The Algebra of Grand Unified Theories

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Oral Exam Presentation

This talk is an introduction to the representation theory used in

- The Standard Model of Particle Physics (SM);
- Certain extensions of the SM, called Grand Unified Theories (GUTs).

There's a lot I won't talk about:

- quantum field theory;
- spontaneous symmetry breaking;
- any sort of dynamics.

This stuff is *essential* to particle physics. What I discuss here is just one small piece.

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There's a loose correspondence between particle physics and representation theory:

- Particles $\rightarrow$ basis vectors in a representation V of a Lie group G.
- Classification of particles $\rightarrow$ decomposition into irreps.
- Unification $\rightarrow$ $G \hookrightarrow H$; particles are "unified" into fewer irreps.
- Grand Unification $\rightarrow$ as above, but H is simple.
- The Standard Model $\rightarrow$ a particular representation V_{SM} of a particular Lie group G_{SM}.

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- Spontaneous symmetry breaking makes the electromagnetic and weak forces look different; at high energies, they're the same.
- $\mathrm{SU}(3)$ corresponds to the *strong force*, which binds quarks together. No symmetry breaking here.

Standard Model Representation		
Name	Symbol	G_{SM}-representation
Left-handed leptons	$\begin{pmatrix} \nu_L \\ e_L^- \end{pmatrix}$	$\mathbb{C}_{-1} \otimes \mathbb{C}^2 \otimes \mathbb{C}$
Left-handed quarks	$\begin{pmatrix} u_L^r, u_L^g, u_L^b \\ d_L^r, d_L^g, d_L^b \end{pmatrix}$	$\mathbb{C}_{\frac{1}{3}} \otimes \mathbb{C}^2 \otimes \mathbb{C}^3$
Right-handed neutrino	ν_R	$\mathbb{C}_0 \otimes \mathbb{C} \otimes \mathbb{C}$
Right-handed electron	e_R^-	$\mathbb{C}_{-2} \otimes \mathbb{C} \otimes \mathbb{C}$
Right-handed up quarks	u_R^r, u_R^g, u_R^b	$\mathbb{C}_{\frac{4}{3}} \otimes \mathbb{C} \otimes \mathbb{C}^3$
Right-handed down quarks	d_R^r, d_R^g, d_R^b	$\mathbb{C}_{-\frac{2}{3}} \otimes \mathbb{C} \otimes \mathbb{C}^3$

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- U is a $\mathrm{U}(1)$ irrep $\mathbb{C}_Y$, where $Y \in \frac{1}{3}\mathbb{Z}$. The underlying vector space is just $\mathbb{C}$, and the action is given by

$$\alpha \cdot z = \alpha^{3Y} z, \quad \alpha \in \mathrm{U}(1), z \in \mathbb{C}$$

- V is an $\mathrm{SU}(2)$ irrep, either $\mathbb{C}$ or $\mathbb{C}^2$.
- W is an $\mathrm{SU}(3)$ irrep, either $\mathbb{C}$ or $\mathbb{C}^3$.

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- The number Y in $\mathbb{C}_Y$ is called the *hypercharge*.
- $\mathbb{C}^2 = \langle u, d \rangle$; u and d are called *isospin up* and *isospin down*.
- $\mathbb{C}^3 = \langle r, g, b \rangle$; r, g, and b are called *red*, *green*, and *blue*.

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For example:

- $u_L^r = 1 \otimes u \otimes r \in \mathbb{C}_{\frac{1}{3}} \otimes \mathbb{C}^2 \otimes \mathbb{C}^3$, say "the red left-handed up quark is the hypercharge $\frac{1}{3}$, isospin up, red particle."
- $e_R^- = 1 \otimes 1 \otimes 1 \in \mathbb{C}_{-2} \otimes \mathbb{C} \otimes \mathbb{C}$, say "the right-handed electron is the hypercharge -2 isospin singlet which is colorless."

- We take the direct sum of all these irreps, defining the reducible representation,

$$F = \mathbb{C}_{-1} \otimes \mathbb{C}^2 \otimes \mathbb{C} \quad \oplus \quad \cdots \quad \oplus \quad \mathbb{C}_{-\frac{2}{3}} \otimes \mathbb{C} \otimes \mathbb{C}^3$$

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  which we'll call the *fermions*.
- We also have the *antifermions*, F^*, which is just the dual of F.
- Direct summing these, we get the *Standard Model representation*

$$V_{\mathrm{SM}} = F \oplus F^*$$

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- Explain the hypercharges!
- Explain other patterns:
  - $\dim V_{\mathrm{SM}} = 32 = 2^5$;
  - symmetry between quarks and leptons;
  - asymmetry between left and right.

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- V is also representation of G_{SM};
- V may break apart into more G_{SM}-irreps than G-irreps.

More precisely, we want:

- A homomorphism $\phi\colon G_{\text{SM}} \to G$.
- A unitary representation $\rho\colon G \to \mathrm{U}(V)$.
- An isomorphism of vector spaces $f\colon V_{\text{SM}} \to V$.
- Such that

$$\begin{array}{ccc} G_{\text{SM}} & \xrightarrow{\phi} & G \\ \downarrow & & \downarrow \rho \\ \mathrm{U}(V_{\text{SM}}) & \xrightarrow{\mathrm{U}(f)} & \mathrm{U}(V) \end{array}$$

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In short: V becomes isomorphic to V_{SM} when we restrict from G to G_{SM}.

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- Take $\mathbb{C}^5 = \langle u, d, r, g, b \rangle$.
- $\mathbb{C}^5$ is a representation of $\mathrm{SU}(5)$, as is the 32-dimensional exterior algebra:

$$\Lambda \mathbb{C}^5 \cong \Lambda^0 \mathbb{C}^5 \oplus \Lambda^1 \mathbb{C}^5 \oplus \Lambda^2 \mathbb{C}^5 \oplus \Lambda^3 \mathbb{C}^5 \oplus \Lambda^4 \mathbb{C}^5 \oplus \Lambda^5 \mathbb{C}^5$$

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- **Theorem** There's a homomorphism $\phi\colon G_{\mathrm{SM}} \to \mathrm{SU}(5)$ and a linear isomorphism $h\colon V_{\mathrm{SM}} \to \Lambda\mathbb{C}^5$ making

$$\begin{array}{ccc} G_{\mathrm{SM}} & \xrightarrow{\ \phi\ } & \mathrm{SU}(5) \\ \downarrow & & \downarrow \\ \mathrm{U}(V_{\mathrm{SM}}) & \xrightarrow{\ \mathrm{U}(h)\ } & \mathrm{U}(\Lambda\mathbb{C}^5) \end{array}$$

commute.

Proof

- Let $\mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(3)) \subseteq \mathrm{SU}(5)$ be the subgroup preserving the $2 + 3$ splitting $\mathbb{C}^2 \oplus \mathbb{C}^3 \cong \mathbb{C}^5$.

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- Can find $\phi \colon G_{\mathrm{SM}} \to \mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(3)) \subseteq \mathrm{SU}(5)$.

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We define ϕ by

$$\phi\colon (\alpha, g, h) \in \mathrm{U}(1)\times\mathrm{SU}(2)\times\mathrm{SU}(3) \longmapsto \begin{pmatrix} \alpha^3 g & 0 \\ 0 & \alpha^{-2} h \end{pmatrix} \in \mathrm{SU}(5)$$

ϕ maps G_{SM} onto $\mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(3))$, but it has a kernel:

$$\ker \phi = \{(\alpha, \alpha^{-3}, \alpha^2) | \alpha^6 = 1\} \cong \mathbb{Z}_6$$

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Thus

$$G_{\mathrm{SM}}/\mathbb{Z}_6 \cong \mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(3))$$

The subgroup $\mathbb{Z}_6 \subseteq G_{\mathrm{SM}}$ acts trivially on V_{SM}.

Because G_{SM} respects the $2 + 3$ splitting

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As a G_{SM}-representation,

$$\Lambda\mathbb{C}^2 \cong \mathbb{C}_0 \otimes \Lambda^0\mathbb{C}^2 \quad \oplus \quad \mathbb{C}_1 \otimes \Lambda^1\mathbb{C}^2 \quad \oplus \quad \mathbb{C}_2 \otimes \Lambda^2\mathbb{C}^2$$

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$$\Lambda\mathbb{C}^3 \cong \mathbb{C}_0 \otimes \Lambda^0\mathbb{C}^3 \quad \oplus \quad \mathbb{C}_{-\frac{2}{3}} \otimes \Lambda^1\mathbb{C}^3 \quad \oplus \quad \mathbb{C}_{-\frac{4}{3}} \otimes \Lambda^2\mathbb{C}^3 \quad \oplus \quad \mathbb{C}_{-2} \otimes \Lambda^3\mathbb{C}^3$$

Then tensor them together, use $\mathbb{C}^2 \cong \mathbb{C}^{2*}$ and $\mathbb{C}_{Y_1} \otimes \mathbb{C}_{Y_2} \cong \mathbb{C}_{Y_1+Y_2}$ to see how

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Thus there's a linear isomorphism $h\colon V_{\mathrm{SM}} \to \Lambda\mathbb{C}^5$ making

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- This creates explicit symmetry between quarks and leptons.
- Unify the $\mathbb{C}^2 \oplus \mathbb{C} \oplus \mathbb{C}$ representations of $\mathrm{SU}(2)$ into the representation $\mathbb{C}^2 \otimes \mathbb{C} \quad \oplus \quad \mathbb{C} \otimes \mathbb{C}^2$ of $\mathrm{SU}(2) \times \mathrm{SU}(2)$.
- This treats left and right more symmetrically.

Standard Model Representation		
Name	Symbol	G_{SM}-representation
Left-handed leptons	$\begin{pmatrix} \nu_L \\ e_L^- \end{pmatrix}$	$\mathbb{C}_{-1} \otimes \mathbb{C}^2 \otimes \mathbb{C}$
Left-handed quarks	$\begin{pmatrix} u_L^r, u_L^g, u_L^b \\ d_L^r, d_L^g, d_L^b \end{pmatrix}$	$\mathbb{C}_{\frac{1}{3}} \otimes \mathbb{C}^2 \otimes \mathbb{C}^3$
Right-handed neutrino	ν_R	$\mathbb{C}_0 \otimes \mathbb{C} \otimes \mathbb{C}$
Right-handed electron	e_R^-	$\mathbb{C}_{-2} \otimes \mathbb{C} \otimes \mathbb{C}$
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The Pati–Salam representation

Name	Symbol	$\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(4)$-representation
Left-handed fermions	$\begin{pmatrix} \nu_L, u_L^r, u_L^g, u_L^b \\ e_L^-, d_L^r, d_L^g, d_L^b \end{pmatrix}$	$\mathbb{C}^2 \otimes \mathbb{C} \otimes \mathbb{C}^4$
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- Write $V_{\mathrm{PS}} = \mathbb{C}^2 \otimes \mathbb{C} \otimes \mathbb{C}^4 \quad \oplus \quad \mathbb{C} \otimes \mathbb{C}^2 \otimes \mathbb{C}^4 \quad \oplus \quad$ dual.

To make the Pati–Salam model work, we need to prove

Theorem There exists maps $\theta\colon G_{\mathrm{SM}} \to G_{\mathrm{PS}}$ and $f\colon V_{\mathrm{SM}} \to V_{\mathrm{PS}}$ which make the diagram

$$\begin{array}{ccc} G_{\mathrm{SM}} & \xrightarrow{\theta} & G_{\mathrm{PS}} \\ \downarrow & & \downarrow \\ \mathrm{U}(V_{\mathrm{SM}}) & \xrightarrow{\mathrm{U}(f)} & \mathrm{U}(V_{\mathrm{PS}}) \end{array}$$

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- Pick θ so G_{SM} maps to a subgroup of $\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(4)$ that preserves the $3 + 1$ splitting

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- $\mathbb{C} \otimes \mathbb{C}^2 \cong \Lambda^{\mathrm{ev}}\mathbb{C}^2 \cong \Lambda^0\mathbb{C}^2 \oplus \Lambda^2\mathbb{C}^2$ has a $1+1$ splitting — the grading!

Build θ so that

- θ maps G_{SM} onto the subgroup $\mathrm{S}(\mathrm{U}(3) \times \mathrm{U}(1)) \subseteq \mathrm{Spin}(6)$ that preserves the $3 + 1$ splitting:

$$(\alpha, x, y) \in \mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3) \mapsto \begin{pmatrix} \alpha y & 0 \\ 0 & \alpha^{-3} \end{pmatrix}$$

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$$(\alpha, x, y) \in \mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3) \mapsto \left(x, \begin{pmatrix} \alpha^3 & 0 \\ 0 & \alpha^{-3} \end{pmatrix} \right)$$

The payoff:

As a G_{SM}-representation,

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Thus there's an isomorphism of vector spaces
$f \colon V_{\mathrm{SM}} \to \Lambda\mathbb{C}^2 \otimes \Lambda\mathbb{C}^3$ such that

$$\begin{array}{ccc} G_{\mathrm{SM}} & \xrightarrow{\theta} & \mathrm{Spin}(4) \times \mathrm{Spin}(6) \\ \downarrow & & \downarrow \\ \mathrm{U}(V_{\mathrm{SM}}) & \xrightarrow{\mathrm{U}(f)} & \mathrm{U}(\Lambda\mathbb{C}^2 \otimes \Lambda\mathbb{C}^3) \end{array}$$

commutes.

Extend the $\mathrm{SU}(5)$ theory to get the $\mathrm{Spin}(10)$ theory, due to Georgi:

- In general,

$$\begin{array}{ccc} \mathrm{SU}(n) & \xrightarrow{\psi} & \mathrm{Spin}(2n) \\ & \searrow & \downarrow \\ & & \mathrm{U}(\Lambda\mathbb{C}^n) \end{array}$$

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- Set $n = 5$:

$$\begin{array}{ccc} \mathrm{SU}(5) & \xrightarrow{\psi} & \mathrm{Spin}(10) \\ \downarrow & & \downarrow \\ \mathrm{U}(\Lambda\mathbb{C}^5) & \xrightarrow{1} & \mathrm{U}(\Lambda\mathbb{C}^5) \end{array}$$

Or extend the Pati–Salam model:

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$$\begin{array}{ccc} \mathrm{Spin}(2n) \times \mathrm{Spin}(2m) & \xrightarrow{\eta} & \mathrm{Spin}(2n+2m) \\ \downarrow & & \downarrow \\ \mathrm{U}(\Lambda\mathbb{C}^n \otimes \Lambda\mathbb{C}^m) & \xrightarrow{\mathrm{U}(g)} & \mathrm{U}(\Lambda\mathbb{C}^{n+m}) \end{array}$$

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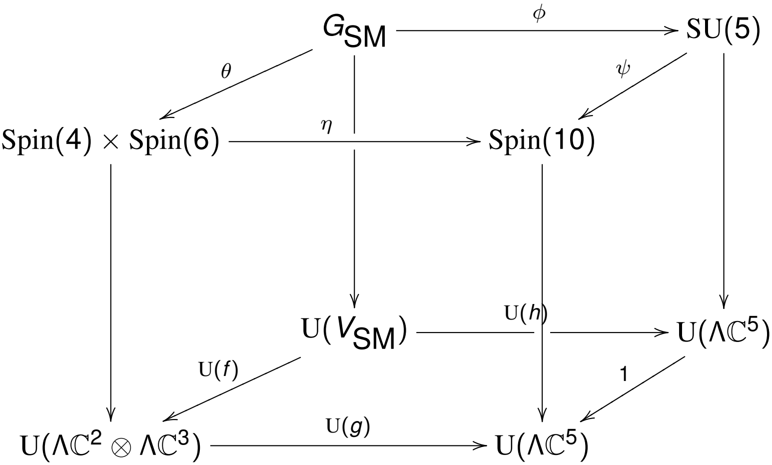
- Set $n = 2$ and $m = 3$:

$$\begin{array}{ccc} \mathrm{Spin}(4) \times \mathrm{Spin}(6) & \xrightarrow{\eta} & \mathrm{Spin}(10) \\ \downarrow & & \downarrow \\ \mathrm{U}(\Lambda\mathbb{C}^2 \otimes \Lambda\mathbb{C}^3) & \xrightarrow{\mathrm{U}(g)} & \mathrm{U}(\Lambda\mathbb{C}^5) \end{array}$$

Theorem The cube of GUTs



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Proof

- The vertical faces of the cube commute.

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\end{array}
$$

- $\Lambda\mathbb{C}^5$ is a faithful representation of $\mathrm{Spin}(10)$ — **the top face commutes:**

$$
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\end{array}
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- The intertwiners commute:

$$\begin{array}{ccc} V_{\mathrm{SM}} & \xrightarrow{h} & \Lambda\mathbb{C}^5 \\ {\scriptstyle f}\downarrow & \nearrow{\scriptstyle g} & \\ \Lambda\mathbb{C}^2 \otimes \Lambda\mathbb{C}^3 & & \end{array}$$

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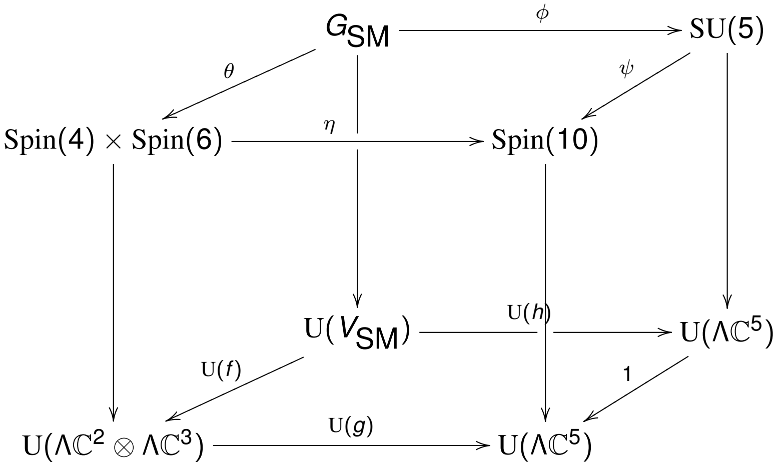
- **The bottom face commutes:**

$$\begin{array}{ccc} \mathrm{U}(V_{\mathrm{SM}}) & \xrightarrow{\mathrm{U}(h)} & \mathrm{U}(\Lambda\mathbb{C}^5) \\ \downarrow \mathrm{U}(f) & & \downarrow 1 \\ \mathrm{U}(\Lambda\mathbb{C}^2 \otimes \Lambda\mathbb{C}^3) & \xrightarrow{\mathrm{U}(g)} & \mathrm{U}(\Lambda\mathbb{C}^5) \end{array}$$

Thus the cube



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