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New Existence Result for Autonomous One-Dimensional Optimal Control Problems

P. Pedregal · J. Tiago

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Abstract The approach to optimal control problems based on a purely variational reformulation may lead to new existence results by using fine, general existence theorems for variational problems without convexity assumptions. We illustrate this perspective here for autonomous one-dimensional problems and defer the study of more complex situations to later work.

Keywords Variational reformulation · Existence under lack of convexity

1 Introduction

We place ourselves in the context of a typical optimal control problem,

$$\text{(P)} \quad \text{minimize the functional} \quad I(y,u) = \int_0^T F(y(t),u(t))dt,$$

with the state $y(t)$ and the control $u(t)$ subject to the state equation

$$y'(t) = f(y(t),u(t)), \quad \text{a.e. } t \in (0,T), \tag{1}$$

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P. Pedregal
Departamento de Matemáticas, ETSI Industriales, Universidad de Castilla-La Mancha, Ciudad Real, Spain

J. Tiago (✉)
EST Barreiro, Instituto Politécnico de Setúbal, Barreiro, Portugal
e-mail: Jorge.Tiago@uclm.es

the initial and/or final conditions

$$y(0) = y_0, \qquad y(T) = y_T, \tag{2}$$

and the viability constraints

$$u(t) \in K, \qquad y(t) \in L, \tag{3}$$

where

$$f : \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^n, \qquad F : \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}, \quad K \subseteq \mathbb{R}^m,\ L \subseteq \mathbb{R}^n.$$

Appropriate typical technical hypotheses are assumed on F and f so that, for instance, the state equation admits absolutely continuous (AC for short) solutions and the state y can be recovered (in a unique way) from the control u. To avoid more technical complications, we assume that the admissibility set for the control is constant throughout time as well as the admissibility set for the state. After all, our main contribution here is for autonomous systems so that we will not assume such time dependence from the start.

As we will see, this optimal control problem is equivalent to the variational problem

$$\text{(VP)} \quad \text{minimize the functional} \quad J(y) = \int_0^T \varphi(y(t), y'(t))dt,$$

with the state $y(t)$ subject to the same initial and final conditions $y(0) = y_0$, $y(T) = y_T$, where the integrand φ is defined by putting

$$\varphi(y, \xi) = \inf_u \{F(y, u) : \xi = f(y, u), u \in K\} \tag{4}$$

if $y \in L$, and $\varphi = +\infty$ if $y \notin L$, where

$$\varphi : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}.$$

Notice in addition that, if the set $\{u \in K : \xi = f(y, u)\}$ turns out to be empty, then $\varphi(y, \xi) = +\infty$. If further assumptions are verified by F and f, this integrand φ is well defined (in the sense that the infimum in (4) is a minimum) and so is the associated variational problem. We will be more precise about this later.

The point is that, whenever we can show existence of optimal solutions for the variational problem determined by φ, we will also have optimal solutions for the original optimal control problem. Indeed, a general situation would be to apply the classical Tonelli's theorem [1] to (VP) and see how coercivity and convexity for φ translate into explicit properties for F and f. This is classical and corresponds to standard existence theorems under convexity [2].

What is more interesting is to apply finer existence theorems for variational problems to our situation and translate them into the ingredients of the optimal control problem (P). Our main contribution here is to explore one such simple, but nontrivial, situation corresponding to autonomous, one-dimensional problems. The most general such theorem, as far as we know, can be found in [3, 4] and reads as follows.

In those papers, convexity is reduced to a minimum as it is only required at the origin, when considering the usual convexification of a function by the bipolar function. To be specific, for a function $f : X \to \bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$, the bipolar function f^{**} is defined [5] by

$$f^{**}(x) = \sup\{A(x) : A(\cdot) \text{ is an affine function, } A(v) \le f(v), \ \forall v \in X\}.$$

When we consider y as a fixed parameter, the bipolar function of φ with respect to the variable ξ is given by

$$\varphi^{**}(y, \xi) = \sup\{A(\xi) : A(\cdot) \text{ is an affine function, } A(v) \le \varphi(y, v), \ \forall v \in \mathbb{R}\}.$$

As just indicated, y is taken here as a parameter.

Requiring the convexity at the origin means that we should have

$$\varphi^{**}(y, 0) = \varphi(y, 0), \quad \forall y.$$

By applying such a result to our problem (VP) when $n = 1$ (although the dimension m could be larger than 1) and reinterpreting appropriately the convexity at the origin, we prove our main contribution in this paper, Theorem 3.2 below. Proceed to Sect. 3 for a rigorous precise statement and proof. Here, we discuss this result more informally.

For the existence of an optimal solution for problem (P), we must ask for f to be continuous, F lower semicontinuous, coercive with respect to K, bounded from below, and verifying the growth condition

$$\lim_{|f(y,u)| \to \infty, \ u \in K} F(y, u)/|f(y, u)| = +\infty, \quad \text{uniformly in } y \text{ (as in (5) below).}$$

In addition, typical convexity conditions required in classical results for existence are based on variational reformulations ([2] or [6]) and amount to the convexity of the epigraph of φ with respect to ξ. In our result, we replace it by the condition

$$\sup_{u \in K, \ f(y,u) < 0} \frac{F(y, u) - m(y)}{f(y, u)} \le \inf_{u \in K, \ f(y,u) > 0} \frac{F(y, u) - m(y)}{f(y, u)},$$

where

$$m(y) = \min_{u \in K}\{F(y, u) : f(y, u) = 0\},$$

which is in fact a necessary and sufficient condition for the convexity at the origin of φ (with respect to ξ).

We will also see (Corollary 3.1 and Corollary 3.2, below) that, when K is bounded, the simple condition

$$\min_u\{F(y, u) : f(y, u) = 0, \ u \in K\} = \min_u\{F(y, u) : u \in K\}$$

will be sufficient to have the existence of optimal solutions.

What is interesting about these results is that we need not care about convexity in any way, and particularly this last condition (Corollary 3.2) can be applied very easily to many situations.

We have organized all of the material in three sections. Section 2 is concerned with some elementary ideas concerning the proof of the equivalence of (P) and (VP) used to specify the main technical assumptions. Section 3 is the main part of the paper and there we prove Theorem 3.2. Finally, Sect. 4 includes several (academic) examples where these results are applied.

It is true that the one-dimensional situation treated here is a very special case and that the higher-dimensional situation is much more complicated. Even so, we would like to address it in the future.

2 Variational Reformulation

We start our analysis by introducing a new format for the Lagrange optimal control problem (P). This approach has been recently examined in different papers, for instance in [6, 7], where a variational reformulation was undertaken for the situation where only boundary conditions are assumed on the state. The process that we will follow here is the same. The viability constraints (3) can also be incorporated in this approach. Additional integral constraints in the form of inequalities can be treated too, but for the sake of simplicity we will not take these into account.

We define the integrand for our reformulation by putting

$$\varphi(y,\xi) = \inf_{u\in K}\{F(y,u) : \xi = f(y,u),\ y \in L\},$$

which is in fact

$$\varphi(y,\xi) = \begin{cases} \inf\limits_{u\in K}\{F(y,u) : \xi = f(y,u)\}, & \text{if } y \in L \text{ and } \{u \in K : \xi = f(y,u)\} \neq \emptyset, \\ +\infty, & \text{else.} \end{cases}$$

Consider the variational problem

$$\text{(VP)} \quad \text{minimize the functional} \quad J(y) = \int_0^T \varphi(y(t), y'(t))dt,$$

with $y(t)$ such that

$$\varphi : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\},$$

and $y(0) = y_0$, $y(T) = y_T$.

We can see that, under certain mild hypotheses, φ is well defined, in the sense that $\varphi > -\infty$ always and the infimum is attained.

Lemma 2.1 *Assume that*:

(i) *K is closed and, if unbounded, F is coercive with respect to K in the sense that, if*

$$U(y,\xi) = \{u \in K : \xi = f(y,u),\ y \in L\},$$

then

$$\lim_{u\in K,\ |u|\to\infty} F(y,u) = +\infty, \tag{5}$$

uniformly in y *inside* $U(y,\xi)$, *which means that*

$$\forall M>0, \qquad \forall R>0, \qquad \exists P_{M,R}, \qquad \forall u,y,\xi, \quad \textit{such that}$$
$$(|u|>P_{M,R},\ \|(y,\xi)\|\le R,\ u\in\{u\in K:\xi=f(y,u)\}\ \textit{then}\ F(y,u)>M).$$

(ii) $F:\mathbb{R}^n\times\mathbb{R}^m\longrightarrow\mathbb{R}$ *is lower semicontinuous and* $f:\mathbb{R}^n\times\mathbb{R}^m\longrightarrow\mathbb{R}$ *is continuous*.

Then, $\varphi:\mathbb{R}^n\times\mathbb{R}^n\longrightarrow\mathbb{R}\cup\{+\infty\}$ *given above is well defined*.

Proof Notice that (5) can also be interpreted in the following fashion:

$$\forall M>0, \qquad \forall R>0, \qquad \exists P_{M,R}, \qquad \forall u,y,\xi, \quad \text{such that}$$
$$(\|(y,\xi)\|\le R,\ u\in\{u\in K:\xi=f(y,u)\},\ F(y,u)\le M,\ \text{then}\ |u|\le P_{M,R}).$$

Using this fact, if we consider by contradiction a minimizing sequence u_n in K such that, for some ξ and y, we have $\xi=f(y,u_n)$ and $F(y,u_n)$ converging to $-\infty$, then $\{u_n\}$ is a bounded sequence, and so it has a subsequence u_{n_k} converging to a certain $\tilde{u}$ such that, due to the lower semicontinuity of F,

$$-\infty=\liminf_n F(y,u_{n_k})\ge F(y,\tilde{u})>-\infty,$$

a contradiction. Therefore, $\varphi(y,\xi)>-\infty$.

In a similar manner, by using the continuity of f, it can be argued easily that the infimum defining φ is a minimum. □

We can now establish the next result which states the equivalence between both problems (P) and (VP).

Theorem 2.1 *Problem* (P) *is equivalent to problem* (VP) *in the sense that the infimum of* I *is the same as that of* J.

Moreover, if (y,u) *is an optimal solution for* (P), *then* y *is an optimal solution for* (VP). *Conversely, if* y *is optimal for* (VP), *then there is a measurable admissible* u *such that* (y,u) *is optimal for* (P).

Proof

Step 1. Let (y,u) be an admissible pair for (P). Then, $F(y(t),u(t))<+\infty$ a.e. t in $[0,T]$. By the definition of φ,

$$\int_0^T \varphi(y(t),y'(t))dt\le\int_0^T F(y(t),u(t))dt<+\infty,$$

and so y is admissible for (VP). Moreover,

$$J(y)\le I(y,u).$$

Step 2. If y is an admissible solution of (VP), then $y \in AC$ and

$$\varphi(y(\cdot), y'(\cdot)) \leq +\infty, \quad \text{a.e. } t \text{ in } [0, T].$$

By definition of φ, $y(t) \in L$, a.e. t in $[0, T]$, and

$$\{u \in K : y'(t) = f(y(t), u)\} \neq \emptyset, \quad \text{a.e. } t \text{ in } [0, T].$$

However, for a.e. t in $[0, T]$, there is $u(t) \in K$ such that

$$\varphi(y(t), y'(t)) \leq F(y(t), u(t)) < +\infty$$

and $y'(t) = f(y(t), u(t))$. Consequently, the set-valued function

$$M(t) = \{u \in K : \varphi(y(t), y'(t)) = F(y(t), u),\ y'(t) = f(y(t), u)\}$$

has nonempty values for a.e. t in $[0, T]$. In addition, $M(t)$ is closed. To this end, consider $\{u_n\} \subset M(t)$ such that $u_n \to \bar{u}$. Then,

$$\begin{aligned} &\varphi(y(t), y'(t)) = F(y(t), u_n), \\ &y'(t) = f(y(t), u_n), \quad \forall n \in \mathbb{N}, \end{aligned}$$

and since F is lower semicontinuous and f is continuous,

$$\begin{aligned} &F(y(t), \bar{u}) \leq \liminf_n F(y(t), u_n) = \varphi(y(t), y'(t)), \\ &y'(t) = f(y'(t), \bar{u}), \quad \forall n \in \mathbb{N}. \end{aligned}$$

But by the definition of φ, we have

$$\varphi(y(t), y'(t)) \leq F(y(t), \bar{u}) \leq \varphi(y(t), y'(t)),$$

and so $\bar{u} \in M(t)$.

In this situation, we can apply to $M(t)$ a typical selection theorem [8] to conclude that there is a measurable selection $u : [0, T] \to K \subseteq \mathbb{R}^m$ such that

$$\varphi(y(t), y'(t)) = F(y(t), u(t)),\ y'(t) = f(y(t), u(t)), \quad \text{a.e. } t \text{ in } [0, T],$$

and

$$\int_0^T \varphi(y(t), y'(t))dt = \int_0^T F(y(t), u(t))dt < +\infty.$$

Hence, (y, u) is admissible for (P) and $I(y, u) = J(y)$.

Step 3. In particular, if y is optimal for (VP), then there is a measurable u such that (y, u) is optimal for (P) (and consequently $\inf I = \inf J$), since otherwise by Step 1 (y, u) would not be optimal. A similar argument give us the reciprocal. □

3 An Existence Theorem in Dimension One

In the particular case when the state y is one-dimensional (the state of the system is a single parameter), then new existence results can be given. One can use recent existence results for (VP) (see [3, 4, 9]) where the full convexity of the associated integrand φ is changed just to the convexity at the origin as it has been indicated in the Introduction.

Theorem 3.1 (*See* [3]) *Suppose that* $\varphi : \mathbb{R} \times \mathbb{R} \longrightarrow [0, +\infty]$ *verifies the following conditions*:

(i) φ *is* $\mathcal{L} \otimes \mathbf{B}$*-measurable and* $\varphi(\cdot, \cdot)$ *is lower semicontinuous*;
(ii) $\varphi^{**}(y, 0) = \varphi(y, 0), \ \forall y$;
(iii) $\varphi(y, \xi)$ *has superlinear growth in* ξ, *in the sense that*

$$\lim_{|\xi| \to +\infty} \frac{\varphi(y, \xi)}{|\xi|} = +\infty, \quad \textit{uniformly in } y.$$

Then, *there is a minimizer for*

$$J(y) = \int_a^b \varphi(y(t), y'(t))dt$$

in the class

$$C(\alpha, \beta) = \{y \in AC((a, b), \mathbb{R}) : y(a) = \alpha, \, y(b) = \beta\}.$$

Here, $\mathcal{L} \otimes \mathbf{B}$-measurable means measurable with respect to $\mathcal{L} \otimes \mathbf{B}$, the smallest σ-algebra of subsets of $\mathbb{R} \times \mathbb{R}$ that contains all the product sets $A \times B$, where A is a Lebesgue set and B a Borel set. We represent by $AC((a, b), \mathbb{R})$ the set of absolutely continuous functions from (a, b) to $\mathbb{R}$.

Theorem 3.1 is a general remarkable result, independent of our application here. Its scope is thus beyond our contribution. It is somehow the final stage of a series of improvements of existence results for scalar, one-dimensional problems. See [3] for the proof and related references.

To apply this result to (VP), we must derive conditions to ensure that $\varphi^{**}(y, 0) = \varphi(y, 0)$, for every y.

Let us consider the following fact.

Proposition 3.1 *For a real function*, $\varphi : \mathbb{R} \longrightarrow [0, +\infty]$, *the condition*

$$\varphi^{**}(0) = \varphi(0) < +\infty$$

is equivalent to

$$\sup_{\xi < 0} \frac{\varphi(\xi) - \varphi(0)}{\xi} \leq \inf_{\xi > 0} \frac{\varphi(\xi) - \varphi(0)}{\xi} \, . \tag{6}$$

Proof Put $m^- \le m^+$ to designate the slopes on the left-hand and right-hand sides of (6), respectively. Notice that we have $m^- = -\infty$ or $m^+ = +\infty$ if and only if $\varphi(\xi) = +\infty$ for all $\xi < 0$ or $\varphi(\xi) = +\infty$ for all $\xi > 0$, respectively. It is easy to see that

$$\forall \xi > 0, \quad m^-\xi + \varphi(0) \le m^+\xi + \varphi(0) \le \varphi(\xi)$$

and

$$\forall \xi < 0, \quad m^+\xi + \varphi(0) \le m^-\xi + \varphi(0) \le \varphi(\xi).$$

Using the definition of φ^{**}, recalled in the Introduction, we conclude that

$$\varphi^{**}(0) = \varphi(0).$$

Conversely, suppose that we have an affine function

$$a(\xi) = m(\xi) + \varphi(0)$$

such that

$$m\xi + \varphi(0) \le \varphi(\xi), \quad \forall \xi.$$

By setting again m^- and m^+ as above, it is easy to check that

$$m^- \le m \le m^+,$$

and consequently

$$\sup_{\xi<0} \frac{\varphi(\xi) - \varphi(0)}{\xi} \le \inf_{\xi>0} \frac{\varphi(\xi) - \varphi(0)}{\xi}.$$ □

We also have a parallel result when $\varphi(0)$ attains the value $+\infty$.

Proposition 3.2 *If* $\varphi(0) = +\infty$, *then*

$$\varphi^{**}(0) = \varphi(0) \quad \Leftrightarrow \quad (\varphi(\xi) = +\infty,\ \forall \xi < 0) \text{ or } (\varphi(\xi) = +\infty,\ \forall \xi > 0).$$

Proof We shall prove the nontrivial implication ($\Leftarrow$). Consider, without loss of generality, that

$$\varphi(\xi) < +\infty$$

for certain $\xi < 0$ (consequently $\varphi(\xi) = +\infty,\ \forall \xi > 0$). Let

$$\hat{\xi} = \max\{\xi : \varphi(\xi) < +\infty\}.$$

Then,

$$\begin{aligned} \varphi^{**}(0) &= \sup\{A(0) : A(\xi) < \varphi(\xi),\ \forall \xi\} \\ &= \sup\{A(0) : A(\hat{\xi}) = \varphi(\hat{\xi})\} = +\infty. \end{aligned}$$ □

To connect these results with our situation, we have the following lemma.

Lemma 3.1 *Let $\varphi : \mathbb{R} \times \mathbb{R} \longrightarrow [0, +\infty]$ be the new density function built using F and f as described in Sect. 2. Then, the fact that*

$$\varphi^{**}(y, 0) = \varphi(y, 0), \quad \text{for all } y,$$

is equivalent to

$$\sup_{u \in K,\ f(y,u)<0} \frac{F(y,u) - m(y)}{f(y,u)} \leq \inf_{u \in K,\ f(y,u)>0} \frac{F(y,u) - m(y)}{f(y,u)}, \tag{7}$$

where $m(y) := \min_{u \in K}\{F(y,u) : f(y,u) = 0\}$, if $\varphi(y,0) < +\infty$, or to

$$f(y,u) < 0, \quad \forall u \in K,$$

or to

$$f(y,u) > 0, \quad \forall u \in K,$$

when $\varphi(y,0) = +\infty$.

Proof The case $y \notin L$ is immediate, since we will have $\varphi(y,\xi) = +\infty$ everywhere.

Let $y \in L$ and $m(y)$ be as indicated above. We realize that, in fact, $m(y) = \varphi(y,0)$, and hence

$$\begin{aligned}
&\max_{\xi<0} \frac{\varphi(y,\xi) - m(y)}{\xi} \\
&\quad = \max_{\xi<0} \frac{1}{\xi}\Big(\min_{u \in K,\ f(y,u)=\xi} (F(y,u) - m(y))\Big) \\
&\quad = \max_{\xi<0}\left(\max_{u \in K,\ f(y,u)=\xi} \frac{F(y,u) - m(y)}{f(y,u)}\right) \\
&\quad = \max_{u \in K,\ f(y,u)<0} \frac{F(y,u) - m(y)}{f(y,u)}
\end{aligned}$$

and

$$\begin{aligned}
&\min_{\xi>0} \frac{\varphi(y,\xi) - m(y)}{\xi} \\
&\quad = \min_{\xi>0} \frac{1}{\xi}\Big(\min_{u \in K,\ f(y,u)=\xi} (F(y,u) - m(y))\Big) \\
&\quad = \min_{\xi<0}\left(\min_{u \in K,\ f(y,u)=\xi} \frac{F(y,u) - m(y)}{f(y,u)}\right) \\
&\quad = \min_{u \in K,\ f(y,u)>0} \frac{F(y,u) - m(y)}{f(y,u)},
\end{aligned}$$

so the first part of the lemma comes directly from an application of Proposition 3.1.

The second part of the lemma is a direct application of the definition of φ in Sect. 2 and Proposition 3.2. □

The following result will also be helpful.

Proposition 3.3
(a) *Let φ be as in Proposition 3.1. If φ has a global minimum at $\xi = 0$, then* $\varphi^{**}(0) = \varphi(0)$.
(b) *Let φ be as in Theorem 3.1. If $\varphi(y, \cdot)$ has a global minimum at $\xi = 0$ for every y, then* $\varphi^{**}(y, 0) = \varphi(y, 0)$.

Proof If φ has a global minimum at $\xi = 0$, then $\varphi(0) \le \varphi(\xi),\ \forall \xi$, and so

$$\xi < 0 \Rightarrow \frac{\varphi(\xi) - \varphi(0)}{\xi} \le 0,$$
$$\xi > 0 \Rightarrow \frac{\varphi(\xi) - \varphi(0)}{\xi} \ge 0,$$

and by Proposition 3.1, (a) follows. When we consider y fixed and $\varphi(0) := \varphi(y, 0)$, like in (a), then we have (b). □

These conclusions lead us to our main result.

Theorem 3.2 *Let F, f, K, L be as above. Suppose that F is lower semicontinuous, f is continuous, K and $L \subseteq \mathbb{R}$ are closed sets. Suppose that the following conditions are verified*:

(i) *F is coercive with respect to K in the sense that*

$$\lim_{|u| \to \infty,\ u \in K} F(y, u) = +\infty, \quad \textit{uniformly in } y.$$

(ii) *For $y \in L$, put $m(y) = \min_{u \in K}\{F(y, u) : f(y, u) = 0\}$. Then,*

$$\sup_{u \in K,\ f(y,u) < 0} \frac{F(y, u) - m(y)}{f(y, u)} \le \inf_{u \in K,\ f(y,u) > 0} \frac{F(y, u) - m(y)}{f(y, u)}.$$

(iii) *F is bounded from below and*

$$\lim_{|f(y,u)| \to \infty,\ u \in K} F(y, u)/|f(y, u)| = +\infty, \quad \textit{uniformly in } y \textit{ (as in (5))}.$$

Then, there is an absolutely continuous function y and a measurable function u, such that the pair (y, u) is an optimal solution for (P).

Proof Our strategy is to apply Theorem 3.1 to the equivalent variational problem (VP). First, let us see that the integrand function φ defined above verifies condition (i) of Theorem 3.1. It will be enough to check if, for all $\alpha \in \mathbb{R}$, the set

$$\begin{aligned} D_\alpha &= \{(y, \xi) : \varphi(y, \xi) \le \alpha\} \\ &= \{(y, \xi) : y \in L, \exists u \in K, F(y, u) \le \alpha, \xi = f(y, u)\} \end{aligned}$$

is closed (see [2] or [10]).

Consider

$$(y_n, \xi_n)_n \subseteq D_\alpha \quad \text{such that} \quad (y_n, \xi_n) \to (\bar{y}, \bar{\xi}).$$

Then, $\bar{y} \in L$ and $\exists u_n \in K : F(y_n, u_n) \leq \alpha$ and $\xi_n = f(y_n, u_n)$. By the coercivity of F (condition (i)), $(u_n)_n$ is bounded and so, it has a convergent subsequence to a certain $\bar{u} \in K$. Since F is lower semicontinuous and f is continuous,

$$F(\bar{y}, \bar{u}) \leq \alpha$$

and

$$\bar{\xi} = f(\bar{y}, \bar{u}), \quad \bar{u} \in K,$$

so $(\bar{y}, \bar{\xi}) \in D_\alpha$.

Condition (ii) in our statement comes directly from Lemma 3.1. The convexity condition (ii) of Theorem 3.1 follows immediately. Notice that, if condition (ii) in our statement is void, when we have

$$f(y, u) < 0, \quad \forall u \in K,$$

or

$$f(y, u) > 0, \quad \forall u \in K,$$

by the second part of Lemma 3.1, the convexity at the origin still follows.

Finally, to check that φ has superlinear growth, consider the growth condition (iii) on F. We have in particular that

$$\forall M > 0,\ \exists L > 0,\ \forall u \in K,\ \forall \xi : |\xi| = |f(y, u)| > L,\ \varphi(y, \xi) = F(y, u)$$
$$\Rightarrow \frac{F(y, u)}{|f(y, u)|} = \frac{\varphi(y, \xi)}{|\xi|} > M,$$

and so

$$\lim_{|\xi| \to +\infty} \frac{\varphi(y, \xi)}{|\xi|} = +\infty.$$

Since F is bounded from below, so is φ, and it is easy to see that φ has superlinear growth in the sense of condition (iii) of Theorem 3.1. We are now in a position to apply Theorem 3.1 to our reformulated problem (VP) and conclude. □

If K is bounded, conditions (i) and (iii) are void, so we can also state the following result.

Corollary 3.1 *Assume that K is bounded and the condition*

$$\sup_{u \in K,\ f(y,u) < 0} \frac{F(y, u) - m(y)}{f(y, u)} \leq \inf_{u \in K,\ f(y,u) > 0} \frac{F(y, u) - m(y)}{f(y, u)}$$

is verified for $y \in L$, where $m(y)$ is as in Theorem 3.2. Then, we have the existence of optimal solutions for our original optimal control problem.

In this situation, if we apply Proposition 3.3(b) and consider the definition of $\varphi(y, 0)$, we also have the useful result that follows.

Corollary 3.2 *Let F, f, K, L be as indicated above. If K is bounded and for all $y \in L$ we have*

$$\min_u\{F(y,u) : f(y,u) = 0,\ u \in K\} = \min_u\{F(y,u) : u \in K\},$$

then the associated optimal control problem admits optimal solutions.

4 Examples

We will next study some examples to illustrate our results.

Example 4.1 Consider the problem

$$\text{minimize} \quad I(y,u) = \int_0^1 |u(t)|^2 dt,$$

subject to

$$y'(t) + y(t) = u(t)$$

and

$$|u(t)| \le 1, \quad \forall t \in [0,1],$$

with boundary conditions

$$y(0) = y_0, \qquad y(1) = y_1.$$

This is an easy example where standard results can be applied since we have a convex function of u in the cost, a linear dependence of the state equation on u, and a convex set for the constraints on u.

If we define

$$\varphi(y,\xi) = \begin{cases} \min\limits_{u\in[-1,1]}\{u^2 : \xi = -y + u\}, & \text{if } \{u \in [-1,1] : \xi = -y + u\} \neq \emptyset, \\ +\infty, & \text{else,} \end{cases}$$
$$= \begin{cases} (\xi + y)^2, & \text{if } -1 \le \xi + y \le 1, \\ +\infty, & \text{else,} \end{cases}$$

we obtain the following reformulated problem:

$$\text{minimize} \quad J(y) = \int_0^1 \varphi(y(t), y'(t))dt,$$

with

$$y(0) = y_0, \qquad y(1) = y_1.$$

To ensure the existence, we must check the conditions of Corollary 3.1, since $K = [-1, 1]$ is compact. We have

$$F(y,u) = u^2, \qquad f(y,u) = u - y, \qquad L = \mathbb{R}.$$

It is easy to see that, for $y \in \mathbb{R} \setminus [-1,1]$, we have

$$\varphi^{**}(y,0) = \varphi(y,0) = +\infty.$$

An interesting case corresponds to $y \in]-1,1[$. One must have

$$m(y) = \min_{u\in[-1,1]}\{u^2 : 0 = u - y\} = y^2,$$

and condition (iii) becomes

$$\begin{aligned}
&\sup_{u\in[-1,1],\ u-y<0} \frac{u^2-y^2}{u-y} \le \inf_{u\in[-1,1],\ u-y>0} \frac{u^2-y^2}{u-y} \\
&\quad\Leftrightarrow \sup_{-1\le u<y}\{u+y\} \le \inf_{y<u\le 1}\{u+y\} \\
&\quad\Leftrightarrow 2y \le 2y,
\end{aligned}$$

which is obviously true and the existence of an optimal pair (y,u) is ensured.

Example 4.2 Let us look at the problem

$$\text{minimize} \quad I(y,u) = \int_0^1 u(t)^4 dt,$$

subject to

$$y'(t) + u(t)y(t) = u^2(t)$$

and

$$|u(t)| \le 1, \quad \forall t \in [0,T],$$

with boundary conditions

$$y(0) = y_0, \qquad y(1) = y_1.$$

The density function is

$$\begin{aligned}
\varphi(y,\xi) &= \begin{cases} \min\limits_{u\in[-1,1]}\left\{u(t)^4 : u(t) = \dfrac{y \pm \sqrt{y^2+4\xi}}{2}\right\}, & \text{if } \xi > -\dfrac{y^2}{4}, \\ +\infty, & \text{else} \end{cases} \\
&= \begin{cases} \left(\dfrac{y+\sqrt{y^2+4\xi}}{2}\right)^4, & \text{if } -1 \le \dfrac{y+\sqrt{y^2+4\xi}}{2} \le 1,\ y \le 0,\ \xi > -\dfrac{y^2}{4}, \\ \left(\dfrac{y-\sqrt{y^2+4\xi}}{2}\right)^4, & \text{if } -1 \le \dfrac{y-\sqrt{y^2+4\xi}}{2} \le 1,\ y > 0,\ \xi > -\dfrac{y^2}{4}, \\ +\infty, & \text{else.} \end{cases}
\end{aligned}$$

In spite of this complicated form of φ, if we consider

$$F(y,u)=u^4, \qquad f(y,u)=u^2-uy,$$
$$K=[-1,1], \quad L=\mathbb{R},$$

we can see easily that

$$\min_{u\in[-1,1]} u^4=0=\min_{u\in[-1,1]}\{u^4 : u=0\vee u=y\},$$

and we apply Corollary 3.2 to conclude the existence of optimal solutions. With a bit of careful calculations, φ above can in fact be checked to be convex not only at 0. But to conclude the existence of optimal solutions, one need not care about the full convexity, but just apply directly Corollary 3.2.

Example 4.3 Let us consider the problem

$$\text{minimize} \quad I(y,u)=\int_0^1 u(t)^2dt,$$

subject to

$$y'(t)=u(t)y(t)$$

and

$$|u(t)|\le 1, \qquad 0\le y(t)\le 1, \quad \forall t\in[0,T],$$

with boundary conditions $y(0)=y_0$, $y(1)=y_1$. We have

$$F(y,u)=u^2, \qquad f(y,u)=uy,$$
$$K=[-1,1], \qquad L=[0,1].$$

Again, we can apply Corollary 3.2 by verifying that

$$\min_{u\in[-1,1]} u^2=0=\min_{u\in[-1,1]}\{u^2 : u=0\vee y=0\}.$$

Example 4.4 Consider the problem

$$\text{minimize} \quad I(y,u)=\int_0^1 u(t)^2dt,$$

subject to

$$y'(t)=-u(t)^2y(t)+u(t)$$

and

$$|u(t)|\le 1, \quad \forall t\in[0,T],$$

with boundary conditions $y(0) = y_0$, $y(1) = y_1$. Once more the condition of convexity is verified easily because

$$F(y,u) = u^2, \qquad f(y,u) = u - u^2 y,$$
$$K = [-1,1], \qquad L = \mathbb{R}$$

and

$$\min_{u\in[-1,1]} u^2 = 0 = \min_{u\in[-1,1]} \{u^2 : u = 0 \vee uy = 1\}.$$

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