The Algebra of Grand Unified Theories

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Joint work with John Baez

Introduction

There's a loose correspondence between particle physics and representation theory:

- Particles → basis vectors in a representation V of a Lie group G.
- Classification of particles → decomposition into irreps.
- Unification → $G \hookrightarrow H$; particles are "unified" into fewer irreps.
- Grand Unification → as above, but H is simple.
- The Standard Model → a particular representation V_{SM} of a particular Lie group G_{SM}.

The Standard Model

- The Standard Model group is $G_{\mathrm{SM}} = \mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3)$.
- The Standard Model representation is made from:

Standard Model Representation

Particles	Symbol	G_{SM}-representation
Left-handed leptons	$\begin{pmatrix} \nu_L \\ e_L^- \end{pmatrix}$	$\mathbb{C}_{-3} \otimes \mathbb{C}^2 \otimes \mathbb{C}$
Left-handed quarks	$\begin{pmatrix} u_L^r, u_L^g, u_L^b \\ d_L^r, d_L^g, d_L^b \end{pmatrix}$	$\mathbb{C}_1 \otimes \mathbb{C}^2 \otimes \mathbb{C}^3$
Right-handed neutrino	ν_R	$\mathbb{C}_0 \otimes \mathbb{C} \otimes \mathbb{C}$
Right-handed electron	e_R^-	$\mathbb{C}_{-6} \otimes \mathbb{C} \otimes \mathbb{C}$
Right-handed up quarks	u_R^r, u_R^g, u_R^b	$\mathbb{C}_4 \otimes \mathbb{C} \otimes \mathbb{C}^3$
Right-handed down quarks	d_R^r, d_R^g, d_R^b	$\mathbb{C}_{-2} \otimes \mathbb{C} \otimes \mathbb{C}^3$

Here, we've written a bunch of $G_{\mathrm{SM}} = \mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3)$ irreps as $U \otimes V \otimes W$, where

- U is a $\mathrm{U}(1)$ irrep $\mathbb{C}_Y$, where $Y \in \mathbb{Z}$. The underlying vector space is just $\mathbb{C}$, and the action is given by
$$\alpha \cdot z = \alpha^Y z, \quad \alpha \in \mathrm{U}(1), z \in \mathbb{C}$$
- V is an $\mathrm{SU}(2)$ irrep, either $\mathbb{C}$ or $\mathbb{C}^2$.
- W is an $\mathrm{SU}(3)$ irrep, either $\mathbb{C}$ or $\mathbb{C}^3$.

The Standard Model representation is

$$V_{\mathrm{SM}} = \mathbb{C}_{-3} \otimes \mathbb{C}^2 \otimes \mathbb{C} \quad \oplus \quad \cdots \quad \oplus \quad \mathbb{C}_{-2} \otimes \mathbb{C} \otimes \mathbb{C}^3 \quad \oplus \quad \text{dual}$$

The GUTs Goal

- $G_{\mathrm{SM}} = \mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3)$ is a mess!
- $V_{\mathrm{SM}} = \mathbb{C}_{-3} \otimes \mathbb{C}^2 \otimes \mathbb{C} \quad \oplus \quad \cdots \quad \oplus \quad \mathbb{C}_{-2} \otimes \mathbb{C} \otimes \mathbb{C}^3 \quad \oplus$ dual is a mess!
- Explain the Y's in the $\mathbb{C}_Y$'s.
- Explain other patterns, like $\dim V_{\mathrm{SM}} = 32 = 2^5$.

Or, *much* more broadly:

- Unify V_{SM} into fewer irreps.

The GUTs Trick

Let V be a representation of some group G, and suppose $G_{\mathrm{SM}} \subseteq G$. Then

- V is also representation of G_{SM};
- V may break apart into more G_{SM}-irreps than G-irreps.

. . . and Its Technicalities

More precisely, we want:

- A **group** G,
- a **representation** V,
- a **map** $G_{\mathrm{SM}} \to G$
- such that V becomes isomorphic to V_{SM} when we restrict back to G_{SM}.
- That is, **prove** there exists a homorphism $G_{\mathrm{SM}} \to G$ and a linear isomorphism $V_{\mathrm{SM}} \to V$ making

$$\begin{CD} G_{\mathrm{SM}} @>>> G \\ @VVV @VVV \\ \mathrm{U}(V_{\mathrm{SM}}) @>>> \mathrm{U}(V) \end{CD}$$

commute.

Three Grand Unified Theories (GUTs)

The $\mathrm{SU}(5)$ Theory (H. Georgi and S. Glashow, 1974)

- The **group** is $\mathrm{SU}(5)$.
- The **representation** is $\Lambda\mathbb{C}^5$.
- The **map** takes G_{SM} onto the subgroup of $\mathrm{SU}(5)$ preserving a splitting: $\mathbb{C}^2 \oplus \mathbb{C}^3 \cong \mathbb{C}^5$.
- $\Lambda\mathbb{C}^5 \cong V_{\mathrm{SM}}$ as a representation of G_{SM}. More precisely:
- **Theorem.** There's a homomorphism $\phi\colon G_{\mathrm{SM}} \to \mathrm{SU}(5)$ and a linear isomorphism $h\colon V_{\mathrm{SM}} \to \Lambda\mathbb{C}^5$ making

$$\begin{CD} G_{\mathrm{SM}} @>\phi>> \mathrm{SU}(5) \\ @VVV @VVV \\ \mathrm{U}(V_{\mathrm{SM}}) @>\mathrm{U}(h)>> \mathrm{U}(\Lambda\mathbb{C}^5) \end{CD}$$

commute.

The Pati–Salam Model (J. Pati and A. Salam, 1974)

- The **group** is $\mathrm{Spin}(4) \times \mathrm{Spin}(6)$.
  Reminder: $\mathrm{Spin}(2n)$ is the double cover of $\mathrm{SO}(2n)$.
- The **representation** is $\Lambda\mathbb{C}^2 \otimes \Lambda\mathbb{C}^3$.
  Reminder: $\mathrm{Spin}(2n)$ has a faithful representation on $\Lambda\mathbb{C}^n$.
- The **map** takes G_{SM} to the subgroup of $\mathrm{Spin}(4) \times \mathrm{Spin}(6)$ preserving the gradings on $\Lambda\mathbb{C}^2$ and $\Lambda\mathbb{C}^3$.
- $\Lambda\mathbb{C}^2 \otimes \Lambda\mathbb{C}^3 \cong V_{\mathrm{SM}}$ as a representation of G_{SM}. More precisely:
- **Theorem.** There's a homomorphism $\theta\colon G_{\mathrm{SM}} \to \mathrm{Spin}(4) \times \mathrm{Spin}(6)$ and linear isomorphism $f\colon V_{\mathrm{SM}} \to \Lambda\mathbb{C}^2 \otimes \Lambda\mathbb{C}^3$ making

$$\begin{CD} G_{\mathrm{SM}} @>\theta>> \mathrm{Spin}(4) \times \mathrm{Spin}(6) \\ @VVV @VVV \\ \mathrm{U}(V_{\mathrm{SM}}) @>\mathrm{U}(f)>> \mathrm{U}(\Lambda\mathbb{C}^2 \otimes \Lambda\mathbb{C}^3) \end{CD}$$

commute.

The $\mathrm{Spin}(10)$ Theory (H. Georgi, 1974)

- The **group** is $\mathrm{Spin}(10)$.
- The **representation** is $\Lambda\mathbb{C}^5$.

For the **map**, we have two choices.

- Either extend the $\mathrm{SU}(5)$ map:

$$\begin{CD} \mathrm{SU}(5) @>\psi>> \mathrm{Spin}(10) \\ @VVV @VVV \\ \mathrm{U}(\Lambda\mathbb{C}^5) @>1>> \mathrm{U}(\Lambda\mathbb{C}^5) \end{CD}$$

- Or extend the Pati–Salam map:

$$\begin{CD} \mathrm{Spin}(4) \times \mathrm{Spin}(6) @>\eta>> \mathrm{Spin}(10) \\ @VVV @VVV \\ \mathrm{U}(\Lambda\mathbb{C}^2 \otimes \Lambda\mathbb{C}^3) @>\mathrm{U}(g)>> \mathrm{U}(\Lambda\mathbb{C}^5) \end{CD}$$

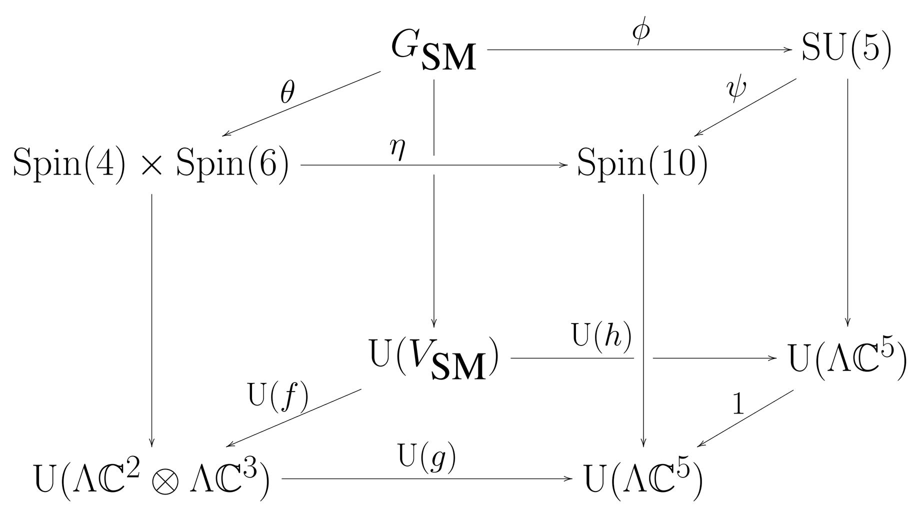
Either way, we get the $\mathrm{Spin}(10)$ theory!

The $\mathrm{Spin}(10)$ theory is well-defined, because of our final result.

Conclusion

The GUTs Cube

- If we put the two routes to the $\mathrm{Spin}(10)$ theory together, we get **the GUTs cube**:



- **Theorem.** We can choose ϕ and θ such that the GUTs cube commutes.

Morals

- The $\mathrm{Spin}(10)$ theory *unites* the $\mathrm{SU}(5)$ theory and the Pati–Salam model.
- The Standard Model is the *compromise* between the $\mathrm{SU}(5)$ theory and the Pati–Salam model.

Reference

John Baez and John Huerta, The algebra of grand unified theories, *Bull. Amer. Math. Soc.* **47** (2010), 483–552. Also available as arXiv:0904.1556.