

## The Strangest Numbers in String Theory

A forgotten number system invented in the mid-19th century may provide the simplest explanation for why our universe could have 10 dimensions

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As children we all learn about numbers. We start with counting, followed by addition and subtraction, multiplication and division. But mathematicians know that the number system we study in school is but one of many possibilities. And indeed, other kinds of numbers are important for understanding geometry and physics. Among the strangest alternatives is the octonions. Largely neglected since their discovery in 1843, in the last few decades they have assumed a curious importance in string theory. And indeed, if string theory is a correct representation of the universe, they may explain why the universe has the number of dimensions it does.

### The Imaginary Made Real

The octonions wouldn't be the first piece of pure mathematics that was later used to enhance our understanding of the cosmos. They wouldn't even form the first alternative number system that was later shown to have practical uses. To understand why, we first have to look at the simplest case of numbers—the number system we learn about in school—which mathematicians call the real numbers. The set of all real numbers forms a line, so we say that the collection of real numbers is one-dimensional. We could also turn this idea on its head: the line is one-dimensional because specifying a point on it requires one real number.

Before the 1500s, the real numbers were the only game in town. Then during the Renaissance, ambitious mathematicians attempted to solve ever more complex forms of equations, even holding competitions to see who could solve the most difficult problems. The square root of  $-1$  was introduced as a kind of secret weapon by the Italian mathematician, physician, gambler and astrologer Gerolamo Cardano. Where others might cavil, he boldly let himself use this mysterious number as part of longer calculations where the answers were ordinary real numbers. He wasn't sure why this trick worked; all he knew is that it gave him the right answers. He published his ideas in

1545, thus beginning a controversy that lasted for centuries: does the square root of  $-1$  really exist, or is it only a trick? Nearly 100 years later, no less a thinker than Rene Descartes rendered his verdict when he gave it the derogatory name “imaginary,” or  $i$ .

Nevertheless, mathematicians followed in Cardano’s footsteps and began working with complex numbers—numbers of the form  $a+bi$ , where  $a$  and  $b$  are ordinary real numbers. Around 1805, Jean-Robert Argand popularized the idea that complex numbers describe points on the plane. How does  $a+bi$  describe a point on the plane? Simple: the number  $a$  tells us how far left or right the point is, while  $b$  tells us how far up or down it is.

In this way, we can think of any complex number as a point in the plane, but Argand went a step further: he showed how to think of the operations one can do with complex numbers—addition, subtraction, multiplication and division—as geometric manipulations on the plane.

As a warm-up for understanding how these operations can be thought of as geometric manipulations, first think about the real numbers. Adding or subtracting any real number slides the real line to the left or right. Multiplying or dividing by any positive number stretches or squashes the line. For example, multiplying by  $2$  stretches the line by a factor of  $2$ , while dividing by  $2$  squashes it down, moving all the points twice as close as they were. Multiplying by  $-1$  flips the line over.

The same story works for complex numbers, with just a few extra twists. Adding any complex number  $a+bi$  to a point in the plane slides that point left (or right) by an amount  $a$ , and up (or down) by an amount  $b$ . Multiplying by a complex number stretches or squashes but also *rotates* the complex plane. In particular, multiplying by  $i$  rotates the plane a quarter turn. Thus, if we multiply  $1$  by  $i$  by  $i$ , we rotate the plane a full half turn from the starting point to arrive at  $-1$ . Division is the opposite of multiplication, so to divide we just shrink instead of stretching, or vice versa, and then rotate in the opposite direction.

Almost everything we can do with real numbers can also be done with complex numbers. In fact, most things work better, as Cardano knew, since we can solve more equations with complex numbers than with real numbers. But if a two-dimensional number system gives the user added calculating power, what about even higher dimensional systems? Unfortunately, a

simple extension turns out to be impossible. An Irish mathematician would uncover the secret to higher-dimensional number systems decades later. And only now, two centuries on, are we beginning to understand how powerful they can be.

## Hamilton's Alchemy

In 1835, at the age of 30, the Irish mathematician and physicist William Rowan Hamilton discovered how to treat complex numbers as pairs of real numbers. At the time mathematicians commonly wrote complex numbers in the form  $a+bi$  that Argand popularized, but Hamilton noted that we are also free, if we like, to think of the number  $a+bi$  as just a peculiar way of writing a list of two real numbers—for instance  $(a, b)$ .

This notation makes it very easy to add and subtract complex numbers—just add or subtract each number in the second (complex) list to the corresponding number in the first (real) list. Hamilton also came up with slightly more involved rules for how to multiply and divide complex numbers so that they maintained the nice geometrical meaning discovered by Argand.

After Hamilton invented this *algebraic* system for complex numbers that had a *geometric* meaning, he tried for many years to invent a bigger algebra of triplets that would play a similar role in three-dimensional geometry, an effort that seemed to give him no end of frustrations. He once wrote to his son, “Every morning ... on my coming down to breakfast, your (then) little brother William Edwin, and yourself, used to ask me: ‘Well, Papa, can you multiply triplets?’ Whereto I was always obliged to reply, with a sad shake of the head: ‘No, I can only add and subtract them.’” Though he could not have known it at the time, the task he had given himself was mathematically impossible.

Hamilton was searching for a three-dimensional number system in which he could add, subtract, multiply and divide. Division is the hard part: a number system where we can divide is called a division algebra. In 1958, three mathematicians proved an amazing fact that had been suspected for decades: any division algebra must have dimension 1 (which is just the real numbers), 2 (the complex numbers), 4 or 8. To succeed, Hamilton had to change the rules of the game.

Hamilton himself figured out a solution on the 16th of October, 1843. He was

walking with his wife along the Royal Canal to a meeting of the Royal Irish Academy in Dublin when he had a sudden revelation. Rotations in three dimensions couldn't be described with just three numbers. He needed a fourth number, thereby generating a four-dimensional set called *quaternions* that take the form  $a+bi+cj+dk$ . Here, the numbers  $i$ ,  $j$  and  $k$  are three *unique* square roots of  $-1$ . Hamilton later wrote: "I then and there felt the galvanic circuit of thought close; and the sparks which fell from it were the fundamental equations between  $i$ ,  $j$ , and  $k$ ; exactly such as I have used them ever since." And in a famous act of mathematical vandalism, he carved these equations into the stone of the Brougham Bridge. They are now buried under other layers of graffiti, but a plaque has been placed there to commemorate the discovery.

It may seem odd that we need points in a four-dimensional space to describe changes in three-dimensional space, but it's true. Three of the numbers come from describing rotations, which we can see most readily if we imagine trying to fly an airplane. To orient the plane however we want, we need to control the pitch, or angle with the horizontal. We also may need to adjust the yaw, by turning left or right, like a car. And finally, we may need to adjust the roll: the angle of the plane's wings. The fourth number we need is used to describe stretching or shrinking.

Hamilton spent the rest of his life obsessed with the quaternions, and found many practical uses for them. Today, in many of these applications the quaternions have been replaced by their simpler cousins: vectors, which can be thought of as quaternions of the special form  $ai+bj+ck$  (the first number is just zero). Yet quaternions still have their niche: they provide an efficient way to represent three-dimensional rotations on a computer, and show up wherever this is needed, from the attitude-control system of a spacecraft to the graphics engine of a video game.

## **Imaginaries Without End**

Despite these applications, we might wonder what, exactly, are  $j$  and  $k$  if we've already defined the square root of  $-1$  as  $i$ . Do these additional square roots of  $-1$  really exist? Can we just keep inventing new square roots of  $-1$  to our heart's content?

These questions were asked by Hamilton's college friend, the lawyer John Graves, whose amateur interest in algebra got Hamilton thinking about

complex numbers and triplets in the first place. The very day after his fateful walk in the fall of 1843, Hamilton sent Graves a letter describing his breakthrough. Graves replied nine days later, complimenting Hamilton on the boldness of the idea, but adding, “There is still something in the system which gravels me. I have not yet any clear views as to the extent to which we are at liberty arbitrarily to create imaginaries, and to endow them with supernatural properties.” And he asked: “If with your alchemy you can make three pounds of gold, why should you stop there?”

Like Cardano before him, Graves set his concerns aside for long enough to conjure some gold of his own. On December 26th, he wrote to Hamilton describing a new 8-dimensional number system that he called the octaves and which are now called *octonions*. Graves was unable to get Hamilton interested in his ideas, however. Hamilton promised to speak about Graves’ octaves at the Irish Royal Society, which is one way mathematical results were published at the time. But Hamilton kept putting it off, and in 1845 the young genius Arthur Cayley rediscovered the octonions and beat Graves to publication. For this reason, the octonions are also sometimes known as Cayley numbers.

Why didn’t Hamilton like the octonions? For one thing, he was obsessed with research on his own discovery, the quaternions. But he also had a purely mathematical reason: the octonions break some cherished laws of arithmetic.

The quaternions were already a bit strange. When you multiply real numbers, it doesn’t matter which order you do it—2 times 3 equals 3 times 2, for example. We say that multiplication *commutes*. The same holds for complex numbers. But quaternions are noncommutative. The order of multiplication matters.

Order is important because quaternions describe rotations in three dimensions, and for such rotations the order makes a difference to the outcome.. You can check this yourself. Take a magazine, turn it a little bit clockwise, then flip it over. Note that it looks like it was rotated a little bit counter clockwise. Now do these two operations in reverse order: flip first, and then rotate a little bit clockwise. The magazine looks like it’s been rotated clockwise. Since the result depends on the order, rotations do not commute.

The octonions are much stranger. Not only are they noncommutative, they are also break another familiar law of arithmetic: the associative law  $(xy)z = x(yz)$ . We have all seen a nonassociative operation in our study of mathematics: subtraction. For example,  $(3-2)-1$  is different from  $3-(2-1)$ . But we are used to multiplication being associative, and most mathematicians still feel this way, even though they have gotten used to noncommutative operations. Rotations are associative, for example, even though they don't commute.

But perhaps most important, it wasn't clear in Hamilton's time just what the octonions would be good for. They are closely related to the geometry of 7 and 8 dimensions, and we can describe rotations in those dimensions using the multiplication of octonions. But for over a century that was a purely intellectual exercise. It would take the development of modern particle physics—and string theory in particular—to see how the octonions might be useful in the real world.

## **Symmetry and Strings**

In the 1970s and '80s theoretical physicists developed a strikingly beautiful idea called *supersymmetry*. (Later, researchers would learn that string theory requires supersymmetry.) It states that at the most fundamental levels, the universe exhibits a symmetry between matter and the forces of nature. Every matter particle (such as an electron) has a partner particle that carries a force. And every force particle (such as a photon, the carrier of the electromagnetic force) has a twin matter particle.

Supersymmetry also encompasses the idea that the laws of physics would remain unchanged if we exchanged all the matter and force particles. Imagine viewing the universe in a strange mirror that, rather than interchanging left and right, traded every force particle for a matter particle, and vice versa. If supersymmetry is true, if it really describes our universe, this mirror universe would act the same as ours. Even though physicists have not yet found any concrete experimental evidence in support of supersymmetry, the theory is so seductively beautiful, and has led to so much enchanting mathematics, that many physicists hope and expect that it is real.

One thing we know to be true, however, is quantum mechanics. And according to quantum mechanics, particles are also waves. In the standard

three-dimensional version of quantum mechanics that physicists use every day, one type of number (called *spinors*) describe the wave motion of matter particles. Another type of number (called *vectors*) to describe the wave motion of force particles. If we want to understand particle interactions we have to multiply these two together, which can only be done in a very careful and peculiar way. While the system we use right now might work, it isn't very elegant at all.

As an alternative, imagine a strange universe with no time, only space. If this universe has dimension 1, 2, 4, or 8, both matter and force particles would be waves described by a single type of number—namely, a number in a division algebra, the only type of system that allows for addition, subtraction, multiplication and division. In other words, in these dimensions the vectors and spinors coincide: they are each just real numbers, complex numbers, quaternions or octonions, respectively. Supersymmetry emerges naturally, providing a unified description of matter and forces. Simple multiplication describes interactions, and all particles—no matter the type—use the same number system.

Yet our plaything universe can't be real, because we need to take time into account. In string theory this consideration has a curious effect. At any moment in time a string is a 1-dimensional thing, like a curve or line. But this string traces out a two-dimensional surface as time passes. This changes the dimensions in which supersymmetry naturally arises, by adding two—one for the string, and one for time. Instead of supersymmetry in dimensions 1, 2, 4 or 8, we get supersymmetry in dimensions 3, 4, 6, or 10.

Coincidentally, string theorists have for years been saying that only 10-dimensional versions of the theory are self-consistent. The rest suffer from glitches called anomalies, where computing the same thing in two different ways gives different answers. In anything other than 10 dimensions, string theory breaks down. But 10-dimensional string theory is, as we have just seen, the version of the theory that uses octonions. So, if string theory is right, the octonions are not a useless curiosity: on the contrary, they provide the deep reason why the universe must have 10 dimensions: in 10 dimensions, matter and force particles are embodied in the same type of numbers—the octonions.

But this is not the end of the story. Recently physicists have started to go beyond strings to consider membranes. For example, a two-dimensional

membrane or 2-brane looks like a soap bubble at any moment in time. As time passes, it traces out a three-dimensional volume in spacetime.

Whereas in string theory we had to add two dimensions to our standard collection 1, 2, 4 and 8, now we must add three. Thus when we're dealing with membranes we would expect supersymmetry to naturally emerge in dimensions 4, 5, 7 and 11. And as in string theory we have a surprise in store: researchers tell us that M-theory (the "M" typically stands for "membrane") requires 11 dimensions—implying that it should naturally make use of octonions. Alas, nobody understands M-theory well enough to even write down its basic equations—that M can also stand for "mysterious." It is hard to tell precisely what shape it might take in the future.

At this point we should emphasize that string theory and M-theory have as of yet made no experimentally testable predictions. They are beautiful dreams—but so far only dreams. The universe we live in does not look 10- or 11-dimensional, and we have not seen any symmetry between matter and force particles. David Gross, one of the world's leading experts on string theory, currently puts the odds of seeing some evidence for supersymmetry at the Large Hadron Collider at 50 percent. Skeptics say they are much less. Only time will tell.

Because of this uncertainty, we are still a long way from knowing if the strange octonions are of fundamental importance in understanding the world we see around us or are merely a piece of beautiful mathematics. Of course mathematical beauty is a worthy end in itself, but it would be even more delightful if the octonions turned out to be built into the fabric of nature. And as the story of the complex numbers and countless other pieces of mathematics demonstrate, it would hardly be the first time that purely mathematical inventions later provided precisely the tools that physicists need.

## **MORE TO EXPLORE**

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