LTTE Q2

From last:

$$Sp(n)$$

$$Ap(n)$$

$$T = \left\{ \begin{pmatrix} e^{i\theta_1} & & \\ & \ddots & \\ & & e^{i\theta_n} \end{pmatrix} \right\} \cong T^n$$

$$\mathfrak{t} = \left\{ \begin{pmatrix} i\theta_1 & & \\ & \ddots & \\ & & i\theta_n \end{pmatrix} : \theta_j \in \mathbb{R} \right\}$$

$$L = \left\{ \begin{pmatrix} i\theta_1 & & \\ & \ddots & \\ & & i\theta_n \end{pmatrix} : \theta_j \in \mathbb{Z} \right\} \cong \mathbb{Z}^n$$

$$= C_n$$

Let's take $K = SU(n)$, $\widetilde{SO(n)} = Spin(n)$, $Sp(n)$ & work out:

1) Dimensions

2) exponential lattices

$$L = \{ x \in \mathfrak{t} : \exp(2\pi x) = 1 \in K \}$$

3) Weight lattices

$$L^* = \{ \ell \in \mathfrak{t}^* : \ell(x) \in \mathbb{Z} \ \forall x \in L \}$$

Recall: Given a rep $\rho : K \to U(H)$, we can restrict it to T & write it as a sum of irreps; every irrep of T corresponds to a point $\ell \in L^*$ as follows: it's $\rho_\ell : T \to U(1)$ where

$$\rho_\ell(\exp(2\pi x)) = e^{2\pi i \ell(x)}, \quad x \in \mathfrak{t}.$$

So any rep ρ of K gives a function, the <u>weighting</u>:

$$d_\rho : L^* \to \mathbb{N}$$

which counts how many times each ρ_ℓ shows up in the direct sum decomposition of ρ.

4) roots. Here take the adjoint rep of K on its own Lie algebra $\mathfrak{k}$ & then complexifying that:

$$\mathbb{C} \otimes \mathfrak{k} = \mathfrak{g}$$

$$\mathrm{Ad} : K \to U(\mathbb{C} \otimes \mathfrak{k})$$

Work out the weighting of this rep

the nonzero $\ell \in L^*$ s.t. $d_{Ad}(\ell) \neq 0$ are called roots.

The lattice L, together w/ the set of roots, determines

Dimensions

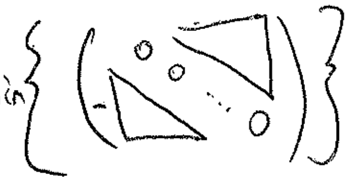
$$\dim SU(n) = \dim \mathfrak{su}(n)$$

$$= \dim \{A \in \mathrm{Mat}_n(\mathbb{C}) : A^* = -A,\ \mathrm{tr}\, A = 0\}$$

$$= n^2 - 1$$

$$\dim SO(n) = \dim AO(n)$$

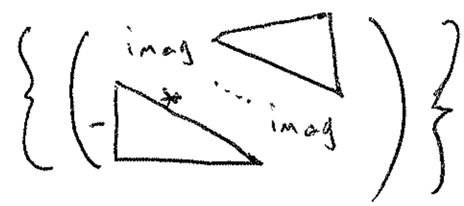
$$= \dim \{ A \in M_n(\mathbb{R}) : A^* = -A, \ \operatorname{tr} A = 0 \}$$

$= \dim$ 

$$= \frac{n(n-1)}{2}$$

$$\dim Sp(n) = \dim Ap(n)$$

$$= \dim \{ A \in M_n(\mathbb{H}) : A^* = -A \}$$

$= \dim$ 

$$= 3n + \frac{n(n-1)}{2} \times 4$$

$$= 2n^2 + n$$

$$= n(2n+1)$$

So the low dimensions are

				$Spin(3)$	3	$Sp(1)$	3
$SU(2)$	3						
$SU(3)$	8	$Spin(4)$	6	$Spin(5)$	10	$Sp(2)$	10
$SU(4)$	15	$Spin(6)$	15	$Spin(7)$	21	$Sp(3)$	21
$SU(5)$	24	$Spin(8)$	28	$Spin(9)$	36	$Sp(4)$	36
$SU(6)$	35	$Spin(10)$	45	$Spin(11)$	55	$Sp(5)$	55

Among the $SU(n)$'s & $Spin(n)$'s, we see the coincidences

$\dim SU(2) = \dim Spin(3)$ & $\dim SU(4) = \dim Spin(6)$. In fact, $SU(2) \cong Spin(3)$ & $SU(4) \cong Spin(6)$, & also $SU(2) \times SU(2) \cong Spin(4)$, the only nonsimple group on this list.

Always have $\dim Spin(2n+1) = \dim Sp(n)$, though only $Sp(1) \cong Spin(3)$ & $Sp(2) \cong Spin(5)$ coincide.

D_2 or $\widetilde{SO}(4)$:

$$T = \left\{ \left(\begin{array}{c|c} e^{\theta_1 J} & 0 \\ \hline 0 & e^{\theta_2 J} \end{array} \right) : \theta_1, \theta_2 \in \mathbb{R} \right\} \subseteq SO(4)$$

where $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ satisfies $J^2 = -1$ &

$$e^{\theta J} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$\mathfrak{t} = \left\{ \begin{pmatrix} \theta_1 J & 0 \\ 0 & \theta_2 J \end{pmatrix} : \theta_i \in \mathbb{R} \right\}$$

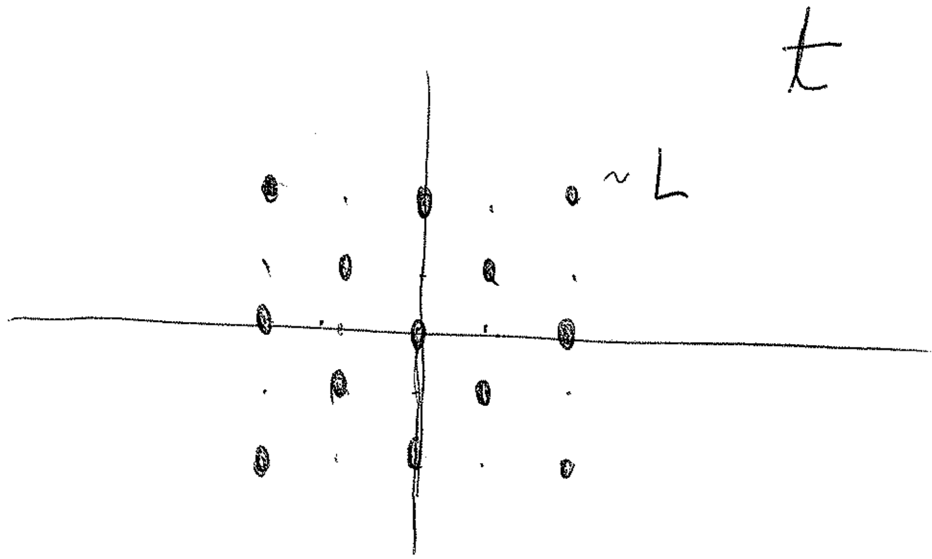
The subset $L' = \{ x \in \mathfrak{t} : \exp(2\pi x) = 1 \in SO(4) \}$

consists of $\begin{pmatrix} \theta_1 J & 0 \\ 0 & \theta_2 J \end{pmatrix}$ w/ $\theta_i \in \mathbb{Z}$.

To lie in $L = \{ x \in \mathfrak{t} : \exp(2\pi x) = 1 \in \widetilde{SO}(4) \}$,

need $\theta_1 + \theta_2 \in 2\mathbb{Z}$.

The Weyl group acts faithfully on $\mathfrak{t}$, preserving L:



So it's a subgroup of the symmetry group of this picture, which is D_4, the dihedral group of the square.

We also have $B_2 = \widetilde{SO(5)}$ w/

$$T = \left\{ \begin{pmatrix} e^{\theta_1 J} & & \\ & e^{\theta_2 J} & \\ & & 1 \end{pmatrix} : \theta_i \in \mathbb{R} \right\}$$

is omorphic to T for $\widetilde{SO(4)}$, & so it's lattice is isomorphic as well.

But W for $\widetilde{SO(5)}$ is a bigger subgroup of D_4 that W for $\widetilde{SO(4)}$.