Supersymmetry and Division Algebras

John Huerta

Department of Mathematics
UC Riverside

2nd Mile High Conference on Nonassociative Mathematics

- The only normed division algebras are $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$. They have dimensions 1, 2, 4, and 8.
- The only Yang–Mills theories with minimal *supersymmetry* occur in dimensions 3, 4, 6, and 10.
- The classical superstring also makes sense only in dimensions 3, 4, 6, and 10.

These are all related. We shall focus on the way division algebras give rise to supersymmetry.

This work has its basis in that of others:

- In 1983, Kugo and Townsend showed that supersymmetric theories were related to the division algebras.
- In 1988, Evans discovered that the existence of a super-Yang–Mills theory in dimension $n+2$ is equivalent to the existence of a normed division algebra in dimension n.
- In 1994, Schray developed an octonionic model for the superparticle.
- More broadly, Dray, Manogue, and Schray have developed an octonionic formulation of spinors in $9+1$-dimensional spacetime.

- Very loosely, **supersymmetry** is a symmetry between bosons (vectors) and fermions (spinors).
- Both super-Yang–Mills theories and classical superstring theories depend on a certain identity between vectors and spinors.

Building Blocks

- Let V be the vectors in D-dimensional spacetime, and $\mathrm{Cliff}(V)$ the associated Clifford algebra.
- The double cover of the Lorentz group, $\mathrm{Spin}(D-1,1)$, is the subgroup generated by pairs of unit vectors in $\mathrm{Cliff}(V)$.
- V forms the **vector representation** of $\mathrm{Spin}(D-1,1)$.

Building Blocks

- Let V be the vectors in D-dimensional spacetime, and $\mathrm{Cliff}(V)$ the associated Clifford algebra.
- The double cover of the Lorentz group, $\mathrm{Spin}(D-1,1)$, is the subgroup generated by pairs of unit vectors in $\mathrm{Cliff}(V)$.
- V forms the **vector representation** of $\mathrm{Spin}(D-1,1)$.
- In Yang–Mills theory, V is used to represent **bosons**, which are roughly the particles transmitting forces.

Building Blocks

- Let $S_\pm$ be **spinor representations** of $\mathrm{Spin}(D-1,1)$, that is representations arising from a module of $\mathrm{Cliff}(V)$.
- Vectors can act on spinors via an intertwiner:

$$\begin{array}{rcl} V \otimes S_\pm & \to & S_\mp \\ (A, \psi) & \mapsto & A\psi \end{array}$$

- Pairs of spinors can be turned into vectors via an intertwiner:

$$\begin{array}{rcl} S_\pm \otimes S_\pm & \to & V \\ (\psi, \phi) & \mapsto & \psi \cdot \phi \end{array}$$

Building Blocks

- Let $S_\pm$ be **spinor representations** of $\mathrm{Spin}(D-1,1)$, that is representations arising from a module of $\mathrm{Cliff}(V)$.
- Vectors can act on spinors via an intertwiner:

$$\begin{array}{ccc} V \otimes S_\pm & \rightarrow & S_\mp \\ (A, \psi) & \mapsto & A\psi \end{array}$$

- Pairs of spinors can be turned into vectors via an intertwiner:

$$\begin{array}{ccc} S_\pm \otimes S_\pm & \rightarrow & V \\ (\psi, \phi) & \mapsto & \psi \cdot \phi \end{array}$$

- In Yang–Mills theory, $S_\pm$ is used to represent **fermions**, which are roughly the particles of matter.

For super-Yang–Mills theories and classical superstring theories, we need the following to hold:

Theorem: In dimensions 3, 4, 6 and 10, let ψ, ϕ, and χ be spinors in S_+. Then the **trilinear term**

$$\mathrm{tri}(\psi, \phi, \chi) = (\psi \cdot \phi)\chi + (\phi \cdot \chi)\psi + (\chi \cdot \psi)\phi$$

vanishes identically.

For super-Yang–Mills theories and classical superstring theories, we need the following to hold:

Theorem: In dimensions 3, 4, 6 and 10, let ψ, ϕ, and χ be spinors in S_+. Then the **trilinear term**

$$\mathrm{tri}(\psi, \phi, \chi) = (\psi \cdot \phi)\chi + (\phi \cdot \chi)\psi + (\chi \cdot \psi)\phi$$

vanishes identically.

We prove this as a consequence of alternativity.

Let $\mathbb{K}$ be a normed division algebra of dimension n. Then

- $\mathbb{K}$ has a **conjugation**, a linear operator $*$ satisfying

$$x^{**} = x, \quad (xy)^* = y^* x^*.$$

- This allows us to define **real** and **imaginary parts** in the same way as for the complex numbers:

$$\mathrm{Re}(x) = \frac{x + x^*}{2}, \quad \mathrm{Im}(x) = \frac{x - x^*}{2}$$

- $\mathbb{K}$ is **alternative**, meaning the subalgebra generated by any two elements is associative.
- In particular, the **associator**

$$[x, y, z] = (xy)z - x(yz)$$

is totally antisymmetric.

Using these facts, we can calculate that

- The associator is purely imaginary.
- For $x, y, z \in \mathbb{K}$, the real part $\mathrm{Re}(xyz)$ is well-defined and *cyclically symmetric*.
- For matrices X, Y, Z with entries in $\mathbb{K}$, the **real trace** $\mathrm{Re}\ \mathrm{tr}(XYZ)$ is well-defined and *cyclically symmetric*.

This last tool is crucial in what follows.

The normed division algebra $\mathbb{K}$ of dimension n gives the vectors and spinors in $n+2$-dimensional spacetime special properties.

- The vectors correspond to 2×2 hermitian matrices:

$$V = \left\{ \begin{pmatrix} t+x & y \\ y^* & t-x \end{pmatrix} \quad : \quad t, x \in \mathbb{R}, \; y \in \mathbb{K} \right\}$$

- The usual formula for the determinant of a matrix gives the Minkowski norm on this $n+2$-dimensional vector space:

$$-\det \begin{pmatrix} t+x & y \\ y^* & t-x \end{pmatrix} = -[(t+x)(t-x)-yy^*] = -t^2+x^2+|y|^2$$

- The Lorentz group $\mathrm{Spin}(n+1,1)$ thus acts on V via determinant-preserving linear transformations.

- The spinors S_+ and S_- both have underlying vector space $\mathbb{K}^2$.
- The vectors act on spinors using their representation as 2×2 matrices.

- The spinors S_+ and S_- both have underlying vector space $\mathbb{K}^2$.
- The vectors act on spinors using their representation as 2×2 matrices.

More precisely,

- $A \in V$ acts on $\psi \in S_+$ via left multiplication, and gives an element S_-:

$$\begin{array}{llcl} \Gamma: & V \otimes S_+ & \to & S_- \\ \Gamma: & (A, \psi) & \mapsto & A\psi \end{array}$$

- $A \in V$ acts on $\psi \in S_-$ via left multplication by its **trace reversal**

$$\tilde{A} = A - \mathrm{tr} A$$

and gives an element of S_+:

$$\begin{array}{llcl} \tilde{\Gamma}: & V \otimes S_- & \to & S_+ \\ \tilde{\Gamma}: & (A, \psi) & \mapsto & \tilde{A}\psi \end{array}$$

- We can combine Γ and $\tilde{\Gamma}$ to have vectors act on $S_+ \oplus S_-$.

$$\begin{array}{rccc} \gamma\colon & V & \rightarrow & \mathrm{End}(S_+ \oplus S_-) \\ \gamma\colon & A & \mapsto & \begin{pmatrix} 0 & \tilde{A} \\ A & 0 \end{pmatrix} \end{array}$$

- This satisfies the Clifford algebra relation, so it induces an action of the Clifford algebra $\mathrm{Cliff}(V)$ on $S_+ \oplus S_-$.
- $\mathrm{Spin}(n+1,1)$ is the subgroup of $\mathrm{Cliff}(V)$ generated by products of pairs of unit vectors.
- S_+ and S_- are representations of $\mathrm{Spin}(n+1,1)$.

We can define a pairing on spinors of opposite chirality:

$$S_{\pm} \otimes S_{\mp} \to \mathbb{R}$$

by

$$\langle \psi, \phi \rangle = \mathrm{Re}(\psi^{\dagger} \phi)$$

Prop: The pairing $\langle \cdot, \cdot \rangle$ is invariant under $\mathrm{Spin}(n+1,1)$

- We use $\langle \cdot, \cdot \rangle$ to turn pairs of spinors into 1-forms.
- Define

$$S_+ \otimes S_+ \to V^*$$

by

$$\psi \cdot \phi(A) = \langle \psi, A\phi \rangle .$$

- When we identify vectors and 1-forms, we can use the cyclic property of the real trace, plus the Clifford relation, to compute:

$$\psi \cdot \phi = \psi \widetilde{\phi^\dagger + \phi} \psi^\dagger$$

Prop: This map is a $\mathrm{Spin}(n+1, 1)$ intertwiner.

- $\psi \cdot \phi = \psi \widetilde{\phi^\dagger + \phi} \psi^\dagger$ is the key formula!

Recall, we are trying to prove:

Theorem: In dimensions 3, 4, 6 and 10, let ψ, ϕ, and χ be spinors in S_+. Then

$$\mathrm{tri}(\psi, \phi, \chi) = (\psi \cdot \phi)\chi + (\phi \cdot \chi)\psi + (\chi \cdot \psi)\phi$$

vanishes identically.

$\mathrm{tri}(\psi, \phi, \chi)$ is totally symmetric in ψ, ϕ, and χ. So it suffices to prove:

Theorem: In dimensions 3, 4, 6 and 10, let ψ be a spinor in S_+. Then

$$(\psi \cdot \psi)\psi = 0.$$

Proof: Indeed, let $\psi \in S_+ = \mathbb{K}^2$. Then

$$(\psi \cdot \psi)\psi = 2\widetilde{(\psi\psi^\dagger)}\psi = 2[(\psi\psi^\dagger)\psi - \psi(\psi^\dagger\psi)] = 0$$

since $\mathbb{K}$ is alternative and ψ involves only two elements of $\mathbb{K}$.