Every compact simply-connected simple Lie group is on this list:

- $SU(n) = \{ n \times n$ complex matrices U w/ $UU^* = 1$, & $\det(U) = 1 \}$

  $n \geq 2$

  Related to $\mathbb{C}$.

- $Spin(n) = \widetilde{SO(n)}$, the universal cover of

  $n \geq 3$

  $SO(n) = \{ n \times n$ real matrices U w/ $UU^* = 1$, & $\det(U) = 1 \}$

  Related to $\mathbb{R}$.

- $Sp(n) = \{ n \times n$ quaternionic matrices U w/ $UU^* = 1 \}$

  $n \geq 1$.

  Related to $\mathbb{H}$.

- G_2, F_4, E_6, E_7, E_8. Related $\mathbb{O}$, though more mysterious.

The infinite families are called the <u>classical</u> Lie groups, & the last groups are called the <u>exceptional</u> Lie groups.

K compact, simply-connected, Lie group	$SU(n)$	$Spin(n) = \widetilde{SO(n)}$ n even
$T \subseteq K$ maximal torus	$\left\{ \begin{pmatrix} e^{i\theta_1} & & 0 \\ & \ddots & \\ 0 & & e^{i\theta_n} \end{pmatrix} : \prod_{j=1}^{n} e^{i\theta_j} = 1 \right\}$ $\cong$ T^{n-1}	double cover of $\left\{ \begin{pmatrix} R_{\theta_1} & & 0 \\ & \ddots & \\ 0 & & R_{\theta_{n/2}} \end{pmatrix} : \theta_i \in \mathbb{R} \right\} \cong T^{n/2}$ where $R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$
Lattice $L \subseteq \mathfrak{t}$ $L = \{ x \in \mathfrak{t} : \exp(2\pi x) = 1 \}$	$\left\{ \begin{pmatrix} i\theta_1 & & 0 \\ & \ddots & \\ 0 & & i\theta_n \end{pmatrix} : \theta_i \in \mathbb{Z}, \sum_j \theta_j = 0 \right\}$ $=$ A_N, where $N = n-1$	$\left\{ \begin{pmatrix} J\theta_1 & & 0 \\ & \ddots & \\ 0 & & J\theta_{n/2} \end{pmatrix} : \begin{array}{l} \theta_i \in \mathbb{Z}, \\ \sum \theta_i \in 2\mathbb{Z} \end{array} \right\}$ $=$ $\quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathfrak{so}(2)$ D_N, where $N = n/2$.

$Spin(n) = \widetilde{SO(n)}$ n odd	$Sp(n)$
double cover of $\left\{ \begin{pmatrix} R_{\theta_1} & & 0 \\ & \ddots & \\ 0 & & R_{\theta_{\frac{n-1}{2}}} & \\ & & & 1 \end{pmatrix} : \theta_i \in \mathbb{R} \right\}$ $\cong$ $T^{\frac{n-1}{2}}$	$\left\{ \begin{pmatrix} e^{i\theta_1} & & 0 \\ & \ddots & \\ 0 & & e^{i\theta_n} \end{pmatrix} : \theta_i \in \mathbb{R} \right\}$ $\cong$? T^n
$\left\{ \begin{pmatrix} J\theta_1 & & \\ & \ddots & \\ & & J\theta_{n-1/2} & \\ & & & 0 \end{pmatrix} : \begin{matrix} \theta_i \in \mathbb{Z} \\ \sum \theta_i \in 2\mathbb{Z} \end{matrix} \right\}$ $\parallel$ $B_N, \quad N = \frac{n-1}{2}$ $B_N \cong D_N$, but something's different!	? $B_N \cong C_N$, but something's different! It's called C_N, $N = n$

The difference between D_n, B_n & C_n is not the lattice, but a choice of certain special vectors in the lattice, called <u>roots</u>.

For A_n, D_n, E_6, E_7, E_8, the roots are just the shortest vectors in the lattice, but this no longer works for B_n, C_n, G_2 & F_4.

The ADE guys are called "simply laced".

The maximal torus $T \subseteq K$ has a <u>normalizer</u>

$$N(T) = \{ g \in K : gTg^{-1} = T \}$$

$N(T)$ acts on T by conjugation, and $T \subseteq N(T)$ acts trivially, so

$$N(T)/T = W$$

acts on T, and W is called the <u>Weyl group</u>. In fact, W is a finite group. Since it acts on T, it acts on $\mathfrak{t}$, thus on $L \subseteq \mathfrak{t}$. It also preserves the inner product on $\mathfrak{t}$ (given by the Killing form), & its action is faithful (only $1 \in W$ fixes everything), so W is included in the permutation group of the shortest vectors in L, hence it must be finite!

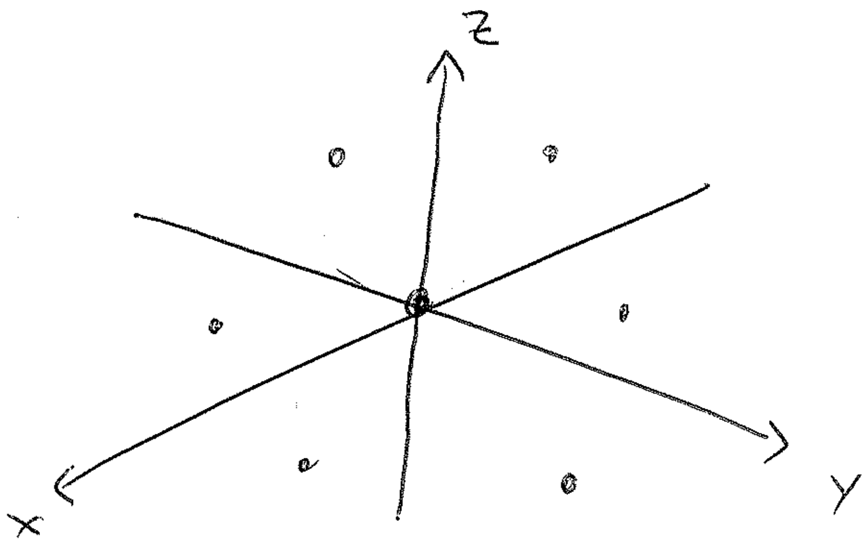
<u>Ex</u> A_2 Here the lattice looks like

$$L = \left\{ \begin{pmatrix} ix & & 0 \\ & iy & \\ 0 & & iz \end{pmatrix} : x, y, z \in \mathbb{Z},\ x+y+z=0 \right\}$$

L is spanned by $\pm(1,-1,0)$,

$$\pm(1,0,-1)$$
$$\pm(0,1,-1)$$

& we draw it as:



$$\underset{g}{\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}} \begin{pmatrix} a & & 0 \\ & b & \\ 0 & & c \end{pmatrix} \underset{g^{-1}}{\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}}$$

$$= \begin{pmatrix} b & & 0 \\ & a & \\ 0 & & c \end{pmatrix}, \quad \text{so } g \in N(T)$$

& gives $[g] \in W = N(T)/T$.

This elt of W acts as a reflection on our picture, switching x & y -axes.

We can also do any other permutation of the three axes, & that turns out to be all of them:

$$W \cong S_3$$

More generally, for A_n, we get

$$W \cong S_{n+1}.$$