

L_∞ -superalgebras for superstrings and 2-branes

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This research began as a puzzle. Explain this pattern:

- ▶ The only normed division algebras are \mathbb{R} , \mathbb{C} , \mathbb{H} and \mathbb{O} .
They have dimensions $k = 1, 2, 4$ and 8 .
- ▶ The classical superstring makes sense only in dimensions $k + 2 = 3, 4, 6$ and 10 .
- ▶ The classical super-2-brane makes sense only in dimensions $k + 3 = 4, 5, 7$ and 11 .

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Pulling on this thread will lead us into higher gauge theory.

Higher gauge theory is a set of ideas generalizing **gauge theory**, which says:

- ▶ The electromagnetic field, A , is a connection on a $U(1)$ -bundle.

We are combining this with **supersymmetry**, mathematics which has a \mathbb{Z}_2 -grading throughout.

String and M-theory have deep connections with geometry:

- ▶ The B field in string theory is a connection on a $U(1)$ -gerbe.
- ▶ The C field in M-theory is a connection on a $U(1)$ -2-gerbe.
- ▶ ... Generalizing the A field, a connection on a $U(1)$ -bundle.

Meaning:

- ▶ The B field is locally a 2-form, but globally has nontrivial transition data on double and triple intersections of an open cover.
- ▶ The C field is locally a 3-form, but globally has nontrivial transition data on double, triple and quadruple intersections of an open cover.
- ▶ ... Just as the A field is locally a 1-form, but globally has nontrivial transition data on double intersections of an open cover.

B and C fields are like connections. In fact, we can generalize the notion of connection to encompass them:

- ▶ Sati, Schreiber and Stasheff: Connections valued in L_∞ -algebras.
- ▶ An L_∞ -algebra is roughly a Lie algebra defined on a chain complex:

$$L_0 \leftarrow L_1 \leftarrow \cdots \leftarrow L_n \leftarrow \cdots$$

but satisfying the Lie algebra axioms only up to chain homotopy.

Definition

More precisely, an **L_∞ -algebra** is a graded vector space L equipped with a system of graded-antisymmetric linear maps

$$[-, \dots, -]: L^{\otimes k} \rightarrow L$$

satisfying a generalization of the Jacobi identity.

So L has:

- ▶ a boundary operator $\partial = [-]$ making it a chain complex,
- ▶ a bilinear bracket $[-, -]$, like a Lie algebra,
- ▶ but also a trilinear bracket $[-, -, -]$ and higher, all satisfying various identities.

We can just as easily consider L_∞ -superalgebras: each term is \mathbb{Z}_2 -graded, and we introduce extra signs.

Connections valued in certain L_∞ -superalgebras describe the *parallel transport* of superstrings and super-2-branes in the appropriate dimension:

$\text{superstring}(V)$	Connection component
\mathbb{R}	\mathbb{R} -valued 2-form, the B field.
\downarrow	
$\mathfrak{so}(V)$	$\mathfrak{so}(V)$ -valued 1-form.

2-brane(\mathcal{V})	Connection component
\mathbb{R}	\mathbb{R} -valued 3-form, the C field.
\downarrow	
0	
\downarrow	
$\mathfrak{so}(\mathcal{V})$	$\mathfrak{so}(\mathcal{V})$ -valued 1-form.

Using *superstring* and *2-brane*, we neatly package these fields with the Levi–Civita connection on spacetime.

Our goal is to construct $\text{superstring}(V)$ and $2\text{-brane}(\mathcal{V})$.

Since they only involve 2 and 3 terms, we call $\text{superstring}(V)$ a Lie 2-superalgebra, and $2\text{-brane}(\mathcal{V})$ a Lie 3-superalgebra.

Let us see where these Lie n -superalgebras come from, starting with the reason superstrings and 2-branes only make sense in certain dimensions.

- ▶ Superstrings and super-2-branes are *exceptional objects*—they only make sense in certain dimensions.
- ▶ The corresponding Lie 2- and Lie 3-superalgebras are similarly exceptional.
- ▶ Like many exceptional objects in mathematics, they are tied to the division algebras, \mathbb{R} , \mathbb{C} , \mathbb{H} and \mathbb{O} .

In the physics literature, the classical superstring and super-2-brane require certain spinor identities to hold.

Recall that:

- ▶ V is a vector space with a nondegenerate quadratic form.
- ▶ $\text{Cliff}(V) = \frac{TV}{v^2 = \|v\|^2}$.
- ▶ V is the vector representation of $\widetilde{\text{Spin}}(V) = \text{SO}_0(V) \subseteq \text{Cliff}(V)$
- ▶ S is a spinor representation, i.e. a representation coming from a module of $\text{Cliff}(V)$.

Superstring In dimensions 3, 4, 6 and 10, we have:

$$[\psi, \psi]\psi = 0$$

for all spinors $\psi \in \mathcal{S}$.

Here:

- ▶ the bracket is a symmetric map from spinors to vectors:

$$[,]: \text{Sym}^2 \mathcal{S} \rightarrow V$$

- ▶ vectors can “act” on spinors via the Clifford action, since $V \subseteq \text{Cliff}(V)$.

Similarly, for the 2-brane:

Super-2-brane In dimensions 4, 5, 7 and 11, the 3- ψ 's rule need not hold:

$$[\Psi, \Psi]\Psi \neq 0$$

Instead, we have the 4- Ψ 's rule:

$$[\Psi, [\Psi, \Psi]\Psi] = 0$$

for all spinors $\Psi \in \mathcal{S}$.

Again:

- ▶ \mathcal{V} and \mathcal{S} are vectors and spinors for these dimensions.
- ▶ $[\cdot, \cdot]: \text{Sym}^2 \mathcal{S} \rightarrow \mathcal{V}$.

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- ▶ We can use \mathbb{K} to build V and S in dimensions 3, 4, 6 and 10, \mathcal{V} and \mathcal{S} in 4, 5, 7 and 11.
- ▶ The 3- ψ 's and 4- Ψ 's rules are consequences of this construction.

In superstring dimensions 3, 4, 6 and 10:

- ▶ The vectors V are the 2×2 Hermitian matrices with entries in \mathbb{K} :

$$V = \left\{ \begin{pmatrix} t+x & \bar{y} \\ y & t-x \end{pmatrix} : t, x \in \mathbb{R}, \quad y \in \mathbb{K} \right\}.$$

- ▶ The determinant is then the norm:

$$-\det \begin{pmatrix} t+x & \bar{y} \\ y & t-x \end{pmatrix} = -t^2 + x^2 + |y|^2.$$

- ▶ This uses the properties of \mathbb{K} :

$$|y|^2 = y\bar{y}.$$

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- ▶ $[-, -]$ has a nice formula using matrix operations:

$$[\psi, \psi] = 2\psi\bar{\psi}^T - 2\bar{\psi}^T\psi\mathbf{1} \in V$$

- ▶ Showing

$$[\psi, \psi]\psi = 0$$

is now an easy calculation!

- ▶ These constructions are originally due to Tony Sudbery, with help from Corinne Manogue, Tevian Dray and Jörg Schray.
- ▶ We have shown how to generalize them to the 2-brane dimensions 4, 5, 7 and 11, taking $\mathcal{V} \subseteq \mathbb{K}[4]$ and $\mathcal{S} = \mathbb{K}^4$.
- ▶ The 4- Ψ 's rule

$$[\Psi, [\Psi, \Psi]\Psi] = 0$$

is then also an easy calculation.

What are the 3- ψ 's and 4- Ψ 's rules?

They are *cocycle conditions*.

- ▶ In 3, 4, 6 and 10, there is a 3-cochain α :

$$\alpha(\psi, \phi, \mathbf{v}) = \langle \psi, \mathbf{v}\phi \rangle.$$

Here, $\langle -, - \rangle$ is a $\text{Spin}(V)$ -invariant pairing on spinors.

- ▶ $d\alpha = 0$ is the 3- ψ 's rule!

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- ▶ $d\alpha = 0$ is the 3- ψ 's rule!
- ▶ In 4, 5, 7 and 11, there is a 4-cochain β :

$$\beta(\Psi, \Phi, V, W) = \langle \Psi, (VW - WV)\Phi \rangle.$$

Here, $\langle -, - \rangle$ is a $\text{Spin}(\mathcal{V})$ -invariant pairing on spinors.

- ▶ $d\beta = 0$ is the 4- Ψ 's rule!

Lie (super)algebra cohomology:

- ▶ Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a Lie superalgebra,
- ▶ which has bracket $[\cdot, \cdot]: \Lambda^2 \mathfrak{g} \rightarrow \mathfrak{g}$,
- ▶ where $\Lambda^2 \mathfrak{g} = \Lambda^2 \mathfrak{g}_0 \oplus \mathfrak{g}_0 \otimes \mathfrak{g}_1 \oplus \text{Sym}^2 \mathfrak{g}_1$ is the graded exterior square.
- ▶ We get a cochain complex:

$$\Lambda^0 \mathfrak{g}^* \rightarrow \Lambda^1 \mathfrak{g}^* \rightarrow \Lambda^2 \mathfrak{g}^* \rightarrow \dots$$

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- ▶ We get a cochain complex:

$$\Lambda^0 \mathfrak{g}^* \rightarrow \Lambda^1 \mathfrak{g}^* \rightarrow \Lambda^2 \mathfrak{g}^* \rightarrow \dots$$

- ▶ where $d = [\cdot, \cdot]^*: \Lambda^1 \mathfrak{g}^* \rightarrow \Lambda^2 \mathfrak{g}^*$, the dual of the bracket.
- ▶ $d^2 = 0$ is the Jacobi identity!

- ▶ In 3, 4, 6 and 10:

$$\mathcal{T} = V \oplus \mathcal{S}$$

is a Lie superalgebra, with bracket

$$[,]: \text{Sym}^2 \mathcal{S} \rightarrow V.$$

- ▶ $\alpha(\psi, \phi, \nu) = \langle \psi, \nu\phi \rangle$ is a 3-cocycle on \mathcal{T} .
- ▶ In 4, 5, 7 and 11:

$$\mathcal{T} = \mathcal{V} \oplus \mathcal{S}$$

is a Lie superalgebra, with bracket

$$[,]: \text{Sym}^2 \mathcal{S} \rightarrow \mathcal{V}.$$

- ▶ $\beta(\Psi, \Phi, V, W) = \langle \Psi, (VW - WV)\Phi \rangle$ is a 4-cocycle on \mathcal{T} .

- ▶ In 3, 4, 6 and 10: we can extend α to a cocycle on

$$\mathfrak{so}(V) = \mathfrak{spin}(V) \ltimes T$$

the Poincaré superalgebra.

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- ▶ In 4, 5, 7 and 11: we can extend β to a cocycle on

$$\mathfrak{so}(\mathcal{V}) = \mathfrak{spin}(\mathcal{V}) \ltimes \mathcal{T}$$

the Poincaré superalgebra.

The spinor identities were cocycle conditions for α and β . What are α and β good for?

Building Lie n -superalgebras!

Theorem (Baez–Crans–Huerta)

If ω is an $(n + 1)$ -cocycle on the Lie superalgebra \mathfrak{g} , then the n term chain complex

$$\mathfrak{g} \leftarrow 0 \leftarrow \cdots \leftarrow 0 \leftarrow \mathbb{R}$$

equipped with

$$\begin{aligned} [-, -] &: \Lambda^2 \mathfrak{g} \rightarrow \mathfrak{g} \\ \omega = [-, \cdots, -] &: \Lambda^{n+1} \mathfrak{g} \rightarrow \mathbb{R} \end{aligned}$$

is a Lie n -superalgebra.

We call the Lie n -superalgebra

$$\mathfrak{g} \leftarrow 0 \leftarrow \dots \leftarrow 0 \leftarrow \mathbb{R}$$

the **extension of \mathfrak{g} by ω** .

Theorem

*In dimensions 3, 4, 6 and 10, there exists a Lie 2-superalgebra, which we call **superstring**(V), formed by extending the Poincaré superalgebra $\mathfrak{siso}(V)$ by the 3-cocycle α .*

Theorem

*In dimensions 4, 5, 7 and 11, there exists a Lie 3-superalgebra, which we call **2-brane**(\mathcal{V}), formed by extending the Poincaré superalgebra $\mathfrak{siso}(\mathcal{V})$ by the 4-cocycle β .*

Theorem

*In dimensions 3, 4, 6 and 10, there exists a Lie 2-supergroup, which we call **Superstring**(\mathcal{V}), integrating $\text{superstring}(\mathcal{V})$.*

Theorem

*In dimensions 4, 5, 7 and 11, there exists a Lie 3-supergroup, which we call **2-Brane**(\mathcal{V}), integrating $\text{2-brane}(\mathcal{V})$.*

These integrations have finite-dimensional models.