)\geq\int_{S_u\cap Q}\Psi_1([u],\nu(u))\,d\mathcal{H}^{N-1}.
$$

Let now $v\in SBV(Q;\mathbb{R}^d)$ be such that $v|_{\partial Q}=0$, $\nabla v=B-A$ and $|D^s v|(Q)\leq C|B-A|$ (cf. Lemma 2.7), and set $w=u+v$. Then by (H_4) and (H_7)

$$
\begin{aligned}
W_1(B) &\leq \int_{S_w\cap Q}\Psi_1([w],\nu(w))\,d\mathcal{H}^{N-1}\\
&\leq \left\{\int_{S_u\cap Q}\Psi_1([u],\nu(u))\,d\mathcal{H}^{N-1}+\int_{S_v\cap Q}\Psi_1([v],\nu(v))\,d\mathcal{H}^{N-1}\right\}\\
&\leq W_1(A)+\epsilon+C_1|B-A|.
\end{aligned}
$$

The reverse inequality is proved in a similar way.

■

Proposition 3.6 *Let* $(g_n, G_n) \in SD(\Omega;\mathbb{R}^d)$ *be such that*

$$g_n \underset{n\to\infty}{\overset{L^1}{\longrightarrow}} g \text{ and } G_n \underset{n\to\infty}{\overset{L^1}{\longrightarrow}} G.$$

Then

$$I_1(g, G, \Omega) \leq \liminf_{n\to\infty} I_1(g_n, G_n, \Omega).$$

The proof of this result easily follows from a diagonalization argument.

Let now E be a set of finite perimeter and $g = \lambda\chi_E$, $\lambda \in \mathbb{R}$. Consider E_n a sequence of polyhedra such that

$$\mathrm{Per}(E_n) \underset{n\to\infty}{\longrightarrow} \mathrm{Per}(E), \tag{3.47}$$

$$\mathcal{L}^N(E_n \Delta E) \underset{n\to\infty}{\longrightarrow} 0,$$

and

$$\chi_{E_n} \underset{n\to\infty}{\overset{L^1}{\longrightarrow}} \chi_E \tag{3.48}$$

(see De Giorgi [15]). By Proposition 3.5 and Proposition 3.6 in Barroso, Bouchitté, Buttazzo and Fonseca [7] we obtain a sequence of functions $\gamma_1^m : \mathbb{R}^N \to [0,\infty)$ which are continuous, homogeneous of degree one and satisfy

$$\gamma_1(\lambda, y) \leq \gamma_1^m(y) \leq C|y|, \quad \forall y \in \mathbb{R}^N, \tag{3.49}$$

$$\gamma_1(\lambda, y) = \inf_m \gamma_1^m(y), \tag{3.50}$$

where $\gamma_1(\lambda, .)$ has been extended as an homogeneous function of degree one to all of $\mathbb{R}^N$. Let

$$g_n = \lambda\chi_{E_n}. \tag{3.51}$$

By (3.48) it is clear that $g_n \underset{n\to\infty}{\overset{L^1}{\longrightarrow}} g$. Given $A \in \mathcal{A}(\Omega)$, from the previous case and Proposition 3.6 we have that

$$\begin{aligned}
I_1(g, G, A) &\leq \liminf_{n\to\infty} I_1(g_n, G, A) \\
&\leq \liminf_{n\to\infty} \left\{ \int_A W_1(G - \nabla g_n)\, dx + \int_{S_{g_n}\cap A} \gamma_1([g_n], \nu(g_n))\, d\mathcal{H}^{N-1} \right\} \\
&= \liminf_{n\to\infty} \left\{ \int_A W_1(G)\, dx + \int_{S_{g_n}\cap A} \gamma_1([g_n], \nu(g_n))\, d\mathcal{H}^{N-1} \right\} \\
&\leq C \int_A |G|\, dx + \lim_{n\to\infty} \int_{S_{g_n}\cap A} \gamma_1([g_n], \nu(g_n))\, d\mathcal{H}^{N-1}
\end{aligned} \tag{3.52}$$

where in the last inequality we have used (3.46) together with the fact that $W_1(0) = 0$. For m fixed by (3.52), the definition of g_n, (3.50) and Theorem 2.2 it follows that

$$\begin{aligned}
I_1(g, G, A) &\leq C \int_A |G|\, dx + \lim_{n\to\infty} \int_{\partial E_n\cap A} \gamma_1^m(\nu(g_n))\, d\mathcal{H}^{N-1} \\
&\leq C \int_A |G|\, dx + \int_{\partial E\cap A} \gamma_1^m(\nu(g))\, d\mathcal{H}^{N-1}.
\end{aligned}$$

Letting now $m \to \infty$ and using the Monotone Convergence Theorem, we obtain

$$I_1(g, G, A) \le C \int_A |G| \, dx + \int_{S_g \cap A} \gamma_1(\lambda, \nu(g)) \, d\mathcal{H}^{N-1}. \tag{3.53}$$

Consider x_0 satisfying (3.41). Then from (3.53) we immediately conclude that for g defined by (3.51)

$$\begin{aligned} \frac{dI_1(g, G, .)}{d\mathcal{H}^{N-1}\lfloor S_g}(x_0) &= \lim_{\delta \to 0} \frac{1}{\delta^{N-1}} I_1(g, G, Q_\nu(x_0, \delta)) \\ &\le \gamma_1([g](x_0), \nu(g)(x_0)). \end{aligned}$$

4 Integral representation in BV

The objective of this section is to derive Theorem 1.2. Its proof relies on Theorem 3.1 and on a representation result of Bouchitté, Fonseca and Mascarenhas [10].

Given $(g, G) \in GSD(\Omega; \mathbb{R}^d))$ we want to derive the integral representation of

$$I(g, G) = \inf_{\{u_n\} \in SBV^2(\Omega; \mathbb{R}^d)} \left\{ \liminf_{n \to \infty} E(u_n), \; u_n \xrightarrow{L^1} g, \; \nabla u_n \xrightarrow{L^1} G \right\},$$

where

$$E(u) = \int_\Omega W(\nabla u, \nabla^2 u) \, dx + \int_{S_u \cap \Omega} \Psi_1([u], \nu(u)) \, d\mathcal{H}^{N-1} + \int_{S_{\nabla u} \cap \Omega} \Psi_2([\nabla u], \nu(\nabla u)) \, d\mathcal{H}^{N-1}.$$

As before we write $I(g, G) = I_1(g, G) + I_2(G)$ (see (3.8) and (3.12)). By Theorem 4.2.2 in Bouchitté, Fonseca and Mascarenhas [10] we have that

$$I_2(G) = \int_\Omega W_2(G, \nabla G) \, dx + \int_{S_G \cap \Omega} \gamma_2(G^+, G^-, \nu(G)) \, d\mathcal{H}^{N-1} + \int_\Omega W_2^\infty \left(G, \frac{dD^c G}{d|D^c G|} \right) \, d|D^c G|.$$

Our objective is then to find an integral representation for $I_1(g, G)$. We divide the argument in three steps and restrict the proof to the derivation of the energy density with respect to the Cantor part $D^c g$ of Dg (see Remark 4.3 below).

Step 1. (Localization) As in Section 3.1 we can see that $I_1(g, G, \cdot)\lfloor \mathcal{A}(\Omega)$ is a Radon measure absolutely continuous with respect to $\mathcal{L}^N + \mathcal{H}^{N-1}\lfloor S_g + d|D^c g|$.

Step 2. (Upper bound) Let us prove that for $|D^c g|$-a.e $x_0 \in \Omega$

$$\frac{dI_1(g, G, .)}{d|D^c g|}(x_0) \le W_1 \left(-\frac{dD^c g}{d|D^c g|}(x_0) \right). \tag{4.54}$$

We start with an auxiliary result.

Lemma 4.1 *Let $(g, G) \in GSD(\Omega; \mathbb{R}^d)$. For all $A \in \mathcal{A}(\Omega)$ define*

$$\tilde{I}_1(g, G, A) = \inf_{\substack{\{g_n\} \subset SBV^2(A; \mathbb{R}^d) \\ \{G_n\} \subset SBV\left(A; \mathbb{R}^{d \times N}\right)}} \left\{ \liminf_{n \to \infty} I_1(g_n, G_n, A), \; g_n \xrightarrow[n \to \infty]{L^1} g, \; G_n \xrightarrow[n \to \infty]{L^1} G \right\}.$$

Then $\tilde{I}_1(g, G, A) = I_1(g, G, A)$.

Proof. Let $(g_n, \nabla g_n) \in SD(\Omega; \mathbb{R}^d)$ with $g_n \underset{n\to\infty}{\overset{L^1}{\longrightarrow}} g$ and $\nabla g_n \underset{n\to\infty}{\overset{L^1}{\longrightarrow}} G$. Then

$$I_1(g_n, \nabla g_n, A) \le \int_{S_{g_n}\cap A} \psi_1([g_n], \nu(g_n))\, d\mathcal{H}^{N-1}$$

for all $n \in \mathbb{N}$. Hence

$$\liminf_{n\to\infty} \int_{S_{g_n}\cap A} \psi_1([g_n], \nu(g_n))\, d\mathcal{H}^{N-1} \ge \liminf_{n\to\infty} I_1(g_n, \nabla g_n, A) \ge \tilde{I}_1(g, G, A).$$

By the arbitrariness of the sequence $\{g_n\}$ it follows that

$$I_1(g, G, A) \ge \tilde{I}_1(g, G, A).$$

To show that the reverse inequality is true let $\tilde{I}_1(g, G, A) = \liminf_{n\to\infty} I_1(g_n, G_n, A)$ with $g_n \in SBV^2(A; \mathbb{R}^d)$, $G_n \in SBV\Big(A; \mathbb{R}^{d\times N}\Big)$, $g_n \underset{n\to\infty}{\overset{L^1}{\longrightarrow}} g$ and $G_n \underset{n\to\infty}{\overset{L^1}{\longrightarrow}} G$. For each $n \in \mathbb{N}$ let $u_n \in SBV^2(A; \mathbb{R}^d)$ be such that

$$I_1(g_n, G_n, A) + \frac{1}{n} \ge \int_{S_{u_n}} \psi_1([u_n], \nu(u_n))\, d\mathcal{H}^{N-1},$$

$|u_n - g_n|_{L^1} \le \frac{1}{n}$ and $|\nabla u_n - G_n|_{L^1} \le \frac{1}{n}$. Therefore

$$\tilde{I}_1(g, G, A) = \liminf_{n\to\infty} I_1(g_n, G_n, A) \ge \liminf_{n\to\infty} \int_{S_{u_n}} \psi_1([u_n], \nu(u_n))\, d\mathcal{H}^{N-1} \ge I_1(g, G, A).$$

■

Let g_n be a sequence of regular functions such that $g_n \underset{n\to\infty}{\overset{L^1}{\longrightarrow}} g$, $||Dg_n||(\Omega) \underset{n\to\infty}{\longrightarrow} ||Dg||(\Omega)$ and in addition consider $G_n \in SBV(\Omega; \mathbb{R}^{d\times N})$ with $G_n \underset{n\to\infty}{\overset{L^1}{\longrightarrow}} G$. Given $A \in \mathcal{A}(\Omega)$ by Theorem 3.1 (see (3.13)), Lemma 4.1 and Proposition 3.5 we get that

$$\begin{aligned} I_1(g, G, A) &\le \liminf_{n\to\infty} I_1(g_n, G_n, A) \\ &= \liminf_{n\to\infty} \int_A W_1(G_n(x) - \nabla g_n(x))\, dx \\ &\le C \int_A |G(x)|\, dx + \lim_{n\to\infty} \int_A W_1(-\nabla g_n(x))\, dx \\ &= C \int_A |G(x)|\, dx + \int_A W_1(-Dg(x))\, dx, \end{aligned} \tag{4.55}$$

where the last equality follows by Theorem 2.2 since W_1 is Lipschitz continuous and homogeneous of degree one.

Let $x_0 \in \operatorname{supp}|D^c g|$. By (4.55) we get that

$$\begin{aligned} \frac{dI_1(g, G, .)}{d|D^c g|}(x_0) &= \lim_{\delta\to 0} \frac{I_1(g, G, Q(x_0, \delta))}{|D^c g|(Q(x_0, \delta))} \\ &\le W_1\left(-\frac{dD^c g}{d|D^c g|}(x_0)\right). \end{aligned}$$

that is, (4.54) holds.

Step 3. (Lower bound) Let us prove that for $|D^c g|$-a.e $x_0 \in \Omega$

$$\frac{dI_1(g, G, .)}{d|D^c g|}(x_0) \geq W_1\left(-\frac{dD^c g}{d|D^c g|}(x_0)\right). \tag{4.56}$$

We start with the following characterization of the density W_1.

Proposition 4.2 *Let $\nu \in S^{N-1}$ and define for all $C \in \mathbb{R}^{d\times N}$*

$$\tilde{W}_1(C) = \inf\left\{\int_{Q_\nu} W_1(C - \nabla v(x))\, dx + \int_{Q_\nu \cap S_v} \gamma_1([v], \nu(v))\, d\mathcal{H}^{N-1},\right.$$

$$\left. v \in SBV^2(Q_\nu; \mathbb{R}^d),\ v|_{\partial Q_\nu}(x) = b(x\cdot\nu),\ b \in SBV^2([-1/2, 1/2]; \mathbb{R}^d),\ b(1/2) = b(-1/2)\right\}.$$

Then $\tilde{W}_1(C) = W_1(C)$.

Proof. Clearly $\tilde{W}_1(C) \leq W_1(C)$. Let us prove the reverse inequality. Fix $\epsilon > 0$ and let $v \in SBV^2(Q_\nu; \mathbb{R}^d)$ with $v|_{\partial Q_\nu}(x) = b(x \cdot \nu)$ for some $b \in SBV^2([-1/2, 1/2]; \mathbb{R}^d)$, with $b(1/2) = b(-1/2)$ be such that

$$\tilde{W}_1(C) \geq \int_{Q_\nu} W_1(C - \nabla v(x))\, dx + \int_{Q_\nu \cap S_v} \gamma_1([v], \nu(v))\, d\mathcal{H}^{N-1} - \epsilon. \tag{4.57}$$

Extend v by periodicity to all of $\mathbb{R}^N$ and define

$$w_n(y) = \frac{v(ny)}{n} - Cy,\ y \in Q_\nu.$$

By Theorem 3.1 it follows that

$$\begin{aligned} I_1(w_n, 0, Q_\nu) &= \int_{Q_\nu} W_1(C - \nabla v(ny))\, dy + \frac{1}{n}\int_{Q_\nu \cap \{y:\ ny \in S_v\}} \gamma_1([v](ny), \nu(v)(ny))\, d\mathcal{H}^{N-1} \\ &= \int_{Q_\nu} W_1(C - \nabla v(ny))\, dy + \frac{1}{n^N}\int_{nQ_\nu \cap S_v} \gamma_1([v](y), \nu(v)(y))\, d\mathcal{H}^{N-1} \\ &= \int_{Q_\nu} W_1(C - \nabla v(ny))\, dy + \int_{Q_\nu \cap S_v} \gamma_1([v](y), \nu(v)(y))\, d\mathcal{H}^{N-1}. \end{aligned}$$

Therefore, by Riemann Lebesgue's Lemma

$$\lim_{n\to\infty} I_1(w_n, 0, Q_\nu) = \int_{Q_\nu} W_1(C - \nabla v(y))\, dy + \int_{Q_\nu \cap S_v} \gamma_1([v](y), \nu(v)(y))\, d\mathcal{H}^{N-1},$$

and consequently, by Proposition 3.6,

$$I_1(-C(.), 0, Q_\nu) \leq \int_{Q_\nu} W_1(C - \nabla v(y))\, dy + \int_{Q_\nu \cap S_v} \gamma_1([v](y), \nu(v)(y))\, d\mathcal{H}^{N-1}.$$

As $I_1(-C(.), 0, Q_\nu) = \int_{Q_\nu} W_1(C)\, dy = W_1(C)$, then by (4.57)

$$\tilde{W}_1(C) \geq W_1(C) - \epsilon$$

and the result follows by letting $\varepsilon \to 0$.

■

Let $x_0 \in \mathrm{supp}|D^c g|$. By Alberti's Rank One Theorem (cf. [3])

$$\frac{dD^c g}{d|D^c g|}(x_0) = a_{x_0} \otimes \nu_{g(x_0)}$$

for $a_{x_0} \equiv a \in \mathbb{R}^d$ and $\nu_{g(x_0)} \equiv \nu \in S^{N-1}$. Therefore showing (4.56) is equivalent to see that

$$\frac{dI_1(g, G, .)}{d|D^c g|}(x_0) \geq W_1\left(-a \otimes \nu\right). \tag{4.58}$$

To prove (4.58) let $(g_n, G_n) \in SBV^2(\Omega; \mathbb{R}^d) \times SBV(\Omega; \mathbb{R}^{d\times N})$ be a sequence with $g_n \underset{n\to\infty}{\overset{L^1(\Omega;\mathbb{R}^d)}{\longrightarrow}} g$ and $G_n \underset{n\to\infty}{\overset{L^1(\Omega;\mathbb{R}^{d\times N})}{\longrightarrow}} G$, and fix $\delta > 0$.

Note that by Proposition 3.5 it follows that

$$\lim_{n\to\infty} \left[I_1(g_n, G_n, Q_\nu(x_0, \delta)) - I_1(g_n, G, Q_\nu(x_0, \delta))\right] = 0$$

By Theorem 3.1 **and** (3.8) **we have**

$$\begin{aligned} I_1(g_n, G, Q_\nu(x_0, \delta)) &= \int_{Q_\nu(x_0,\delta)} W_1(G - \nabla g_n)\, dx + \int_{Q_\nu(x_0,\delta)\cap S_{g_n}} \gamma_1([g_n], \nu(g_n))\, d\mathcal{H}^{N-1} \\ &= \delta^N \int_{Q_\nu} W_1(G(x_0 + \delta\, y) - \nabla g_n(x_0 + \delta\, y))\, dy \\ &\quad + \delta^{N-1} \int_{Q_\nu \cap \{y:\ x_0 + \delta\, y \in S_{g_n}\}} \gamma_1\big([g_n](x_0 + \delta\, y), \nu(g_n)(x_0 + \delta\, y)\big)\, d\mathcal{H}^{N-1}. \end{aligned}$$

Defining

$$t_\delta = \frac{|D^c g|(Q_\nu(x_0, \delta))}{\delta^N} \tag{4.59}$$

we have that

$$\begin{aligned} \frac{I_1(g_n, G_n, Q_\nu(x_0, \delta))}{|D^c g|(Q_\nu(x_0, \delta))} &= \frac{1}{t_\delta} \int_{Q_\nu} W_1(G(x_0 + \delta\, y) - \nabla g_n(x_0 + \delta\, y))\, dy \\ &\quad + \frac{1}{\delta t_\delta} \int_{Q_\nu \cap \{y:\ x_0 + \delta\, y \in S_{g_n}\}} \gamma_1\big([g_n](x_0 + \delta\, y), \nu(g_n)(x_0 + \delta\, y)\big)\, d\mathcal{H}^{N-1}. \end{aligned}$$

Set now

$$w_{n,\delta}(y) = \frac{g_n(x_0 + \delta y) - \displaystyle\int_{Q_\nu} g_n(x_0 + \delta y)\ dy}{\delta\, t_\delta}, \quad y \in Q_\nu.$$

Hence

$$\frac{I_1(g_n, G_n, Q_\nu(x_0,\delta))}{|D^c g|(Q_\nu(x_0,\delta))} = \frac{1}{t_\delta}\int_{Q_\nu} W_1(G(x_0+\delta\, y) - t_\delta \nabla w_{n,\delta}(y)))\, dy + \int_{Q_\nu \cap S_{w_{n,\delta}}} \gamma_1\big([w_{n,\delta}](y), \nu(w_{n,\delta})(y)\big)\, d\mathcal{H}^{N-1}$$

from where

$$\liminf_{\delta\to 0}\liminf_{n\to\infty} \frac{I_1(g_n, G_n, Q_\nu(x_0,\delta))}{|D^c g|(Q_\nu(x_0,\delta))} = \liminf_{\delta\to 0}\liminf_{n\to\infty}\left[\frac{1}{t_\delta}\int_{Q_\nu} W_1(G(x_0+\delta\, y) - t_\delta \nabla w_{n,\delta}(y)))\, dy + \int_{Q_\nu \cap S_{w_{n,\delta}}} \gamma_1\big([w_{n,\delta}](y), \nu(w_{n,\delta})(y)\big)\, d\mathcal{H}^{N-1}\right].$$

Using Alberti's result on the blow-up of the Cantor part (see [3], Theorem 2.3 in [4] and Lemma 5.1 in [25]), there exists a nondecreasing function $\zeta \in BV[-1/2, 1/2]$ such that

$$\zeta\,(1/2) - \zeta\,(-1/2) = 1, \quad \int_{-\frac{1}{2}}^{\frac{1}{2}} \zeta(s)\, ds = 0 \tag{4.60}$$

and

$$\lim_{\delta\to 0}\lim_{n\to\infty}\int_{Q_\nu} |w_{n,\delta}(y) - a\zeta(y.\nu)|\, dy = 0.$$

Consequently, passing to a diagonalizing sequence $\{w_k\}$ and using the homogeneity property of W_1

$$\liminf_{\delta\to 0}\liminf_{n\to\infty} \frac{I_1(g_n, G_n, Q_\nu(x_0,\delta))}{|D^c g|(Q_\nu(x_0,\delta))} = \lim_{k\to\infty}\left[\int_{Q_\nu} W_1\left(\frac{G(x_0+\delta_k\, y)}{t_{\delta_k}} - \nabla w_k(y)\right) dy + \int_{Q_\nu \cap S_{w_k}} \gamma_1\big([w_k](y), \nu(w_k)(y)\big)\, d\mathcal{H}^{N-1}\right].$$

Now set

$$v_k(y) = \frac{a(\rho_k * \zeta)(y.\nu)}{c_k}, \ y \in Q_\nu, \tag{4.61}$$

where ρ_k denotes the standard mollifier sequence, and

$$c_k = (\rho_k * \zeta)\big(1/2\big) - (\rho_k * \zeta)\big(-1/2\big).$$

It is clear, by (4.60), that $c_k \to 1$ as $k \to \infty$. Since

$$w_k - v_k \ \underset{k\to\infty}{\overset{L^1}{\longrightarrow}} 0$$

with a similar argument to the one used in Lemma 3.3 we can assume that $w_k\lfloor_{\partial Q_\nu} = v_k\lfloor_{\partial Q_\nu}$. Thus defining

$$\bar{w}_k(y) = w_k(y) - (a \otimes \nu)y, \ y \in Q_\nu,$$

we have that

$$\liminf_{\delta\to 0}\liminf_{n\to\infty}\frac{I_1(g_n,G_n,Q_\nu(x_0,\delta))}{|D^cg|(Q_\nu(x_0,\delta))} \geq \lim_{k\to\infty}\int_{Q_\nu} W_1\left(\frac{G(x_0+\delta_k y)}{t_{\delta_k}} - \nabla\bar{w}_k(y)) - (a\otimes\nu)\right)\,dy + \int_{Q_\nu\cap S_{\bar{w}_k}} \gamma_1([\bar{w}_k](y),\nu(\bar{w}_k)(y))\,d\mathcal{H}^{N-1}.$$

Since by Proposition 3.5 and (4.59)

$$\begin{aligned}
\lim_{k\to\infty}\int_{Q_\nu}&\left|W_1\left(\frac{G(x_0+\delta_k y)}{t_{\delta_k}} - \nabla\bar{w}_k(y) - (a\otimes\nu)\right) - W_1\left(-\nabla\bar{w}_k(y) - (a\otimes\nu)\right)\right|\,dy\\
&\leq \lim_{k\to\infty} C\int_{Q_\nu}\left|\frac{G(x_0+\delta_k y)}{t_{\delta_k}}\right|\,dy\\
&= \lim_{k\to\infty}\frac{C}{t_{\delta_k}}\int_{Q_\nu}|G(x_0+\delta_k y)|\,dy\\
&= \lim_{k\to\infty}\frac{1}{t_{\delta_k}\delta_k^N}\int_{Q_\nu(x_0,\delta_k)}|G(x)|\,dx = 0. \qquad (4.62)
\end{aligned}$$

From (4.61) it is easy to see that for each $k\in\mathbb{N}$ the function $\bar{w}_k(y)$ is admissible for $\tilde{W}_1$. Therefore from (4.62) and Proposition 4.2 we conclude that

$$\begin{aligned}
\liminf_{\delta\to 0}\liminf_{n\to\infty}\frac{I_1(g_n,G_n,Q_\nu(x_0,\delta))}{|D^cg|(Q_\nu(x_0,\delta))} &\geq \lim_{k\to\infty}\int_{Q_\nu} W_1\big(-\nabla\bar{w}_k(y)) - (a\otimes\nu)\big)\,dy\\
&\quad + \int_{Q\cap S_{\bar{w}_k}}\gamma_1([\bar{w}_k](y),\nu(\bar{w}_k)(y))\,d\mathcal{H}^{N-1}\\
&\geq \tilde{W}_1\big(-a\otimes\nu\big) = W_1\big(-a\otimes\nu\big).
\end{aligned}$$

Finally the lower bound (4.58) follows from the arbitrariness of the sequence $(g_n,\,G_n)\in SBV^2(\Omega;\mathbb{R}^d)\times SBV(\Omega;\mathbb{R}^{d\times N})$ considered and from the characterization of I_1 given in Lemma 4.1.

Remark 4.3 We note that the densities $\frac{dI_1(g,G,\cdot)}{d\mathcal{L}^N}$ and $\frac{dI_1(g,G,\cdot)}{d\mathcal{H}^{N-1}\lfloor S_g}$ are the ones derived in Theorem 3.1. The proof of the upper bounds can be obtained in a similar way than in (4.54). Indeed, it is enough to choose sequences $g_n \underset{n\to\infty}{\overset{L^1}{\longrightarrow}} g$ with $|Dg_n|(\Omega) \underset{n\to\infty}{\longrightarrow} |Dg|(\Omega)$ (regular functions for the case of $\frac{dI_1(g,G,\cdot)}{d\mathcal{L}^N}$ and piecewise constant functions to address $\frac{dI_1(g,G,\cdot)}{d\mathcal{H}^{N-1}\lfloor S_g}$). Since both W_1 and γ_1 are homogeneous functions of degree one, the result follows from Lemma 4.1 and Theorem 2.2. It is also easy to check that the lower bounds hold since the proof of their counterparts in Theorem 3.1 is still valid in the BV setting.

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