COUPLING ESTIMATION AND CONTROL FOR A TWO DIMENSIONAL BURGERS TYPE EQUATION

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Abstract. The aim of this paper is to study the boundary feedback stabilization of semilinear parabolic type systems in the case of partial information coming from boundary measurements. We study the well possedness of infinite dimensional compensators for linearized parabolic systems in the case of highly unbounded control and observation operators. In particular, we study the case of a two dimensional linearized Burgers type equation with a Dirichlet boundary control and boundary measurements. We study the system obtained by coupling the linear estimator with the linear feedback control law applied to the nonlinear equation.

Key words. Burgers equation, feedback law, estimation, boundary control, compensator, boundary measurements, semilinear parabolic equations

AMS subject classifications. 93B52, 93C20, 93E10.

1. Introduction.

1.1. Motivation.

For a given controlled system

$$z' = Az + Bu, \quad z(0) = z_0,$$

with the observation Hz, let us consider the 'Observer-Based Dynamical Controller'

$$\begin{aligned} &z' = Az + BKz_e, \quad z(0) = z_0, \\ &z_e' = (A + BK + LH)z_e - LHz, \quad z_e(0) = z_{e,0}, \end{aligned} \tag{1.1}$$

and the associated system coupling the state z and the error $e = z - z_e$:

$$\begin{aligned} &z' = (A + BK)z - BKe, \quad z(0) = z_0, \\ &e' = (A + LH)e, \quad e(0) = e_0. \end{aligned} \tag{1.2}$$

When B and H are unbounded operators, it may happen that system (1.2) is well posed, while (1.1) is not. To the best of our knowledge, this surprising fact has not been noticed in the literature. In the case of parabolic systems, we give sufficient conditions for the well posedness of both systems (1.2) and (1.1), with particular emphasis on a Burgers type equation. This is explained in details in the second part of the introduction.

1.2. Detailed introduction.

In this paper, we are interested in stabilizing parabolic systems in the case of partial information. We consider control systems of the form

$$z' = Az + F(z) + Bu + \mu, \quad z(0) = z_0 + \mu_0, \tag{1.3}$$

with the observation

$$y_{obs}(t) = Hz(t) + \eta(t), \tag{1.4}$$

where H is an observation operator, μ is a model error which takes into account different kinds of uncertainties (see section 5), μ_0 is an error in the intial condition, and η is a measurement error. In this setting, z is the state variable, u is the control variable, A is the infinitesimal generator of the evolution

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equation (see below the precise assumptions), B is the control operator, and F is a nonlinear term of the model.

We assume that the state space Z, the control space U and the observation space Y are real Hilbert spaces. The assumptions on this model are as follows.

(H_1) The operator A, with domain $\mathcal{D}(A)$, is the infinitesimal generator of an analytic semigroup on Z. The resolvent of A is compact.

(H_2) The operator B is assumed to be bounded from U into $(\mathcal{D}(A^*))'$ and to satisfy

$$(\lambda_0 I - A)^{-\alpha} B \in \mathcal{L}(U, Z), \quad \text{for some } 0 \leq \alpha < 1, \tag{1.5}$$

where $\lambda_0 > 0$ is chosen so that $\lambda_0 I - A$ is a positive operator.

(H_3) The operator H is defined on $\mathcal{D}(H)$, a subspace of Z. We assume that $\mathcal{D}(A) \subset \mathcal{D}(H)$ and that

$$H(\lambda_0 I - A)^{-\beta} \in \mathcal{L}(Z, Y), \quad \text{for some } 0 \leq \beta < 1. \tag{1.6}$$

We do not state the precise assumptions on the nonlinear term F in a full generality. We shall see in section 4, that the condition (H_F), given below, is convenient. Our goal is mainly to treat a particular nonlinear system corresponding to a Burgers equation.

(H_F) The nonlinear term F is assumed to be of class C^2 from $[Z, \mathcal{D}(A)]_{1/4+\varepsilon/2}$ into $(\mathcal{D}(\lambda_0 I - A^*)^{1/2-\varepsilon/2})'$ for some $0 < \varepsilon < 1/2$. Moreover, we assume that $F(0) = 0$ and $F'(0) = 0$. ($F'(0)$ is the Fréchet derivative of F at 0.)

Our goal is to determine an estimation $z_e(t)$ of $z(t)$, based on the measurement $y_{obs}(t)$, and a feedback control law $K \in \mathcal{L}(U, Z)$ such that the control $u(t) = K z_e(t)$ stabilizes locally the solution to equation (1.3). (Here, locally means under smallness conditions on z_0, η, and on the error $z_e(0) - z_0$ in the initial condition of the estimated state z_e.)

A particular example corresponding to system (1.3) is the following two dimensional Burgers type equation written down below and for which we shall provide numerical tests. The two dimensional domain Ω in which we consider the Burgers equation is the rectangle $\Omega = (0, L_1) \times (0, L_2)$, with $L_1 > 0$ and $L_2 > 0$. Its boundary Γ is split into two parts. Neumann boundary conditions are prescribed on $\Gamma_n = \{0\} \times]0, L_2[$, while Dirichlet boundary conditions are imposed on $\Gamma_d = \Gamma \setminus \Gamma_n$. More precisely, we consider the following partial differential equation

$$\begin{aligned} &\frac{\partial w}{\partial t} - \nu \Delta w + (\partial_1 w + \partial_2 w) w = f_s \quad \text{in } \Omega \times (0, \infty), \\ &\nu \frac{\partial w}{\partial n} = g_{n,s} \quad \text{on } \Gamma_n \times (0, \infty), \quad w = g_{d,s} + Mu + M_\zeta \zeta \quad \text{on } \Gamma_d \times (0, \infty), \\ &w(0) = w_0 + \mu_0 \quad \text{in } \Omega. \end{aligned} \tag{1.7}$$

In this setting, ν is a positive constant, for $j = 1$ or $j = 2$, ∂_j stands for $\dfrac{\partial}{\partial x_j}$, f_s, $g_{n,s}$ and $g_{d,s}$ are stationary data (independent of t), while u is the control variable, M is a truncation function used to localize the control on a part of Γ_d, ζ in an uncertainty in the Dirichlet boundary condition (see section 5), and M_ζ is a truncation function which localizes the uncertainty ζ on a part of Γ_d.

We assume that $w_s \in H^2(\Omega)$ is a solution to the following stationary equation

$$\begin{aligned} &-\nu \Delta w_s + (\partial_1 w_s + \partial_2 w_s) w_s = f_s \quad \text{in } \Omega, \\ &\nu \frac{\partial w_s}{\partial n} = g_{n,s} \quad \text{on } \Gamma_n, \quad w_s = g_{d,s} \quad \text{on } \Gamma_d. \end{aligned} \tag{1.8}$$

We are interested in the local feedback stabilization of equation (1.7) about the stationary solution w_s, with a prescribed exponential decay rate $-\omega$, in the case of partial information. This means that we look for a control u of the form

$$u(t) = K(w_e(t) - w_s), \tag{1.9}$$

where w_e is an estimation of the state variable w, determined thanks to the equation satisfied by w and some measurements of the state w, such that the solution w of equation (1.7), corresponding to u defined by (1.9), obeys

$$\|e^{\omega t}(w(t)-w_s)\|_{H^\varepsilon(\Omega)} \le C(\|w_0\|_{H^\varepsilon(\Omega)}) \quad \text{for all } t \ge 0,$$

provided that $\|w_0 - w_s\|_{H^\varepsilon(\Omega)}$ is small enough for some $\varepsilon > 0$ (here $C(\|w_0\|_{H^\varepsilon(\Omega)})$ denotes a function of $\|w_0\|_{H^\varepsilon(\Omega)}$).

It is convenient to write the nonlinear equation satisfied by $z = w - w_s$. The uncertainty corresponding to ζ in equation (1.7) can be transformed as a source term μ in the equation satisfied by z (see section 5):

$$\begin{aligned}
&\frac{\partial z}{\partial t} - \nu\Delta z + (\partial_1 w_s + \partial_2 w_s)z + (\partial_1 z + \partial_2 z)w_s + (\partial_1 z + \partial_2 z)z = \mu \quad \text{in } \Omega\times(0,\infty),\\
&\nu\frac{\partial z}{\partial n} = 0 \quad \text{on } \Gamma_n\times(0,\infty), \quad z = Mu \quad \text{on } \Gamma_d\times(0,\infty),\\
&z(0) = w_0 + \mu_0 - w_s = z_0 + \mu_0 \quad \text{in } \Omega.
\end{aligned} \tag{1.10}$$

The corresponding linearized equation is

$$\begin{aligned}
&\frac{\partial z}{\partial t} - \nu\Delta z + (\partial_1 w_s + \partial_2 w_s)z + (\partial_1 z + \partial_2 z)w_s = \mu \quad \text{in } \Omega\times(0,\infty),\\
&\nu\frac{\partial z}{\partial n} = 0 \quad \text{on } \Gamma_n\times(0,\infty), \quad z = Mu \quad \text{on } \Gamma_d\times(0,\infty),\\
&z(0) = z_0 + \mu_0 \quad \text{in } \Omega.
\end{aligned} \tag{1.11}$$

In section 2, we shall explain that equation (1.10) may be rewritten in the form (1.3). We shall choose some measurements depending on $w(t)|_{\Gamma_n}$, or similarly on $z(t)|_{\Gamma_n}$. For the numerical tests, we will take an obervation of the form

$$Hz(t) = \left(\frac{1}{|\Gamma_1|}\int_{\Gamma_1} z(t), \cdots, \frac{1}{|\Gamma_{N_o}|}\int_{\Gamma_{N_o}} z(t)\right) \in \mathbb{R}^{N_o}, \tag{1.12}$$

where $\Gamma_1, \cdots, \Gamma_{N_o}$ are nonempty open intervals in Γ_n.

We have already studied similar problems, in the case of full information, for the Burgers equation in [29], and for the Navier-Stokes equations in [21], [22] and [25]. In [21] and [25], the feedback control law is determined by stabilizing the linearized model and this feedback law is next applied to the nonlinear system. Here we are going to follow the same approach but in the case of partial information. Thus, assuming that $(A+\omega I, B)$ is stabilizable and that $(A+\omega I, H)$ is detectable with $\omega > 0$ given fixed, our goal is to find $K \in \mathcal{L}(Z,U)$ and $L \in \mathcal{L}(Y,Z)$ such that the system

$$\begin{aligned}
&z' = Az + F(z) + BKz_e + \mu, \quad z(0) = z_0 = z_{e,0} + \mu_0,\\
&z_e' = Az_e + BKz_e + L(Hz_e - y_{obs}), \quad z_e(0) = z_{e,0},\\
&y_{obs}(t) = Hz(t) + \eta(t),
\end{aligned} \tag{1.13}$$

is well posed and that its solution satisfies

$$\|e^{\omega t}z(t)\|_Z \longrightarrow 0 \quad \text{as} \quad t \longrightarrow \infty, \tag{1.14}$$

provided that z_0, $z_0 - z_{e,0}$, η, and μ are small enough in appropriate norms.

The corresponding linearized compensator is

$$\begin{aligned}
&z' = Az + BKz_e + \mu, \quad z(0) = z_{e,0} + \mu_0,\\
&z_e' = Az_e + BKz_e + L(Hz_e - y_{obs}), \quad z_e(0) = z_{e,0},\\
&y_{obs}(t) = Hz(t) + \eta(t).
\end{aligned} \tag{1.15}$$

In that case we may have an exponential decay as in (1.14), without smallness conditions on the initial data. Only some boundedness condition is required on η and μ.

In the linear case, this type of problem has been studied by many authors in the case when the degrees of unboundeness of the control and observation operators obey $\alpha+\beta \leq 1/2$, see e.g. [27], [6], [8]. Next, Lasiecka [16] has proposed a finite dimensional estimator in the case when $\alpha+\beta<1$. With Li [12], she has extended this result to the case when $\alpha+\beta>1$.

Here, we are interested in the case when $\alpha+\beta>1$, but our goal is different from the approximation process studied in [12]. Even if we are aware that, for pratical applications, we need compensators of finite dimension, we look for a compensator of infinite dimension of the form (1.15). First, we are interested in proving that the estimation equation

$$z_e' = Az_e + BKz_e + L(Hz_e - y_{obs}), \quad z_e(0) = z_{e,0}, \tag{1.16}$$

is well posed. We are going to see that the well posedness of this equation in the case when $\alpha+\beta>1$ is not obvious, and that additional assumptions are needed. Next, we want to prove that the systems (1.13) and (1.15) are well posed and that the stability condition (1.14) holds true in both cases.

Our future goal will be to construct a finite dimensional estimator. But for the applications we have in view (stabilization of fluid flows), an estimator based on the approximation of A by a finite element method (or another classical approximation method) leads to a finite dimensional estimator of too large dimension to be useful. We shall present in a future work new estimators of finite dimension. But their analysis will be based on the well posedness of systems of the form (1.15) and of equations of the form (1.16).

As far as we know, these issues have not been studied in the literature in the case when $\alpha+\beta>1$. We are going to see that the assumptions $(H_1)-(H_3)$ are not sufficient to establish the well posedness of system (1.15) and of equation (1.16) (see e.g. Example 1 in section 3).

Additional references on compensators for infinite dimensional systems are [1], [3], [7], and [15].

The plan of the paper is as follows. We study the feedback control law and the estimation equation for the linearized Burgers equation (1.11) in section 2. In section 3, we consider system (1.15) when the assumptions $(H_1)-(H_3)$ are satisfied. To establish the well posedness of this system, we shall see that these assumptions are not sufficient and that two additional assumptions are needed. A compensator of the form (1.15) for the Burgers equation is studied in section 4. In section 5 we explain the finite element approximation used in the numerical tests. Some numerical experiments in which we test the efficiency of the compensator both for the linearized Burgers equation and for the corresponding nonlinear model are provided in section 5.

2. Feedback law and filtering operator. The aim of this section is to study the feedback law and the estimation of the linearized Burgers equation (1.11). We are going to see that assumptions $(H_1)-(H_3)$, which are satisfied for the evolution equation corresponding to equation (1.11), are not sufficient to define correctly the estimation equation. An additional assumption will be needed (see assumption (H_4) in section 2.4).

2.1. Assumptions and preliminary results for the Burgers equation. From now on, we assume that the function M is equal to zero on $[0,L_1]\times\{0,L_2\}$. The restriction of M to $\{L_1\}\times[0,L_2]$ is a nonnegative C^∞-function, positive in $\{L_1\}\times(L_2/4,3L_2/4)$, and equal to zero at $\{L_1\}\times\{0,L_2\}$.

In order to write equation (1.11) as a control system, we set

$$\begin{aligned}
&\mathcal{D}(A) = \Big\{ z\in H^2(\Omega) \mid z=0 \text{ on } \Gamma_d \text{ and } \frac{\partial z}{\partial n} = 0 \text{ on } \Gamma_n \Big\},\\
&\text{and}\\
&Az = \nu\Delta z - (\partial_1 w_s + \partial_2 w_s)z - (\partial_1 z + \partial_2 z)w_s \quad \text{for all } z\in\mathcal{D}(A).
\end{aligned} \tag{2.1}$$

We also introduce the operator A_0 corresponding to the particular case when $w_s = 0$:

$$\begin{aligned} &\mathcal{D}(A_0) = \Big\{ z \in H^2(\Omega) \mid z = 0 \text{ on } \Gamma_d \text{ and } \frac{\partial z}{\partial n} = 0 \text{ on } \Gamma_n \Big\}, \\ &\text{and} \\ &A_0 z = \nu \Delta z \quad \text{for all } z \in \mathcal{D}(A_0). \end{aligned} \tag{2.2}$$

THEOREM 2.1. *The operator $(A, \mathcal{D}(A))$ defined in (2.1) is the infinitesimal generator of an analytic semigroup on $L^2(\Omega)$. Its resolvent is compact.*

Proof. The proof relies on the following inequality

$$\big((\lambda_0 I - A)z, z\big)_{L^2(\Omega)} \geq \frac{\nu}{2} \|z\|^2_{H^1(\Omega)} \quad \forall z \in \mathcal{D}(A), \tag{2.3}$$

with $\lambda_0 > 0$ large enough. See e.g. [29], or [21] where a similar estimate is established for the Oseen operator. □

In this section we set $Z = L^2(\Omega)$ and $U = L^2(\Gamma_d)$. We introduce the Dirichlet operator $D \in \mathcal{L}(L^2(\Gamma_d), L^2(\Omega))$, by setting $Du = \xi$, where ξ is the solution to the following elliptic equation

$$\begin{aligned} &\lambda_0 \xi - \nu \Delta \xi + (\partial_1 \xi + \partial_2 \xi) w_s + (\partial_1 w_s + \partial_2 w_s) \xi = 0 \quad \text{in } \Omega, \\ &\nu \frac{\partial \xi}{\partial n} = 0 \quad \text{on } \Gamma_n, \quad \zeta = u \quad \text{on } \Gamma_d, \end{aligned} \tag{2.4}$$

where $\lambda_0 > 0$ is chosen so that the inequality (2.3) is true. Notice that $(A^*, \mathcal{D}(A^*))$ is defined by

$$\begin{aligned} &\mathcal{D}(A^*) = \Big\{ \phi \in H^2(\Omega) \mid \phi = 0 \text{ on } \Gamma_d \text{ and } \nu \frac{\partial \phi}{\partial n} + w_s \phi = 0 \text{ on } \Gamma_n \Big\}, \\ &\text{and} \\ &A^* \phi = \nu \Delta \phi + (\partial_1 \phi + \partial_2 \phi) w_s \quad \text{for all } \phi \in \mathcal{D}(A^*). \end{aligned} \tag{2.5}$$

We define $B \in \mathcal{L}(U, (\mathcal{D}(A^*))')$ by

$$Bu = (\lambda_0 I - A) D M u \quad \text{for all } u \in U.$$

Let us notice that

$$(\lambda_0 I - A)^{-\alpha} B \in \mathcal{L}(U, Z) \quad \text{for all } 3/4 < \alpha \leq 1.$$

When $u \in L^2(0, \infty; L^2(\Gamma_d))$, solutions to equation (1.11) may be defined by transposition [19], or they can also be defined as solutions to the evolution equation

$$z' = Az + Bu, \quad z(0) = z_0. \tag{2.6}$$

2.2. Feedback control law. Up to the end of section 2, $(A, \mathcal{D}(A))$ and B are the operators defined in the above section. We need to introduce the following spaces

$$H^1_{\Gamma_d}(\Omega) = \{ z \in H^1(\Omega) \mid z = 0 \text{ on } \Gamma_d \} \quad \text{and} \quad H^{-1}_{\Gamma_d}(\Omega) = (H^1_{\Gamma_d}(\Omega))'.$$

We have the following continuous and dense imbeddings

$$H^1_{\Gamma_d}(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow H^{-1}_{\Gamma_d}(\Omega).$$

In order to obtain the feedback stabilization of equation (2.6) with a given exponential decay rate $\omega > 0$, we consider the equation

$$z' = (A + \omega I) z + Bu, \quad z(0) = z_0. \tag{2.7}$$

Since $M > 0$ on $\{L_1\} \times (L_2/4, 3L_2/4)$, it is well known that the pair $(A + \omega I, B)$ is stabilizable. Indeed the stabilizability reduces to verify some unique continuation property for the eigenfunctions associated

to the eigenvalues λ_j of A^* satisfying $\mathrm{Re}\lambda_j + \omega \geq 0$ (see [23]). This unique continuation property can be established by using results from Fursikov and Imanuvilov, see e.g. [11]. A way for finding a stabilizing feedback operator consists in looking for the solution to the following algebraic Riccati equation

$$\begin{aligned} &P \in \mathcal{L}(Z),\ P^* = P,\ P \geq 0, \quad (A+\omega I) - BR^{-1}B^*P \quad \text{is stable,} \\ &P(A+\omega I) + (A^*+\omega I)P - PBR^{-1}B^*P + Q = 0, \end{aligned} \tag{2.8}$$

where $R \in \mathcal{L}(U)$, $R = R^* > 0$, $Q \in \mathcal{L}(Z)$, $Q = Q^* \geq 0$. If P is the solution to (2.8), then $K = -R^{-1}B^*P$ is a stabilizing feedback. The operator $K = -R^{-1}B^*P$ depends on the choice of R and Q. Since R does not play any critical role let us choose $R = I_U$ (I_U is the identity in U).

It is clear that the feedback gain K depends on Q. The main objective of this section is to prove that there exists a feedback gain K such that

$$\mathcal{D}((-A-BK)^{1/2}) = [Z, \mathcal{D}(A+BK)]_{1/2} = H^1_{\mathcal{F}}(\Omega), \tag{2.9}$$

where

$$H^1_{\mathcal{F}}(\Omega) := \{z \in H^1(\Omega) \mid (A+BK)z \in H^{-1}_{\Gamma_d}(\Omega),\ z = Kz \text{ on } \Gamma_d\}. \tag{2.10}$$

This result will be proved in Corollary 2.5, and it will be used in Section 2.4 to study the estimation equation. To prove (2.9), we have to consider $A+BK$ as an unbounded operator on Z and also as an unbounded operator on $H^{-1}_{\Gamma_d}(\Omega)$. Thus we have to introduce $\mathcal{D}(A+\omega I+BK; Z)$, the domain of the operator in Z, and $\mathcal{D}(A+\omega I+BK; H^{-1}_{\Gamma_d}(\Omega))$, the domain of the operator in $H^{-1}_{\Gamma_d}(\Omega)$. When we introduce the domain of an unbounded operator without specifying in which space we consider it, by default we shall consider it as an unbounded operator in Z.

When $R = I_U$ and $Q = I_Z$ (I_Z is the identity in Z), it is shown in [21] that $P \in \mathcal{L}(Z, \mathcal{D}(A))$. (In [21], the proof, which is given for the linearized Navier-Stokes system, may be rewritten word for word for system (2.6)). We can improve these regularity properties by choosing $R = I_U$ and $Q = (\lambda_0 I - A_0)^{-1}$ (see [22]), or by choosing Q of finite rank (see [25]) or even $Q = 0$ (see [24]). More precisely we have the following theorem.

THEOREM 2.2. *Let us assume that the pair $(A+\omega I, B)$ is stabilizable in $Z = L^2(\Omega)$ and $H^{-1}_{\Gamma_d}(\Omega)$, with controls in U. There exists an operator $K \in \mathcal{L}(H^{-1}_{\Gamma_d}(\Omega), U)$ such that the following properties are satisfied*

(i) $A+\omega I+BK$, with domain $\mathcal{D}(A+\omega I+BK; Z) = \{z \in Z \mid (A+BK)z \in Z\}$, is the generator of an analytic semigroup on Z, exponentially stable on Z.

(ii) $A+\omega I+BK$, with domain $\mathcal{D}(A+\omega I+BK; H^{-1}_{\Gamma_d}(\Omega)) = \{z \in H^{-1}_{\Gamma_d}(\Omega) \mid (A+BK)z \in H^{-1}_{\Gamma_d}(\Omega)\}$, is the generator of an analytic semigroup on $H^{-1}_{\Gamma_d}(\Omega)$, exponentially stable on $H^{-1}_{\Gamma_d}(\Omega)$.

Proof. The existence of an operator $K \in \mathcal{L}(Z, U)$ for which (i) is satisfied is well known, see e.g. [17]. What is new in this statement is the existence of $K \in \mathcal{L}(H^{-1}_{\Gamma_d}(\Omega), U)$ for which (i) and (ii) are satisfied for the same operator K.

Let us notice that, due to (2.5), we have $H^{-1}_{\Gamma_d}(\Omega) = [(\mathcal{D}(A^*))', Z]_{1/2}$. The existence of a feedback $K \in \mathcal{L}(H^{-1}_{\Gamma_d}(\Omega), U)$ for which (i) and (ii) are satisfied simultaneously is established in [22] for the Oseen equation. The adaptation to the linearized Burgers equation is straightforward. Next, other feedback operators satisfying the properties (i) and (ii) have been constructed in [25] by using control spaces of finite dimension. The same approach can be followed here. □

COROLLARY 2.3. *Let us assume that the pair $(A+\omega I, B)$ is stabilizable in $L^2(\Omega)$ and $H^{-1}_{\Gamma_d}(\Omega)$, with controls in U, and let $K \in \mathcal{L}(H^{-1}_{\Gamma_d}(\Omega), U)$ satisfy the conclusions of Theorem 2.2. The operator $A+BK$ is an isomorphism from $H^1_{\mathcal{F}}(\Omega)$, equipped with the $H^1(\Omega)$ norm, into $H^{-1}_{\Gamma_d}(\Omega)$.*

Proof. First notice that if $z \in H^1(\Omega)$, then $z = Kz$ on Γ_d is meaningful. Moreover Az belongs to $(\mathcal{D}(A^*))'$ (because $z \in L^2(\Omega)$), and BKz belongs to $(\mathcal{D}(A^*))'$. Since $H^{-1}_{\Gamma_d}(\Omega) \subset (\mathcal{D}(A^*))'$, the condition $(A+BK)z \in H^{-1}_{\Gamma_d}(\Omega)$ is also meaningful.

Assume that $z \in H^1_{\mathcal{F}}(\Omega)$, then $(A+BK)z \in H^{-1}_{\Gamma_d}(\Omega)$. Thus, by definition of $H^1_{\mathcal{F}}(\Omega)$, $A+BK$ is a mapping from $H^1_{\mathcal{F}}(\Omega)$ to $H^{-1}_{\Gamma_d}(\Omega)$. Let us show that it is invertible and that its inverse is continuous from $H^{-1}_{\Gamma_d}(\Omega)$ into $H^1_{\mathcal{F}}(\Omega)$. We have to solve the equation

$$(A+BK)z = f \in H^{-1}_{\Gamma_d}(\Omega). \tag{2.11}$$

From Theorem 2.2(ii), we know that this equation admits a unique solution $z_f \in H^{-1}_{\Gamma_d}(\Omega)$ defined by

$$z_f = -\int_0^\infty e^{\tau(A+BK)} f \, d\tau.$$

Thus

$$\|z_f\|_{H^{-1}_{\Gamma_d}(\Omega)} \leq C\|f\|_{H^{-1}_{\Gamma_d}(\Omega)}.$$

Equation (2.11) is equivalent to

$$\begin{aligned} &-\lambda_0 z_f + \nu\Delta z_f - (\partial_1 w_s + \partial_2 w_s) z_f - (\partial_1 z + \partial_2 z) w_s = f - \lambda_0 z_f \quad \text{in } \Omega, \\ &\nu \frac{\partial z_f}{\partial n} = 0 \quad \text{on } \Gamma_n, \quad z_f = K z_f \quad \text{on } \Gamma_d. \end{aligned} \tag{2.12}$$

It is easy to verify that $z_f \in H^1_{\mathcal{F}}(\Omega)$ and that

$$\|z_f\|_{H^1(\Omega)} \leq C(\|f\|_{H^{-1}_{\Gamma_d}(\Omega)} + \|z_f\|_{H^{-1}_{\Gamma_d}(\Omega)}) \leq C\|f\|_{H^{-1}_{\Gamma_d}(\Omega)}.$$

We have shown that $A + BK$ is a bijection from $H^1_{\mathcal{F}}(\Omega)$ to $H^{-1}_{\Gamma_d}(\Omega)$. Since $(A + BK)^{-1}$ is continuous from $H^{-1}_{\Gamma_d}(\Omega)$ to $H^1_{\mathcal{F}}(\Omega)$, $A + BK$ is continuous from $H^1_{\mathcal{F}}(\Omega)$ to $H^{-1}_{\Gamma_d}(\Omega)$. Thus

$$\|f\|_{H^{-1}_{\Gamma_d}(\Omega)} \leq C\|z_f\|_{H^1(\Omega)},$$

which completes the proof.
□

REMARK 2.4. *Let us notice that* $\mathcal{D}((A + BK)^*) = \mathcal{D}(A^*)$. *Indeed,* $(A + BK)^* = A^* + K^*B^*$ *and due to* (1.5), K^*B^* *is* A^**-relatively bounded with relative bound equal to zero (see [14]). Thus, the operator* $A + BK$ *can be extended by extrapolation to* $(\mathcal{D}((-A - BK)^*))' = (\mathcal{D}(A^*))'$ *and to* $H^{-1}_{\Gamma_d}(\Omega) = (\mathcal{D}(\lambda_0 I - A^*)^{1/2})' = (\mathcal{D}(((-A - BK)^*)^{1/2}))'$.

COROLLARY 2.5. *Let us assume that the assumptions of Corollary 2.3 are satisfied, then* $H^1_{\mathcal{F}}(\Omega) = \mathcal{D}((-A - BK)^{1/2}) = [Z, \mathcal{D}(A + BK)]_{1/2}$, *where* $H^1_{\mathcal{F}}(\Omega)$ *is the space introduced in* (2.10).

Proof. Step 1. We first prove that $\lambda_0 I - A - BK$ is a maximal dissipative operator in $H^{-1}_{\Gamma_d}(\Omega)$ for $\lambda_0 > 0$ large enough. We equip $H^{-1}_{\Gamma_d}(\Omega)$ with the norm

$$z \longmapsto \left(\left\langle (-A_0)^{-1} z, z \right\rangle_{H^1_{\Gamma_d}(\Omega), H^{-1}_{\Gamma_d}(\Omega)} \right)^{1/2},$$

where A_0 is defined in (2.1). We have to show that

$$\left((-A_0)^{-1} (\lambda_0 I - A - BK) z, z \right)_Z \geq 0 \quad \forall z \in H^{-1}_{\Gamma_d}(\Omega). \tag{2.13}$$

For that, we write

$$\begin{aligned} &\left((-A_0)^{-1} (\lambda_0 I - A - BK) z, z \right)_Z = \left(z, (\lambda_0 I - A^* - K^*B^*)(-A_0)^{-1} z \right)_Z \\ &= \left(z, (\lambda_0 I - A^*)(-A_0)^{-1} z \right)_Z - \left(z, K^*B^*(-A_0)^{-1} z \right)_Z. \end{aligned}$$

We have

$$\begin{aligned} &\left(z, (\lambda_0 I - A^*)(-A_0)^{-1} z \right)_Z \\ &\geq \lambda_0 \left(z, (-A_0)^{-1} z \right)_Z + (z, z)_Z - C\|(-A_0)^{-1} z\|_Z^{1/2} \|w_s\|^{3/2}_{H^{3/2}(\Omega)} \|z\|_Z, \end{aligned}$$

for some $C > 0$, and

$$-\left(z, K^*B^*(-A_0)^{-1} z \right)_Z = -\left(z, K^*(B^*(-A_0)^{-\alpha})(-A_0)^{\alpha-1} z \right)_Z \geq -C\|z\|_Z \|(-A_0)^{\alpha-1} z\|_Z.$$

Thus, the dissipativity condition (2.13) will be satisfied if $\lambda_0 > 0$ is large enough. Next, it is easy to show that $\lambda_0 I - A - BK$ is maximal dissipative in $H^{-1}_{\Gamma_d}(\Omega)$.

Step 2. Since $A + BK$ is an isomorphism from $H^1_{\mathcal{F}}(\Omega) := \{z \in H^1(\Omega) \mid (A + BK)z \in H^{-1}_{\Gamma_d}(\Omega),\ z = Kz \text{ on } \Gamma_d\}$ into $H^{-1}_{\Gamma_d}(\Omega)$, from Step 1 and [2, Part II, Chapter 1, proposition 6.1], it follows that $(-A - BK)^{1/2}$ is an isomorphism from $H^1_{\mathcal{F}}(\Omega)$ into $[H^1_{\mathcal{F}}(\Omega), H^{-1}_{\Gamma_d}(\Omega)]_{1/2}$. We can verify that $[H^1_{\mathcal{F}}(\Omega), H^{-1}_{\Gamma_d}(\Omega)]_{1/2} = Z$. Thus $H^1_{\mathcal{F}}(\Omega)$ is the domain of $(-A-BK)^{1/2}$ considered as an unbounded operator in Z. The proof is complete. □

2.3. Filtering operator. We do not precisely define the observation operator, but we assume that it obeys the two following conditions.

(i) The pair $(A + \omega I, H)$ is detectable.

(ii) The operator H obeys

$$\|Hz\|_Y \le C\|z|_{\Gamma_n}\|_{L^2(\Gamma_n)}. \tag{2.14}$$

This last condition is obviously satisfied when H is defined by (1.12) with $Y = \mathbb{R}^{N_o}$. The case of an observation depending on $\dfrac{\partial z}{\partial n}$ can be considered too, but for this type of observation the condition (2.14) has to be modified. In section 5, the numerical tests correspond to an observation of the trace of the form (1.12). In particular $H^1(\Omega)$ is included in $\mathcal{D}(H)$. Moreover, due to (2.14), we have

$$H(\lambda_0 I - A)^{-\beta} \in \mathcal{L}(Z, Y), \quad \text{for all } 1/4 < \beta < 1.$$

We look for $L \in \mathcal{L}(Y, Z)$ such that $A + \omega I + LH$, with domain $\mathcal{D}(A + \omega I + LH) = \{z \in Z \mid (A + LH)z \in Z\} = \mathcal{D}(A)$, is the generator of an analytic semigroup on Z, exponentially stable on Z.

Under the above conditions, if $R_o \in \mathcal{L}(Y)$, $R_o = R_o^* > 0$, $Q_o \in \mathcal{L}(Z)$, $Q_o = Q_o^* \ge 0$, the equation

$$\begin{aligned} &P_e \in \mathcal{L}(Z),\ P_e^* = P_e,\ P_e \ge 0, \quad A - P_e H^* R_o^{-1} H \quad \text{is stable,} \\ &(A + \omega I)P_e + P_e(A^* + \omega I) - P_e H^* R_o^{-1} H P_e + Q_o = 0, \end{aligned} \tag{2.15}$$

admits a unique solution (see e.g. [17]). We choose $L \in \mathcal{L}(Y, Z)$ as

$$L = -P_e H^* R_o^{-1}. \tag{2.16}$$

2.4. The estimation equation. In this section, we want to study the equation

$$z_e' = Az_e + L(Hz_e - y_{obs}) + BKz_e, \quad z_e(0) = z_{e,0}. \tag{2.17}$$

We already know that $A + \omega I + BK$, with domain $\mathcal{D}(A + \omega I + BK) = \{z \in Z \mid (A + BK)z \in Z\}$, is the infinitesimal generator of an analytic semigroup exponentially stable on Z.

In order to study equation (2.17), we need the additional condition

$$(H_4) \qquad LH\,(-A - BK)^{-\gamma} \quad \text{is bounded from } Z \text{ into } Z,$$

for some $0 \le \gamma < 1$.

PROPOSITION 2.6. *Let us assume that the pair $(A+\omega I, B)$ is stabilizable and that the pair $(A+\omega I, H)$ is detectable. In addition, we assume that* (2.14) *is satisfied.*

(i) If (H_4) is satisfied, then the operator $A + BK + LH$, with domain $\mathcal{D}(A + BK + LH) = \mathcal{D}(A + BK)$, generates an analytic semigroup on Z. If $y_{obs} \in L^2(0, \infty; Y)$ and $z_{e,0} \in Z$, then the solution z_e to equation (2.17) *belongs to $L^2_{\text{loc}}([0, \infty); \mathcal{D}((-A - BK)^{1/2})) \cap H^1_{\text{loc}}([0, \infty); \mathcal{D}((\lambda_0 I - A^*)^{-1/2}))$, where $\mathcal{D}((\lambda_0 I - A^*)^{-1/2}) = (\mathcal{D}((\lambda_0 I - A^*)^{1/2}))' = H^{-1}_{\Gamma_d}(\Omega)$.*

(ii) If condition (2.14) *is satisfied, if K satisfies the conclusions of Theorem 2.2, and if L is defined by* (2.16)*, then assumption (H_4) is satisfied.*

Proof. Step 1. Proof of (i). If (H_4) is true, then, from [20, Chapter 3, Corollary 2.4], it follows that $A+BK+LH$, with domain $\mathcal{D}(A+BK+LH) = \mathcal{D}(A+BK)$ is also the infinitesimal generator of an analytic semigroup on Z. Thus equation (2.17) will be well posed for any $y_{obs} \in L^2(0, \infty; Y)$. The existence of

the solution in $L^2_{\rm loc}([0,\infty);\mathcal{D}((-A-BK)^{1/2}))\cap H^1_{\rm loc}([0,\infty);(\mathcal{D}((\lambda_0 I-A^*-K^*B^*-H^*L^*)^{1/2}))')$ follows from [2, Chapter 3, Theorem 2.2]. Once we know that z_e belongs to $L^2_{\rm loc}([0,\infty);\mathcal{D}((-A-BK)^{1/2}))$, the term LHz_e can be considered as a source term in equation (2.17), and we can prove that z_e also belongs to $L^2_{\rm loc}([0,\infty);\mathcal{D}((-A-BK)^{1/2}))\cap H^1_{\rm loc}([0,\infty);\mathcal{D}((\lambda_0 I-A^*)^{-1/2}))$.

Step 2. Proof of (ii). We are going to prove (H_4) in the case when H depends continuously on $z|_{\Gamma_n}$, that is when H obeys estimate (2.14). To prove (H_4) we consider the elliptic equation

$$(A+BK)z=f,\quad \text{with } f\in Z. \tag{2.18}$$

This equation admits a unique solution defined by

$$z=-\int_0^\infty e^{\tau(A+BK)}f\,d\tau.$$

It is easy to prove that, when $f\in Z=L^2(\Omega)$, equation (2.18) is equivalent to the elliptic boundary value problem

$$\begin{aligned}&\nu\Delta z-(\partial_1 w_s+\partial_2 w_s)z-(\partial_1 z+\partial_2 z)w_s=f \quad \text{in } \Omega,\\ &\nu\frac{\partial z}{\partial n}=0\quad \text{on } \Gamma_n,\quad z=Kz\quad \text{on } \Gamma_d.\end{aligned} \tag{2.19}$$

From regularity results for elliptic equations, it follows that

$$\|z|_{\Gamma_n}\|_{L^2(\Gamma_n)}\le C(\|z\|_{L^2(\Omega)}+\|f\|_{H^{-1}_{\Gamma_d}(\Omega)}+\|Kz\|_{L^2(\Gamma_d)}).$$

Since $A+BK$ is an isomorphism from $H^{-1}_{\Gamma_d}(\Omega)$ into $\{z\in H^1(\Omega)\mid (A+BK)z\in H^{-1}_{\Gamma_d}(\Omega),\ z=Kz \text{ on } \Gamma_d\}=H^1_{\mathcal{F}}(\Omega)$ (see Corollary 2.3), we have

$$\|f\|_{H^{-1}_{\Gamma_d}(\Omega)}\le C\|z\|_{H^1_{\mathcal{F}}(\Omega)}.$$

Knowing that $H^1_{\mathcal{F}}(\Omega)=[Z,\mathcal{D}(A+BK)]_{1/2}$ (see Corollary 2.5), we have

$$\|z\|_{H^1_{\mathcal{F}}(\Omega)}\le C\|(-A-BK)^{1/2}z\|_{L^2(\Omega)}.$$

Due to the choice of the feedback law, we also have

$$\|Kz\|_{L^2(\Gamma_d)}\le C\|z\|_{L^2(\Omega)}.$$

Collecting these different results, we obtain

$$\|z|_{\Gamma_n}\|_{L^2(\Gamma_n)}\le C(\|z\|_{L^2(\Omega)}+\|(-A-BK)^{1/2}z\|_{L^2(\Omega)})\le C\|(-A-BK)^{1/2}z\|_{L^2(\Omega)}.$$

Which implies that (H_4) is satisfied for $\gamma=1/2$, because $(-A-BK)^{1/2}$ is an isomorphism from $\mathcal{D}((-A-BK)^{1/2})$ onto $L^2(\Omega)$.
□

3. Extension to other models and examples. In this section, we assume that the operators A, B and H obey assumptions $(H_1)-(H_3)$, and we want to study the two following systems

$$\begin{aligned}&z'=(A+BK)z-BKe,\quad z(0)=z_0,\\ &e'=(A+LH)e,\quad e(0)=e_0,\end{aligned} \tag{3.1}$$

and

$$\begin{aligned}&z'=Az+BKz_e,\quad z(0)=z_0,\\ &z_e'=(A+BK+LH)z_e-LHz,\quad z_e(0)=z_{e,0}.\end{aligned} \tag{3.2}$$

We are going to see that assumptions $(H_1)-(H_3)$ are not sufficient for the well posedness of system (3.2). More precisely, we have to assume that assumption (H_4) of section 2 and the following assumption are satisfied:

(H_5) $\qquad H(\lambda_0 I - A)^{-1}B$ is a bounded operator from U into Y.

REMARK 3.1. *If we combine* (1.5) *and* (1.6)*, we arrive at*

$$H(\lambda_0 I - A)^{-(\alpha+\beta)}B \in \mathcal{L}(U,Y).$$

Since $\alpha+\beta>1$, (H_5) is stronger than what we can obtain by combining (H_2) and (H_3). We are going to see, in Examples 1 and 2 below, that (H_5) can be satisfied even if (H_2) and (H_3) are valid only for $\alpha>3/4$ and $\beta>3/4$.

THEOREM 3.2. *We assume that $(A+\omega I,B)$ is stabilizable and that $(A+\omega I,H)$ is detectable. In addition, if assumptions $(H_1)-(H_3)$ are satisfied, then there exist $K\in\mathcal{L}(Z,U)$ and $L\in\mathcal{L}(Y,Z)$ such that*

$$\left\|e^{t(A+BK)}\right\|_{\mathcal{L}(Z)} \le M_\omega\, e^{-\omega t} \quad \text{and} \quad \left\|e^{t(A+LH)}\right\|_{\mathcal{L}(Z)} \le M_\omega\, e^{-\omega t}, \tag{3.3}$$

for some $M_\omega\ge 1$. Moreover, the operator $\mathcal{A}_e$ defined in $Z\times Z$ by

$$\begin{aligned}
&\mathcal{D}(\mathcal{A}_e) = \Big\{(z,e)\in Z\times Z \mid Az+BKz-BKe\in Z,\ (A+LH)e\in Z\Big\},\\
&\mathcal{A}_e = \begin{pmatrix} A+BK & -BK\\ 0 & A+LH\end{pmatrix},
\end{aligned}$$

generates an analytic semigroup exponentially stable on $Z\times Z$.

Proof. Under assumptions $(H_1)-(H_3)$ and the stabilizability of $(A+\omega I,B)$ and the detectability of $(A+\omega I,H)$, the existence of K and L for which the semigroups $(e^{t(A+BK)})_{t>0}$ and $(e^{t(A+BK)})_{t>0}$ are analytic and satisfy (3.3) is well known in the literature (see e.g. [17]).

Since $(e^{t(A+BK)})_{t>0}$ and $(e^{t(A+BK)})_{t>0}$ are analytic and exponentially stable, it is clear that $(\mathcal{A}_e,\mathcal{D}(\mathcal{A}_e))$ generates an exponentially stable semigroup on $Z\times Z$, because BK is bounded from Z into $(\mathcal{D}(\lambda_0 I-A^*)^\alpha)'$ and $(\mathcal{D}(((-A-BK)^*)^\alpha))'=(\mathcal{D}(\lambda_0 I-A^*)^\alpha)'$ (indeed $\mathcal{D}((A+BK)^*)=\mathcal{D}(A^*)$). We can verify that the semigroup generated by $\mathcal{A}_e$ is analytic. We can use [2, Part II, Chapter 3, Theorem 2.2] to obtain the existence of a unique solution to system (3.1). □

From Theorem 3.2, it follows that the family of operators

$$((z_0,z_{e,0})\mapsto(z(t),z_e(t)))_{t\ge 0}\,, \tag{3.4}$$

where $z_e(t)=z(t)-e(t)$ and (z,e) is the solution to (3.1) corresponding to the initial data $(z_0,e_0)=(z_0,z_0-z_{e,0})$, is the infinitesimal generator of an exponentially stable semigroup on $Z\times Z$. We want to characterize its infinitesimal generator. We would like to write that it is the operator $\mathcal{A}$ defined in $Z\times Z$ by

$$\begin{aligned}
&\mathcal{D}(\mathcal{A}) = \Big\{(z,z_e)\in Z\times Z \mid Az+BKz_e\in Z,\ -LHz+(A+BK+LH)z_e\in Z\Big\}\\
&\text{and}\\
&\mathcal{A} = \begin{pmatrix} A & BK\\ -LH & A+BK+LH\end{pmatrix}.
\end{aligned} \tag{3.5}$$

We show that this is true if assumptions $(H_1)-(H_5)$ are satisfied.

THEOREM 3.3. *Assume that $(A+\omega I,B)$ is stabilizable, $(A+\omega I,H)$ is detectable, and that hypotheses $(H_1)-(H_5)$ are satisfied. Then (z,e) belongs to $\mathcal{D}(\mathcal{A}_e)$ if and only if $(z,z_e)=(z,z-e)$ belongs to $\mathcal{D}(\mathcal{A})$.*

Proof. Assume that (z,e) belongs to $\mathcal{D}(\mathcal{A}_e)$. We already know that $Az+BKz-BKe=Az+Bkz_e\in Z$. Since $Az+BKz_e\in Z$, we have $z=(\lambda_0 I-A)^{-1}(BKz_e+f)$, for some $f\in Z$. Due to assumption (H_5), it follows that $LHz=LH(\lambda_0 I-A)^{-1}(BKz_e+f)\in Z$. Moreover $e=z-z_e\in\mathcal{D}(A+LH)=\mathcal{D}(A)$. Thus LHe belongs to Z. Since LHz belongs to Z, we deduce that $LHz_e\in Z$.

From the equality $(A+BK)z - BKe = Az + BKz_e$, it follows that $Az + BKz_e \in Z$. Moreover, we have $-LHz + (A+BK+LH)z_e = -LHz + (A+BK+LH)(z-e) = Az + BK(z-e) - (A+LH)e \in Z$. Thus (z, z_e) belongs to $\mathcal{D}(\mathcal{A})$.

Let us show the converse statement. Assume that (z, z_e) belongs to $\mathcal{D}(\mathcal{A})$. First we have that $Az + BKz_e \in Z$. In particular $(A+BK)z - BKz_e$ belongs to Z. Writing $z = (\lambda_0 I - A)^{-1}(BKz_e + f)$, for some $f \in Z$, with (H_5) we deduce that $LHz \in Z$.

Since $-LHz + (A+BK+LH)z_e \in Z$, we have $(A+BK+LH)z_e \in Z$. Writing $\lambda I - A - BK - LH = \Big(I - LH(-A-BK)^{-\gamma}(-A-BK)^{-\gamma}(\lambda I - A - BK)^{-1}\Big)(\lambda I - A - BK)$, using (H_4) and the analitycity of the semigroup generated by $(A+BK)$, it follows that $(I - LH(-A-BK)^{-\gamma}(-A-BK)^{-\gamma}(\lambda I - A - BK)^{-1})$ is bounded and invertible for $\lambda > 0$ large enough (see [20, proof of Theorem 2.1, Chapter 3] or [17, page 151]). Thus $(A+BK+LH)z_e$ belongs to Z if and only if $(A+BK)z_e$ belongs to Z. It follows that $(A+BK)z_e \in Z$. If we set $(A+BK)z_e = f$, we have $(\lambda_0 I - A)z_e = -f + BKz_e$ and $z_e = (\lambda_0 I - A)^{-1}(-f + BKz_e)$. With (H_5), we deduce that $LHz_e = (\lambda_0 I - A)^{-1}(-f + BKz_e)$ belongs to Z. Thus $(A+LH)e = (A+LH)(z - z_e) = Az + BKz_e - (A+LH+BK)z_e + LHz \in Z$. The proof is complete. □

THEOREM 3.4. *A pair (z, e) is the solution to equation (3.1) if, and only if, $(z, z_e) = (z, z-e)$ is the solution to equation (3.2) with $z_{e,0} = z_0 - e_0$. Moreover the operator $(\mathcal{A}, \mathcal{D}(\mathcal{A}))$ is the infinitesimal generator of an exponentially stable semigroup on $Z \times Z$.*

Proof. Assume that (z, e) is the solution to equation (3.1). Due to regularity results for parabolic equations, we know that $(z, e) \in C((0,\infty); \mathcal{D}(\mathcal{A}_e)) \cap C^1((0,\infty); Z))$. From Theorem 3.3, it follows that $(z, z_e) \in C((0,\infty); \mathcal{D}(\mathcal{A})) \cap C^1((0,\infty); Z))$. It is easy to prove that $z'(t) = Az(t) + BKz_e(t)$ and $z_e'(t) = (A+BK+LH)z_e(t) - LHz(t)$ for all $t > 0$. Knowing that $(\mathcal{A}_e, \mathcal{D}(\mathcal{A}_e))$ is the generator of the semigroup

$$((z_0, e_0) \mapsto (z(t), e(t)))_{t \geq 0},$$

where (z, e) is the solution to equation (3.1), with Theorem 3.3, it is easy to verify that $(\mathcal{A}, \mathcal{D}(\mathcal{A}))$ generates the semigroup defined by (3.4). Now, assuming that (z, z_e) is the solution to equation (3.2), we can easily verify that $(z, e) = (z, z-e)$ is the solution to equation (3.1). The proof is complete.
□

REMARK 3.5. *If assumptions $(H_1) - (H_5)$ are satisfied, then*

$$\mathcal{D}(\mathcal{A}) = \Big\{(z, z_e) \in Z \times Z \mid Az + BKz_e \in Z,\ (A+BK)z_e \in Z\Big\}.$$

Notice that assumptions (H_4) and (H_5) are not made in [16] and in [12]. Without these assumptions, we do not known how to prove Theorem 3.4.

REMARK 3.6. *In section 2, we have already seen that assumption (H_4) is needed for the well posedness of equation (2.17). Moreover assumption (H_5) is needed to define $Hz(t)$ when z is the solution to*

$$z' = Az + BKz_e, \quad z(0) = z_0,$$

and z_e is the solution to equation (2.17). Without this additional assumption, we do not know what is the meaning of $Hz(t)$.

Example 1. Let us now consider the first example studied in [12, Section 4]. It corresponds to the heat equation with a Dirichlet boundary control:

$$\begin{aligned} &\frac{\partial z}{\partial t} - \Delta z - c^2 z = 0 \quad \text{in } \Omega \times (0,\infty), \\ &z(x,t) = u(t)\, g(x) \quad \text{on } \Gamma \times (0,\infty), \\ &z(0) = z_0 \quad \text{in } \Omega, \end{aligned} \tag{3.6}$$

where $\Gamma = \partial\Omega$, and a Neumann boundary observation:

$$Hz(t) = \frac{\partial z(t)}{\partial n} \quad \text{on } \Gamma, \quad t \in (0,\infty).$$

We assume that Ω is a bounded regular domain. The coefficient c^2 is added to the heat operator to obtain an unstable system. In this example $Z = L^2(\Omega)$, $Y = L^2(\Gamma)$, $U = \mathbb{R}$, and we choose $g \in L^2(\Gamma)$. Thus the control space U is of dimension one. We can rewrite equation (3.6) in the form

$$z' = Az + Bu,$$

with $\mathcal{D}(A) = H^2(\Omega) \cap H_0^1(\Omega)$, $Az = \Delta z + c^2 z$, $B = (\lambda_0 I - A)Dg$, with $\lambda_0 - c^2 > 0$, and where $Dg = \xi$ is the solution to

$$\lambda_0 \xi - \Delta \xi - c^2 \xi = 0 \quad \text{in } \Omega, \quad \xi = g \quad \text{on } \Gamma.$$

The operator H is well defined in $\mathcal{D}(A)$. It is easy to verify that assumptions $(H_1) - (H_3)$ are satisfied for any $3/4 < \alpha < 1$ and $3/4 < \beta < 1$. Let us show that there exist K and L for which the conclusions of Theorem 3.2 are true, but for which assumptions (H_4) and (H_5) are not necessarily satisfied.

For simplicity we may suppose that the unstable eigenvalues of A are simple. Let $\lambda_1 > \cdots > \lambda_{N_u}$, be the unstable eigenvalues of A, and $\Phi_1, \cdots, \Phi_{N_u}$ be a family of associated eigenfunctions. Let us choose a function $g \in L^2(\Gamma)$ such that

$$\int_\Gamma g \frac{\partial \Phi_i}{\partial n} \neq 0, \quad \text{for all } 1 \leq i \leq N_u.$$

With this condition, the pair (A, B) is stabilizable. It is well known that we can find an operator $K \in \mathcal{L}(L^2(\Omega), U)$, of the form

$$Kz = \int_\Omega k(\xi) z(\xi)\, d\xi,$$

such that $A + BK$ generates an analytic semigroup exponentially stable on Z. For example, this operator K can be obtained by solving a Riccati equation of the form (2.8) and by choosing $K = -R^{-1}B^*P$.

Similarly, by solving a Riccati equation of the form (2.15), we can find $L \in \mathcal{L}(L^2(\Gamma), Z)$ of the form

$$Lh(x) = \int_\Gamma \ell(x, \xi) h(\xi)\, d\xi \quad \text{for } x \in \Omega,$$

such that $A + LH$ is exponentially stable. With [4, Lemma 9.6], we can prove that, if $g \in H^1(\Gamma)$, then (H_4) and (H_5) are satisfied. If $g \in L^2(\Gamma)$ but if $g \notin H^1(\Gamma)$, then $(H_1) - (H_3)$ are satisfied, but not (H_4) and (H_5). Indeed, if (H_5) were satisfied then

$$H(\lambda_0 I - A)^{-1} B r = r\, HDg = r\, \frac{\partial Dg}{\partial n} \in L^2(\Gamma) \quad \text{for all } r \in \mathbb{R}.$$

But since $g \in L^2(\Gamma)$ and $g \notin H^1(\Gamma)$, we cannot hope to have $\dfrac{\partial Dg}{\partial n} \in L^2(\Gamma)$ and we have a contradiction. The assumption (H_4) is not satisfied for similar reasons.

This example shows that $(H_1) - (H_3)$ may be satisfied while the domain of $\mathcal{A}$, introduced in (3.5), is not necessarily well defined, and therefore the equivalence stated in Theorem 3.4 is not necessarily satisfied.

Example 2. We now consider a variant of the previous example. It corresponds to the heat equation with a Dirichlet boundary control localized on a part of the boundary Γ and a Neumann boundary observation localized on another part:

$$\begin{aligned}
&\frac{\partial z}{\partial t} - \Delta z - c^2 z = 0 \quad \text{in } \Omega \times (0, \infty), \\
&z = u(t) g(x) \quad \text{on } \Gamma \times (0, \infty), \\
&z(0) = z_0 \quad \text{in } \Omega, \\
&Hz = M_o \frac{\partial z}{\partial n} \quad \text{on } \Gamma \times (0, \infty),
\end{aligned} \tag{3.7}$$

where M_o is a C^∞-function with values in $[0,1]$, equal to 1 on a relatively open part Γ_o, $g \in L^2(\Gamma)$ is equal to 1 on a relatively open part Γ_c, and we assume that

$$\mathrm{dist}(\mathrm{supp}(g), \mathrm{supp}(M_o)) > 0. \tag{3.8}$$

As in example 1, we have $U = \mathbb{R}$, $Y = L^2(\Gamma)$, and we denote by $\Phi_1, \cdots, \Phi_{N_u}$ a family of eigenfunctions associated with the unstable eigenfunctions of A (which are assumed to be simple as in the previous example). We can write (3.7) in the form

$$z' = Az + Bu, \quad z(0) = z_0,$$

where A and B are defined as in Example 1. We choose g and M_o such that

$$\int_\Gamma g \frac{\partial \Phi_i}{\partial n} \neq 0 \quad \text{and} \quad \int_\Gamma M_o \frac{\partial \Phi_i}{\partial n} \neq 0, \quad \text{for all } 1 \leq i \leq N_u.$$

Under these conditions, the pair (A, B) is stabilizable and the pair (A, H) is detectable. We do not need that g belongs to $H^1(\Gamma)$, we may assume only that $g \in L^2(\Gamma)$. Let us check (H_5). Let us set $z = (\lambda_0 I - A)^{-1} Bu$. Then z is the solution to

$$\lambda_0 z - \Delta z - c^2 z = 0 \quad \text{in } \Omega, \quad z = u\, g \quad \text{on } \Gamma.$$

Let $V(M_o)$ be a neighbourhood in $\overline{\Omega}$ of $\mathrm{supp}(M_o)$ such that $\mathrm{dist}(\mathrm{supp}(g), V(M_o)) > 0$. By using a truncation method, it is easy to prove that $z|_{V(M_o)}$ belongs to $H^2(V(M_o))$ (actually $z|_{V(M_o)}$ belongs to any $H^s(V(M_o))$, $s \in \mathbb{N}^*$, if Γ is of class C^s). Thus (H_5) is satisfied. We can choose $K = -R^{-1}B^*P$, where P is the solution of a Riccati equation of the form (2.8) for the pair (A, B), and we can choose $L = -P_e H^* M_o\, R_o^{-1}$, where P_e is the solution to a Riccati equation of the form (2.15), in such a way that (H_4) is satisfied.

Example 3. Let us consider equation (1.11) with the measure operator H introduced in (1.12). For this example, we have $\Omega = (0, L_1) \times (0, L_2)$, $Z = L^2(\Omega)$, $U = L^2(\Gamma_c)$, $\Gamma_c = \{L_1\} \times (0, L_2)$, and $Y = \mathbb{R}^{N_o}$. Due to the particular choice of H, there is no hope to prove that the pair $(A + \omega I, H)$ is detectable. Actually the detectability of this pair will depend on the choice of the open subsets $\Gamma_1, \cdots, \Gamma_{N_o}$. A bad choice may lead to the existence of undetectable unstable eigenfunctions. Indeed, if $N_o = 1$ and if there exists an eigenfunction z of A satisfying $Az = \lambda z$ with $\mathrm{Re}\,\lambda + \omega > 0$ and such that $\displaystyle\int_{\Gamma_1} z|_{\Gamma_1} = 0$, then the eigenfunction z is undetectable by H and the pair $(A + \omega I, H)$ is not detectable. This means that the sensor locations $\Gamma_1, \cdots, \Gamma_{N_o}$ cannot be chosen arbitrarily. But, for a particular choice of $\Gamma_1, \cdots, \Gamma_{N_o}$ it is very easy to check either exactly or numerically whether the detectability condition is fulfilled or not (see Proposition 5.1).

The operators A and B have already been introduced in section 2, and we have seen that we can choose K and L so that assumption (H_4) is satisfied. As in the above example, we can use a truncation method to show that assumption (H_5) is also satisfied.

4. Local stabilization of the nonlinear system. In this section, we want to prove a local stabilization result for the system

$$\begin{aligned} &z' = Az + BKz_e + F(w_\zeta + z) + \mu, \quad z(0) = z_0 = z_{e,0} + \mu_0, \\ &z_e' = Az_e + L(Hz_e - y_{obs}) + BKz_e, \quad z_e(0) = z_{e,0}, \end{aligned} \tag{4.1}$$

with the observation

$$y_{obs}(t) = Hz(t) + \eta(t). \tag{4.2}$$

The model error in equation (4.1) is more general than the one considered in equation (1.13). Indeed, in addition to the model error μ, we have introduced a function w_ζ in the nonlinear term of the equation. We shall see in section 5 that the introduction of w_ζ in the nonlinear term is useful to take into account some boundary uncertainties.

For simplicity and clarity, we only consider the nonlinear system corresponding to the Burgers equation. The possibility to extend the stabilization result to other systems depends on the behavior of the nonlinear term F. Thus A and B are the operators defined in section 2, H is an operator satisfying the condition (2.14), K and L are the operators introduced in Theorem 2.2 and in section 2.3, and $F(z) = (\partial_1 z + \partial_2 z)z$.

Since we look for solutions satisfying a prescribed exponential decay rate $-\omega$, we make the following change of unknowns

$$\hat{z} = e^{\omega t} z, \quad \hat{z}_e = e^{\omega t} z_e, \quad \hat{w}_\zeta = e^{\omega t} w_\zeta, \quad \hat{\mu} = e^{\omega t} \mu,$$

and we denote by $\hat{y}_{obs}$ the noisy observation of $\hat{z}$, that is

$$\hat{y}_{obs}(t) = H\hat{z} + \hat{\eta}(t), \quad \text{with } \hat{\eta}(t) = e^{\omega t}\eta(t).$$

Thus equation (4.1) may be rewritten in the form

$$\begin{aligned} \hat{z}' &= (A + \omega I)\hat{z} + BK\hat{z}_e + e^{-\omega t}F(\hat{w}_\zeta + \hat{z}) + \hat{\mu}, \quad \hat{z}(0) = z_0, \\ \hat{z}_e' &= (A + \omega I)\hat{z}_e + L(H\hat{z}_e - \hat{y}_{obs}) + BKz_e, \quad \hat{z}_e(0) = z_{e,0}. \end{aligned} \tag{4.3}$$

If we write the system satisfied by $\hat{z}$ and $\hat{e} = \hat{z} - \hat{z}_e$, we obtain

$$\begin{aligned} \hat{z}' &= (A + \omega I)\hat{z} + BK\hat{z} - BK\hat{e} + e^{-\omega t}F(\hat{w}_\zeta + \hat{z}) + \hat{\mu}, \quad \hat{z}(0) = z_0, \\ \hat{e}' &= (A + \omega I)\hat{e} + LH\hat{e} - LH\hat{\eta} + e^{-\omega t}F(\hat{w}_\zeta + \hat{z}) + \hat{\mu}, \quad \hat{e}(0) = z_0 - z_{e,0}. \end{aligned} \tag{4.4}$$

Remark 4.1. *We are going to assume that $\hat{\mu}$ belongs to $L^2(0,\infty; H^{-1+\varepsilon}_{\Gamma_d}(\Omega))$. Since $H^{-1}_{\Gamma_d}(\Omega) = (\mathcal{D}(\lambda_0 I - A^*)^{1/2})' = (\mathcal{D}(((-A-BK)^*)^{1/2}))'$ (see the proof of Theorem 2.2), with the reiteration theorem in interpolation, we can also show that $H^{-1+\varepsilon}_{\Gamma_d}(\Omega) = (\mathcal{D}(\lambda_0 I - A^*)^{1/2-\varepsilon/2})' = (\mathcal{D}(((-A-BK)^*)^{1/2-\varepsilon/2}))'$. This result will be very helpful to study the first equation in system* (4.4).

Remark 4.2. *To study the second equation in system* (4.4)*, we need an additional assumption on $A + LH$, see (H_6) below, in order to take into account the nonlinear term $e^{-\omega t}F(\hat{w}_\zeta + \hat{z})$ of the equation. This assumption can be easily verified when the measure oparator is defined by* (1.12)*, see Proposition 4.5.*

(H_6) The following imbedding are satisfied

$$\mathcal{D}(((-A-LH)^*)^{1/2-\varepsilon/2}) \hookrightarrow L^{1/\varepsilon}(\Omega) \quad \text{and} \quad L^{1/(1-\varepsilon)}(\Omega) \hookrightarrow (\mathcal{D}(((-A-LH)^*)^{1/2-\varepsilon/2}))',$$

for all $\varepsilon \in (0, 1/2)$.

We rewrite system (4.4) in the form

$$\begin{pmatrix} \hat{z} \\ \hat{e} \end{pmatrix}' = \begin{pmatrix} A + \omega I + BK & -BK \\ 0 & A + \omega I + LH \end{pmatrix} \begin{pmatrix} \hat{z} \\ \hat{e} \end{pmatrix} + \begin{pmatrix} e^{-\omega t}F(\hat{w}_\zeta + \hat{z}) + \hat{\mu} \\ e^{-\omega t}F(\hat{w}_\zeta + \hat{z}) + \hat{\mu} - LH\hat{\eta} \end{pmatrix}. \tag{4.5}$$

Theorem 4.3. *We assume that assumptions (H_1) to (H_6) are satisfied. Let $\varepsilon \in (0, 1/2)$ given fixed for which (H_6) holds to be true. Assume that z_0 and $z_{e,0}$ belong to $H^\varepsilon(\Omega)$, and that $\hat{w}_\zeta$ belongs to $H^{1+\varepsilon,1/2+\varepsilon/2}(\Omega \times (0,\infty))$. There exists a constant $C_0 > 0$ and a nondecreasing function θ from $\mathbb{R}^+$ into itself, such that, if $C \in (0, C_0)$ and*

$$\|z_0\|_{H^\varepsilon(\Omega)} + \|\mu_0\|_{H^\varepsilon(\Omega)} + \|e^{\omega \cdot}\mu\|_{L^2(0,\infty;H^{-1+\varepsilon}_{\Gamma_d}(\Omega))} + \|e^{\omega \cdot} w_\zeta\|_{H^{1+\varepsilon,1/2+\varepsilon/2}(\Omega\times(0,\infty))} + \|e^{\omega \cdot}\eta\|_{L^2(0,\infty;Y)} \leq \theta(C),$$

then, the system (4.4) *admits a unique solution in the space*

$$\Big\{(z,e) \in (H^{1+\varepsilon,1/2+\varepsilon/2}(\Omega \times (0,\infty)))^2 \mid \|(z,e)\|_{(H^{1+\varepsilon,1/2+\varepsilon/2}(\Omega\times(0,\infty)))^2} \leq C\Big\}.$$

In particular the solution to system (4.1) *obeys*

$$\|z(t)\|_{H^\varepsilon(\Omega)} + \|z(t) - z_e(t)\|_{H^\varepsilon(\Omega)} \leq Ce^{-\omega t}. \tag{4.6}$$

Proof. The proof is based on the Banach fixed point Theorem. Let ξ belong to $H^{1+\varepsilon,1/2+\varepsilon/2}(\Omega\times(0,\infty))$. We assume that $\hat{w}_\zeta$ belongs to $H^{1+\varepsilon,1/2+\varepsilon/2}(\Omega\times(0,\infty))$. We want to estimate $F(\hat{w}_\zeta+\xi)=(\partial_1+\partial_2)(\hat{w}_\zeta+\xi)(\hat{w}_\zeta+\xi)$ in $L^2(0,\infty;H^{-1+\varepsilon}_{\Gamma_d}(\Omega))$. We have

$$\begin{aligned}
&\|\partial_1(\hat{w}_\zeta+\xi)(\hat{w}_\zeta+\xi)\|_{L^2(0,\infty;H^{-1+\varepsilon}_{\Gamma_d}(\Omega))}\le C\|\partial_1(\hat{w}_\zeta+\xi)(\hat{w}_\zeta+\xi)\|_{L^2(0,\infty;L^{2/(2-\varepsilon)}(\Omega))}\\
&\le C\|\partial_1(\hat{w}_\zeta+\xi)(\hat{w}_\zeta+\xi)\|_{L^2(0,\infty;L^{1/(1-\varepsilon)}(\Omega))}\\
&\le C\|\partial_1(\hat{w}_\zeta+\xi)\|_{L^2(0,\infty;L^{2/(1-\varepsilon)}(\Omega))}\,\|\hat{w}_\zeta+\xi\|_{L^\infty(0,\infty;L^{2/(1-\varepsilon)}(\Omega))}\\
&\le C\|\partial_1(\hat{w}_\zeta+\xi)\|_{L^2(0,\infty;H^{\varepsilon}(\Omega))}\,\|\hat{w}_\zeta+\xi\|_{L^\infty(0,\infty;H^{\varepsilon}(\Omega))}\\
&\le C\|\hat{w}_\zeta+\xi\|_{L^2(0,\infty;H^{1+\varepsilon}(\Omega))}\,\|\hat{w}_\zeta+\xi\|_{H^{1+\varepsilon,1/2+\varepsilon/2}(\Omega\times(0,\infty))}.
\end{aligned}$$

The first inequality follows from the imbeddings $H^{1-\varepsilon}_{\Gamma_d}(\Omega)\hookrightarrow L^{2/\varepsilon}(\Omega)$ and $L^{2/(2-\varepsilon)}(\Omega)\hookrightarrow H^{-1+\varepsilon}_{\Gamma_d}(\Omega)$, the fourth one follows from the imbedding $H^{\varepsilon}(\Omega)\hookrightarrow L^{2/(1-\varepsilon)}(\Omega)$, and the last one from the imbedding $H^{1+\varepsilon,1/2+\varepsilon/2}(\Omega\times(0,\infty))\hookrightarrow L^\infty(0,\infty;H^\varepsilon(\Omega))$. Thus $e^{-\omega t}F(\hat{w}_\zeta+\xi)$ belongs to $L^2(0,\infty;H^{-1+\varepsilon}_{\Gamma_d}(\Omega))$. It also belongs to $L^2(0,\infty;L^{1/(1-\varepsilon)}(\Omega))$.

Let us denote by (z_ξ,e_ξ) the solution to the system

$$\begin{aligned}
z'_\xi&=(A+\omega I)z_\xi+BKz_\xi-BKe_\xi+e^{-\omega t}F(\hat{w}_\zeta+\xi),\quad z_\xi(0)=z_0,\\
e'_\xi&=(A+\omega I)e_\xi+LHe_\xi-LC\hat{\eta}+e^{-\omega t}F(\hat{w}_\zeta+\xi),\quad e_\xi(0)=z_0-z_{e,0}.
\end{aligned}\tag{4.7}$$

Since $H^{-1+\varepsilon}_{\Gamma_d}(\Omega)\hookrightarrow(\mathcal{D}(((-A-BK)^*)^{1/2}))'$ and $L^{1/(1-\varepsilon)}(\Omega)\hookrightarrow(\mathcal{D}(((-A-LH)^*)^{1/2-\varepsilon/2}))'$, we have the estimate (see [2, Chapter 3, page 165])

$$\begin{aligned}
&\|(z_\xi,e_\xi)\|_{(H^{1+\varepsilon,1/2+\varepsilon/2}(\Omega\times(0,\infty)))^2}\\
&\le C_1(\|e^{-\omega t}F(\hat{w}_\zeta+\xi)\|_{L^2(0,\infty;H^{-1+\varepsilon}_{\Gamma_d}(\Omega))}+\|e^{-\omega t}F(\hat{w}_\zeta+\xi)\|_{L^2(0,\infty;L^{1/(1-\varepsilon)}(\Omega))}+\|\hat{\mu}\|_{L^2(0,\infty;H^{-1+\varepsilon}_{\Gamma_d}(\Omega))}\\
&\qquad+\|\hat{\eta}\|_{L^2(0,\infty;Y)}+\|z_0\|_{H^\varepsilon(\Omega)}+\|\mu_0\|_{H^\varepsilon(\Omega)})\\
&\le C_1(\|\hat{w}_\zeta+\xi\|^2_{H^{1+\varepsilon,1/2+\varepsilon/2}(\Omega\times(0,\infty))}+\|\hat{\mu}\|_{L^2(0,\infty;H^{-1+\varepsilon}_{\Gamma_d}(\Omega))}+\|\hat{\eta}\|_{L^2(0,\infty;Y)}+\|z_0\|_{H^\varepsilon(\Omega)}+\|\mu_0\|_{H^\varepsilon(\Omega)}).
\end{aligned}$$

As in [29], we can find $C_0>0$ and θ such that the nonlinear mapping

$$\xi\longmapsto(z_\xi,e_\xi)$$

is a mapping from

$$V_C=\Big\{(z,e)\in(H^{1+\varepsilon,1/2+\varepsilon/2}(\Omega\times(0,\infty)))^2\mid\|(z,e)\|_{(H^{1+\varepsilon,1/2+\varepsilon/2}(\Omega\times(0,\infty)))^2}\le C\Big\},$$

into itself, if $0<C\le C_0$, and provided that

$$\|z_0\|_{H^\varepsilon(\Omega)}+\|z_0-z_{e,0}\|_{H^\varepsilon(\Omega)}+\|e^{\omega\cdot}\eta\|_{L^2(0,\infty;Y)}\le\theta(C).$$

Next, we can show that it is also a contraction for C_0 suitably chosen. Thus system (4.4) admits a unique solution in V_C. From Theorem 3.4, it follows that (4.3) admits a unique solution in the metric space

$$\Big\{(z,z_e)\in(H^{1+\varepsilon,1/2+\varepsilon/2}(\Omega\times(0,\infty)))^2\mid\|(z,z-z_e)\|_{(H^{1+\varepsilon,1/2+\varepsilon/2}(\Omega\times(0,\infty)))^2}\le C\Big\},$$

equipped with the distance corresponding to the norm in $(H^{1+\varepsilon,1/2+\varepsilon/2}(\Omega\times(0,\infty)))^2$. The existence of a unique solution to system (4.1) follows from the fact that $(\widehat{z},\widehat{z}_e)$ is a solution to system (4.3) if and only if $(z,z_e)=e^{-\omega t}(\widehat{z},\widehat{z}_e)$ is a solution to system (4.3). The proof is complete.
□

REMARK 4.4. *Let us notice that assumption (H_F) of the introduction is satisfied by the nonlinear term of our model and that this is exactly what has been used in the above proof.*

Let us now verify that assumption (H_6) is satisfied by the operator A is defined by (2.1) and the operator H by (1.12).

PROPOSITION 4.5. *Assume that A is defined by* (2.1)*, and H by* (1.12)*. Assume that $A+LH$, with $L \in \mathcal{L}(\mathbb{R}^{N_o}, L^2(\Omega))$ and $\mathcal{D}(A+LH) = \mathcal{D}(A)$, is exponentially stable on $Z = L^2(\Omega)$. Then assumption (H_6) is satisfied.*

Proof. Step 1. We first show that

$$\|\phi\|_{H^{2-\tilde{\varepsilon}}(\Omega)} \leq C_{\tilde{\varepsilon}} \|\phi\|_{\mathcal{D}(A^*+H^*L^*)}, \quad \forall \phi \in \mathcal{D}(A^*+H^*L^*), \quad \text{and all } 0 < \tilde{\varepsilon} < 1/2.$$

The operator L^* belongs to $\mathcal{L}(L^2(\Omega), \mathbb{R}^{N_o})$. Let us set

$$L^*\phi = (r_1^\phi, \cdots, r_{N_o}^\phi) \in \mathbb{R}^{N_o}.$$

We have

$$\|(r_1^\phi, \cdots, r_{N_o}^\phi)\|_{\mathbb{R}^{N_o}} \leq C\|\phi\|_Z.$$

Since $A^*+H^*L^*$ is an isomorphism from $\mathcal{D}(A^*+H^*L^*)$ into Z, we set $\|\phi\|_{\mathcal{D}(A^*+H^*L^*)} = \|(A^*+H^*L^*)\phi\|_Z$. The equation $(A^*+H^*L^*)\phi = f$ is equivalent to the system

$$\begin{aligned}
&\lambda_0\phi - \nu\Delta\phi - (\partial_1\phi + \partial_2\phi)w_s = \lambda_0\phi - f \quad \text{in } \Omega,\\
&\nu\frac{\partial\phi}{\partial n} + w_s\phi = \sum_{i=1}^{N_o} \frac{r_i^\phi}{|\Gamma_i|}\chi_{\Gamma_i} \text{ on } \Gamma_n\\
&(r_1^\phi, \cdots, r_{N_o}^\phi) \in \mathbb{R}^{N_o} = L^*\phi.
\end{aligned}$$

Since $\sum_{i=1}^{N_o} \frac{r_i^\phi}{|\Gamma_i|}\chi_{\Gamma_i}$ belongs to $H^{1/2-\tilde{\varepsilon}}(\Gamma_n)$ for all $\tilde{\varepsilon} \in (0, 1/2)$, from elliptic regularity results it follows that

$$\|\phi\|_{H^{2-\tilde{\varepsilon}}(\Omega)} \leq \tilde{C}_{\tilde{\varepsilon}} \left(\|(r_1^\phi, \cdots, r_{N_o}^\phi)\|_{\mathbb{R}^{N_o}} + \|\lambda_0\phi - f\|_Z \right) \leq C_{\tilde{\varepsilon}}\|f\|_Z = C_{\tilde{\varepsilon}}\|\phi\|_{\mathcal{D}(A^*+H^*L^*)},$$

for all $\tilde{\varepsilon} \in (0, 1/2)$ and all $\phi \in \mathcal{D}(A^*+H^*L^*)$.

Step 2. The identity mapping is linear and continuous from $\mathcal{D}(A^*+H^*L^*)$ into $H^{2-\tilde{\varepsilon}}(\Omega)$ and from Z into Z. Thus it is also continuous from $[Z, \mathcal{D}(A^*+H^*L^*)]_{1/2-\varepsilon/2} = \mathcal{D}(((-A-LH)^*)^{1/2-\varepsilon/2})$ into $[Z, H^{2-\tilde{\varepsilon}}(\Omega)]_{1/2-\varepsilon/2} = H^{1-\varepsilon-\tilde{\varepsilon}/2+\varepsilon\tilde{\varepsilon}/2}(\Omega)$. Since $H^{1-\varepsilon-\tilde{\varepsilon}/2+\varepsilon\tilde{\varepsilon}/2}(\Omega) \hookrightarrow L^{1/\varepsilon}(\Omega)$ for $\tilde{\varepsilon} > 0$ small enough, and since $\mathcal{D}(((-A-LH)^*)^{1/2-\varepsilon/2})$ is dense in $L^{1/\varepsilon}(\Omega)$, we also have $L^{1/(1-\varepsilon)}(\Omega) \hookrightarrow (\mathcal{D}(((-A-LH)^*)^{1/2-\varepsilon/2}))'$. The proof is complete.
□

5. Approximate finite dimensional model.

5.1. The model. For the numerical simulations, we choose $L_1 = L_2 = 1$, $\Omega = (0,1)\times(0,1)$. Dirichlet boundary conditions are imposed on $\Gamma_d = \Gamma_0 \cup \Gamma_c \cup \Gamma_\zeta$, see Figure 5.1. The stationary solution is $w_s = 0$. We slightly modify equation (1.7) by adding a reaction term $-\gamma w$ with $\gamma > 0$ in order to have an unstable dynamical system. A model error ζ is introduced in the boundary condition on $\Gamma_\zeta = (0,1)\times\{1\}$ in order to study the robustness of the compensator with respect to this type of perturbation. The space of controls is of dimension 1, that is $U = \mathbb{R}$. The control acts on $\Gamma_c = \{1\}\times(0,1)$. To localise the control on Γ_c we set $M(x) = \sin(\pi x_2)$ on Γ_c and $M = 0$ on $\Gamma \setminus \Gamma_c$. The homogeneous Dirichlet condition is imposed on $\Gamma_0 = (0,1)\times\{0\}$. Thus the controlled system is

$$\begin{aligned}
&\frac{\partial w}{\partial t} - \nu\Delta w - \gamma w + (\partial_1 w + \partial_2 w)w = 0 \quad \text{in } \Omega\times(0,\infty),\\
&\nu\frac{\partial w}{\partial n} = 0 \quad \text{on } \Gamma_n\times(0,\infty), \quad w(1,x_2,t) = \sin(\pi x_2)u(t) \quad \forall (x_2,t) \in (0,1)\times(0,\infty),\\
&w = 0 \quad \text{on } \Gamma_0\times(0,\infty),\\
&w(x_1,1,t) = \cos(\tfrac{\pi}{2}x_1)\zeta(t) \quad \forall (x_1,t) \in (0,1)\times(0,\infty)\\
&w(0) = w_0 + \mu_0 \quad \text{in } \Omega.
\end{aligned} \tag{5.1}$$

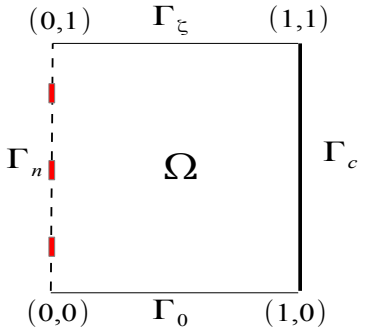


FIG. 5.1. *Domain with these boundaries.*

As explained in Section 2 for system (1.11), the above equation can be written in the form

$$w' = A_\gamma w + Bu + B_\zeta \zeta + F(w), \quad w(0) = w_0 + \mu_0,$$

where

$$A_\gamma = \nu\Delta + \gamma I \quad \text{with} \quad D(A_\gamma) = \left\{ z \in H^2(\Omega) \mid z = 0 \text{ on } \Gamma_d, \ \frac{\partial z}{\partial n} = 0 \text{ on } \Gamma_n \right\},$$
$$B = -A_0 DM, \quad B_\zeta = -A_0 DM_\zeta, \quad F(w) = (\partial_1 w + \partial_2 w)w,$$

A_0 is nothing but A_γ for $\gamma = 0$ (see (2.3)), the operator D is the one defined in (2.4) with $w_s = 0$ and $\lambda_0 = 0$, $M_\zeta(x) = \cos(\frac{\pi}{2}x_1)$ on $\Gamma_\zeta = (0,1) \times \{1\}$ and $M_\zeta = 0$ on $\Gamma \setminus \Gamma_\zeta$. We are going to explain at the end of section 5.2 that system (5.1) may be written in the form $(4.1)_1$, and therefore that stability results stated in Theorem 4.3 apply.

The measurements are of the form (1.12) and are precisely defined by

$$\begin{aligned} y_{obs}(t) &= (y_{obs,1}(t), y_{obs,2}(t), y_{obs,3}(t)) \\ &= Hw(t) + \eta(t) \\ &= \left(\frac{1}{0.05}\int_{0.2}^{0.25} w(t), \frac{1}{0.05}\int_{0.5}^{0.55} w(t), \frac{1}{0.05}\int_{0.8}^{0.85} w(t) \right) + (\eta_1(t), \eta_2(t), \eta_3(t)) \end{aligned} \tag{5.2}$$

where η represents a measurement error. The sensor locations are drawn in red in Figure 5.1.

5.2. Spectrum of the operator $(A_\gamma, D(A_\gamma))$. We notice that $(A_\gamma, D(A_\gamma))$ is a selfadjoint operator in $L^2(\Omega)$. For all for $(k,\ell) \in \mathbb{N}^* \times \mathbb{N}^*$, the eigenvectors and the corresponding eigenvalues of $(A_\gamma, D(A_\gamma))$ are

$$\phi_{k,\ell}(x_1, x_2) = \cos\left((2k-1)\frac{\pi}{2}x_1\right)\sin(\ell\pi x_2), \quad \lambda_{k\ell} = -\nu\left(\frac{(2k-1)^2}{4} + \ell^2\right)\pi^2 + \gamma.$$

Throughout what follows, we choose

$$\nu = \frac{1}{50} \quad \text{and} \quad \gamma = \frac{2}{5} = 0.4.$$

Thus, we have one positive eigenvalue $\lambda_{1,1} = -\frac{\pi^2}{40} + \frac{2}{5} \approx 0.1532$.

PROPOSITION 5.1. *The pair (A_γ, H) is detectable.*

Proof. Since we have only one unstable eigenvalue $\lambda_{1,1}$, it is sufficient to verify that the vector

$$\left(\frac{1}{0.05}\int_{0.2}^{0.25} \phi_{1,1}, \frac{1}{0.05}\int_{0.5}^{0.55} \phi_{1,1}, \frac{1}{0.05}\int_{0.8}^{0.85} \phi_{1,1} \right),$$

with $\phi_{1,1}(x_1,x_2) = \cos\left(\frac{\pi}{2}x_1\right)\sin(\pi x_2)$, is different from $(0,0,0)$. □

Let us recall that $B = -A_0 DM \in \mathcal{L}(\mathbb{R}, (D(A_\gamma^*))')$ is the control operator corresponding to the system (5.1). We can easily verify that $B^* \in \mathcal{L}(D(A_\gamma^*), \mathbb{R})$ is defined by

$$B^*\phi = -\int_{\Gamma_c} M\,\frac{\partial \phi}{\partial n}.$$

PROPOSITION 5.2. *The pair (A_γ, B) is stabilizable.*

Proof. It is sufficient to verify that if $\phi = \lambda\,\phi_{1,1}$ and if $B^*\phi = 0$, then $\lambda = 0$. This is obvious since

$$B^*\phi = -\lambda \int_{\Gamma_c} M\,\frac{\partial \phi_{1,1}}{\partial n} = -\lambda\frac{\pi}{2}\int_0^1 \sin^2(\pi x_2)\,dx_2 = -\frac{\pi}{4}\lambda.$$

□

To take into account the model error $w(x,t) = \cos(\frac{\pi}{2}x_1)\zeta(t)$ on $\Gamma_\zeta \times (0,\infty)$, we may introduce the function

$$w_\zeta(x_1,x_2,t) = \zeta(t)\,\cos\left(\frac{\pi}{2}x_1\right)\,\sin\left(\frac{\pi}{2}x_2\right),$$

and we set $w = w_\zeta + z$. If we set $\mu = (-\zeta' - \nu\,\zeta\pi^2/2 + \gamma\,\zeta)\,\cos\left(\frac{\pi}{2}x_1\right)\,\sin\left(\frac{\pi}{2}x_2\right)$, the equation satisfied by z is

$$\begin{aligned}
&\frac{\partial z}{\partial t} - \nu\Delta z - \gamma z + (\partial_1(w_\zeta + z) + \partial_2(w_\zeta + z))(w_\zeta + z) = \mu \quad \text{in } \Omega\times(0,\infty),\\
&\nu\frac{\partial z}{\partial n} = 0 \quad \text{on } \Gamma_n\times(0,\infty), \quad z(1,x_2,t) = \sin(\pi x_2)u(t) \quad \forall (x_2,t)\in(0,1)\times(0,\infty),\\
&z = 0 \quad \text{on } (\Gamma_d\setminus\Gamma_c)\times(0,\infty),\\
&z(0) = w_0 - w_\zeta(0) + \mu_0 \quad \text{in } \Omega.
\end{aligned} \tag{5.3}$$

Thus we have a dynamical system of the form

$$z' = Az + Bu + F(w_\zeta + z) + \mu, \quad z(0) = z_0 + \mu_0,$$

as the one introduced in $(4.1)_1$. If we assume that $\zeta \in H^1(0,\infty)$, then μ belongs to $L^2(0,\infty;L^2(\Omega))$ and w_ζ belongs to $H^1(0,\infty;H^2(\Omega))$. In that case the assumptions on μ and w_ζ stated in Theorem 4.3 are satisfied for $\omega = 0$. If $e^{\omega t}\mu$ belongs to $L^2(0,\infty;L^2(\Omega))$ with $\omega > 0$, the assumptions are satisfied for $\omega > 0$.

5.3. Finite dimensional system. To compute numerically the feedback gains and the solution of compensator, we use a finite element method with P_1 basis functions on a structured triangular mesh [10]. The mesh size h is equal to $1/n$, which corresponds to a subdivision of each side of Ω into n segments. We denote by $\mathcal{T}_h$ the corresponding partition of Ω into triangles. We introduce the two following finite dimensional spaces

$$W_h = \{\psi \in C^0(\bar\Omega) \;\mid\; \psi|_K \in P_1,\ \forall K\in\mathcal{T}_h,\ \psi = 0 \text{ on } \Gamma_0\}$$

and

$$W_{0h} = \{\psi \in C^0(\bar\Omega) \;\mid\; \psi|_K \in P_1,\ \forall K\in\mathcal{T}_h,\ \psi = 0 \text{ on } \Gamma_d\}.$$

We have $N := \dim(W_{0h}) = (n-1)(n-2)$ and $\dim(W_h) = n(n-1)$. We denote by $(\varphi_j)_{j\in\mathcal{N}}$ the $n(n-1)$ basis functions of W_h, where $\mathcal{N}$ is the set of nodes of the triangulation $\mathcal{T}_h$. We look for an approximate solution $w_h \in H^1(0,\infty;W_h)$ of equation (5.1) of the form

$$w_h(x,t) = \sum_{j\in\mathcal{N}_{\Omega\cup\Gamma_n}} w_j(t)\varphi_j(x) + \sum_{j\in\mathcal{N}_{\Gamma_c}} \sin(\pi a_j)\varphi_j(x)u(t) + \sum_{j\in\mathcal{N}_{\Gamma_\zeta}} \cos\left(\frac{\pi}{2}a_j\right)\varphi_j(x)\zeta(t), \tag{5.4}$$

where $\mathcal{N}_{\Omega\cup\Gamma_n}$ is the set of nodes of $\mathcal{T}_h$ belonging to $\Omega\cup\Gamma_n$, $\mathcal{N}_{\Gamma_c}$ is the set of nodes of $\mathcal{T}_h$ belonging to Γ_c and $\mathcal{N}_{\Gamma_\zeta}$ is the set of nodes of $\mathcal{T}_h$ belonging to Γ_ζ. Thus the discrete evolution equation associated with (5.1) is

$$\begin{aligned}&\text{Find } w_h \in H^1(0,\infty;W_h) \quad \text{of the form (5.4)} \quad \text{such that}\\ &\frac{d}{dt}\int_\Omega w_h(t)\,\varphi = -\nu\int_\Omega \nabla w_h\cdot\nabla\varphi + \alpha\int_\Omega w_h\varphi - \int_\Omega\left(w_h\frac{\partial w_h}{\partial x_1} + w_h\frac{\partial w_h}{\partial x_2}\right)\varphi, \quad \text{for all } \varphi\in W_{0h},\\ &w_h(0) = w_{0h},\end{aligned} \tag{5.5}$$

where w_{0h} is the finite element approximation of w_0. By substituting w_h defined by (5.4) in (5.5), we obtain two terms involving the time derivative of the control u and the noise ζ:

$$\left(\sum_{j\in\mathcal{N}_{\Gamma_c}}\sin(\pi a_j)\int_\Omega\varphi_j\varphi\right)u'(t), \quad \left(\sum_{j\in\mathcal{N}_{\Gamma_\zeta}}\cos\left(\frac{\pi}{2}a_j\right)\int_\Omega\varphi_j\varphi\right)\zeta'(t). \tag{5.6}$$

Hence the corresponding system of ordinary differential equations is not written in the form (1.3). To overcome this difficulty, we use a mass lumping method that is well known to compute, with high order elements, the solution of the wave equation [13] or parabolic equations [5]. For the space P_1, this method consists in calculating the integral on each triangle with a trapezoidal formula:

$$\int_K\psi \approx \frac{|K|}{3}(\psi(a_1)+\psi(a_2)+\psi(a_3)), \tag{5.7}$$

where the nodes a_1, a_2, a_3 are the vertices of triangle K. It follows that the mass matrix becomes diagonal since

$$(\varphi_i,\varphi_j)_h = \delta_{ij}\frac{|K|}{3},$$

where $(\cdot,\cdot)_h$ is an approximation of the $L_2(\Omega)$ inner product. Using this integration formula, we approximate the two integrals in (5.6) by zero, because $\varphi\in W_{0h}$ and $j\in\mathcal{N}_{\Gamma_c}$ for the first integral, while $j\in\mathcal{N}_{\Gamma_\zeta}$ for the second one. Thus, system (5.5) is of the form

$$\mathbf{E}\dot{\mathbf{w}} = \mathbf{A}\mathbf{w} + \mathbf{B}u + \mathbf{B}_\zeta\zeta + \mathbf{F}(\mathbf{w}), \quad \mathbf{w}(0) = \mathbf{w}_0, \tag{5.8}$$

where $\mathbf{A}\in\mathbb{R}^{N\times N}$ is the stiffness matrix, $\mathbf{E}\in\mathbb{R}^{N\times N}$ is the diagonal mass matrix, $\mathbf{B}\in\mathbb{R}^{N\times 1}$ is the approximation of the control operator B, $\mathbf{B}_\zeta\in\mathbb{R}^{N\times 1}$ is the approximation of the operator B_ζ, $\mathbf{w}\in\mathbb{R}^N$ is the vector of coordinates of w_h in the finite element basis and $\mathbf{F}:\mathbb{R}^N\to\mathbb{R}^N$ stands for the nonlinear term in (5.1). Let us notice that using a finite difference method, we would have obtained a similar system. Substituting w by w_h in (5.2) and using the trapezoidal formula (5.7), we obtain the observation for the discrete system (5.8):

$$\mathbf{y}_{obs} = \mathbf{H}\mathbf{w} + \eta \quad \text{with } \mathbf{H}\in\mathbb{R}^{3\times N} \quad \text{and} \quad \eta = \begin{bmatrix}\eta_1 & \eta_2 & \eta_3\end{bmatrix}^T.$$

For the numerical simulations, we shall simulate the measurement error $\eta(t)$ and the model error $\zeta(t)$ with the Matlab function '*randn*' corresponding to uncorrelated zero-mean Gaussian random noises of covariance matrices $\mathbf{R}_\eta$ and $\mathbf{R}_\zeta$ respectively. We shall use these matrices to determine the filtering gain in the estimation equation.

5.4. Feedback gains. Using the separation principle, we design the LQG controller in two steps. First, we solve a classical LQR problem to find the feedback gain K. We minimize the functional J defined by

$$J(u) = \frac{1}{2}\int_0^\infty\left(\|Cw(t)\|^2_{L^2(\Omega)} + R|u(t)|^2\right)dt, \tag{5.9}$$

where $R > 0$ is a penalization parameter and $Cw(t) = z(t)$ is solution to equation

$$-\nu\Delta z(t) = w(t) \quad \text{in } \Omega, \quad \nu\frac{\partial z(t)}{\partial n} = 0 \quad \text{on } \Gamma_n, \quad z(t) = 0 \quad \text{on } \Gamma_d, \tag{5.10}$$

and w is solution of linearized system corresponding to (5.1) with $\zeta = 0$. Let us recall that the control space $U = \mathbb{R}$ is of dimension 1. The operator C is nothing else than $(-A_0)^{-1}$. Thus if we choose Q in equation (2.8) as $Q = C^*C = (-A_0)^{-2}$, we are in the case where $K = -R^{-1}B^*P$ (with P solution of equation (2.8) corresponding to $A + \omega I = A_\gamma$) satisfies the conclusions of Theorem 2.2. This is why we have chosen C in this form. Let $K \in \mathcal{L}(L^2(\Omega); \mathbb{R})$ be a feedback gain satisfying the conclusions of Theorem 2.2. From the Riesz theorem, we deduce that exists a function $k \in L^2(\Omega)$ such that

$$u(t) = Kw(t) = \int_\Omega k(x)w(x,t)\,dx.$$

This function k is the kernel of the feedback operator K. Next, the filtering operator $L \in \mathcal{L}(\mathbb{R}^3, L^2(\Omega))$ can be obtained by solving another optimal control problem. If we denote by ℓ_1, ℓ_2 and ℓ_3 the corresponding filtering gains, the compensator for system (5.1) can be written in the form

$$\begin{aligned}
&\frac{\partial w}{\partial t} - \nu\Delta w - \gamma w + (\partial_1 w + \partial_2 w)w = 0 \quad \text{in } \Omega \times (0,\infty), \quad w(0) = w_0 + \mu_0 \quad \text{in } \Omega,\\
&\nu\frac{\partial w}{\partial n} = 0 \text{ on } \Gamma_n \times (0,\infty), \quad w(x,t) = \sin(\pi x_2)Kw_e(t) \quad \forall (x,t) \in \Gamma_c \times (0,\infty),\\
&w = 0 \quad \text{on } \Gamma_0 \times (0,\infty), \quad w(x,t) = \zeta(t)\cos(\pi x_1/2) \quad \forall (x,t) \in \Gamma_\zeta \times (0,\infty),\\
&\frac{\partial w_e}{\partial t} - \nu\Delta w_e - \gamma w_e + \sum_{i=1}^{3} \ell_i(y_{e,i} - y_{obs,i}) = 0 \quad \text{in } \Omega \times (0,\infty), \quad w_e(0) = w_0 \quad \text{in } \Omega,\\
&\nu\frac{\partial w_e}{\partial n} = 0 \quad \text{on } \Gamma_n \times (0,\infty), \quad w_e(x,t) = \sin(\pi x_2)Kw_e(t) \quad \forall (x,t) \in \Gamma_c \times (0,\infty),\\
&w_e = 0 \quad \text{on } (\Gamma_0 \cup \Gamma_\zeta) \times (0,\infty).
\end{aligned} \tag{5.11}$$

For the discrete system (5.8), the feedback gain $\mathbf{K} \in \mathbb{R}^{1\times N}$ is defined by

$$\mathbf{K} = -R^{-1}\mathbf{B}^T\mathbf{P}\mathbf{E}, \tag{5.12}$$

where $\mathbf{P}$ is the solution to the following Generalized Algebraic Riccati equation (GARE in brief)

$$\begin{aligned}
&\mathbf{P} \in \mathbb{R}^{N\times N},\ \mathbf{P}^T = \mathbf{P},\ \mathbf{P} \geq 0,\\
&\mathbf{A}^T\mathbf{P}\mathbf{E} + \mathbf{E}\mathbf{P}\mathbf{A} - \mathbf{E}\mathbf{P}\mathbf{B}\,R^{-1}\,\mathbf{B}^T\mathbf{P}\mathbf{E} + \mathbf{C}^T\mathbf{E}\mathbf{C} = 0,
\end{aligned} \tag{5.13}$$

with $\mathbf{C} = \mathbf{A}_0^{-1}\mathbf{E}$ and $\mathbf{A}_0$ is the stiffness matrix corresponding to (5.10). The operator $\mathbf{C}$ is the discrete approximation of the operator C in the functional J. As the matrix $\mathbf{E}$ is diagonal and the matrix $\mathbf{A}_0$ is a band matrix, their inversion are easily performed.

The filtering gain $\mathbf{L} \in \mathbb{R}^{N\times 3}$, the discrete approximation of $L \in \mathcal{L}(L^2(\Omega), \mathbb{R}^3)$, is defined by

$$\mathbf{L} = -\mathbf{E}\mathbf{P}_e\mathbf{H}^T\mathbf{R}_\eta^{-1}, \tag{5.14}$$

where $\mathbf{P}_e$ is the solution to the following Riccati equation

$$\begin{aligned}
&\mathbf{P}_e \in \mathbb{R}^{N\times N},\ \mathbf{P}_e^T = \mathbf{P}_e,\ \mathbf{P}_e \geq 0,\\
&\mathbf{A}\mathbf{P}_e\mathbf{E} + \mathbf{E}\mathbf{P}_e\mathbf{A}^T - \mathbf{E}\mathbf{P}_e\mathbf{H}^T\mathbf{R}_\eta^{-1}\mathbf{H}\mathbf{P}_e\mathbf{E} + \mathbf{B}_\zeta R_\zeta \mathbf{B}_\zeta^T = 0,
\end{aligned} \tag{5.15}$$

where the weight matrix $\mathbf{R}_\eta > 0$ is the covariance of the noise η, and $R_\zeta > 0$ is the variance of ζ. ($\mathbf{R}_\eta > 0$ and $R_\zeta > 0$ are assumed to be known.)

The solution of the GAREs (5.13) and (5.15) are accomplished using a classical Schur decomposition of the corresponding Hamiltonian matrix [18]. The corresponding semi-discrete version of the compensator (5.11) is

$$\begin{aligned}
&\mathbf{E}\dot{\mathbf{w}} = \mathbf{A}\mathbf{w} + \mathbf{B}\mathbf{K}\mathbf{w}_e + \mathbf{F}(\mathbf{w}) + \mathbf{B}_\zeta\zeta, \quad \mathbf{w}(0) = \mathbf{w}_0 + \mu_0,\\
&\mathbf{E}\dot{\mathbf{w}}_e = \mathbf{A}\mathbf{w}_e + \mathbf{L}(\mathbf{y}_e - \mathbf{y}_{obs}) + \mathbf{B}\mathbf{K}\mathbf{w}_e, \quad \mathbf{w}_e(0) = \mathbf{w}_0,\\
&\mathbf{y}_{obs} = \mathbf{H}\mathbf{w} + \eta, \quad \mathbf{y}_e = \mathbf{H}\mathbf{w}_e,
\end{aligned} \tag{5.16}$$

where $\mathbf{w}_0$ is the vector of coordinates of w_{0h} in the FEM basis.

6. Numerical experiments. In this section, we present numerical results for the stabilization of system (5.16). To define the mesh size we set $n = 60$, which corresponds to $N = 3422$ degrees of freedom for the system (5.1). We choose the initial condition $w_0 = 0$ and μ_0 as an eigenfunction corresponding to the unstable eigenvalue of A_γ

$$\mu_0 = \delta \cos\left(\frac{\pi}{2}x_1\right) \sin\left(\pi x_2\right).$$

where $\delta > 0$ is used as a parameter to prescribe the magnitude of μ_0. The time integration is performed with the BDF scheme of order 2 [26]. The system (5.16) is solved up to $T = 100$, a choice large enough to capture the effect of both feedback and filtering operators in the nonlinear coupled system.

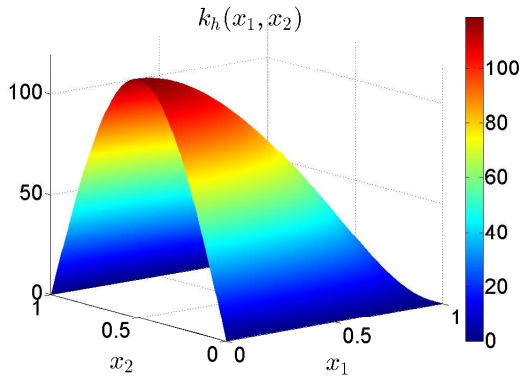


FIG. 6.1. *Finite element approximation of functional gain k for $R = 10^{-2}$.*

6.1. Computation of functional gains. To obtain a fast stabilization of the closed-loop system, we set $R = 10^{-2}$ in the cost function (5.9). Since this parameter is small, we allow large control amplitudes. From the expression of feedback gain K (5.12), we deduce the coordinates of the finite element approximation k_h of the functional gain k. In Figure 6.1, we have plotted the graph of k_h. We notice that the location of maximal control action is near of Γ_n. In fact, we see that k_h has the same behaviour as the unstable eigenvector $\phi_{1,1}$ of A_γ. Similar stabilization results will be obtained if we set $y_{out}(t) = Cw(t) = \left(\int_\Omega w(t)\phi_{1,1}\right)\phi_{1,1}$ that is the orthogonal projection of z onto the unstable eigenspace of A_γ.

The covariance matrice of measurement is $\mathbf{R}_\eta = 4 \times 10^{-6}\mathbf{I}_3$ and $R_\zeta = 4 \times 10^{-4}$. In Figure 6.2, the three kernels of L are plotted. We note that the values of ℓ_{h3} are globally greater than ℓ_{h1} and ℓ_{h2} and more particularly near of Γ_ζ. Recall that theses functions are involved in the differential equation (5.11) for the estimator w_e in the expression $\sum_{i=1}^{3} \ell_i(y_{e,i} - y_{obs,i})$. Consequently, a large difference between $y_{obs,3}(t)$ and $y_{e,3}(t)$ will have an impact on the solution of compensator larger than $(y_{e,1}(t) - y_{obs,1}(t))$ or $(y_{e,2}(t) - y_{obs,2}(t))$. We can expect than $y_{e,3}$ will converge to $y_{obs,3}$ faster than $y_{e,1}$ to $y_{obs,1}$ or $y_{e,2}$ to $y_{obs,2}$.

6.2. Numerical tests for the linearized system. First, we present the results for the integration of the linear closed-loop system associated to problem (5.16), that is, by setting $\mathbf{F} = 0$. We choose $\delta = 0.1$

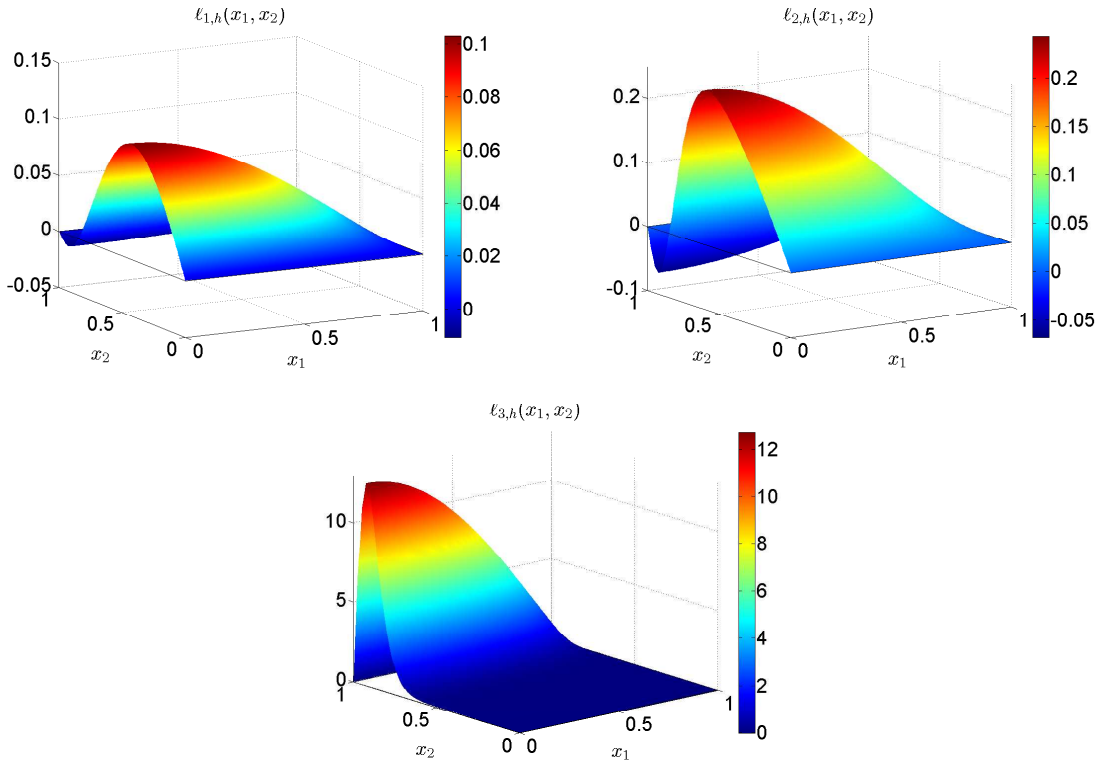


FIG. 6.2. *Finite element approximations of functional filter gains for* $R_\eta = 4\,10^{-6} I_3$ *and* $R_\zeta = 4\,10^{-4}$.

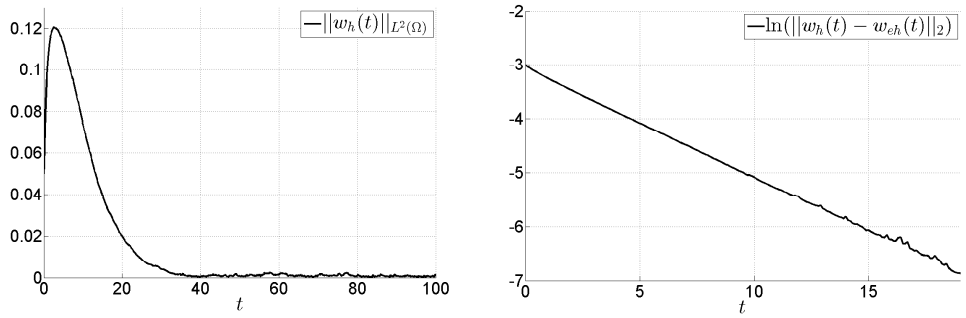


FIG. 6.3. *Evolution of* $||w_h(t)||_2$ *and* $\ln(||w_h(t) - w_{eh}(t)||_2)$ *with respect to* t *for the linear closed-loop system.*

as the magnitude parameter for the initial perturbation. For this problem, the uncontrolled solution increases exponentially making impossible a meaningful comparison with the controlled solution. On the left side of Figure 6.3 we have represented the L^2 norm of the solution $w_h(t)$ for the linear closed-loop system. We see that, after an initial increase of the norm, it decreases quickly to 0. On the right side, the logarithm of the L^2 norm of the error function $e_h = w_h - w_{eh}$ is plotted. We observe an exponential decrease of rate $-\omega = -0.2$.

In Figure 6.4, the control u with respect to time is plotted. It shows a control that starts to act strongly for initial times then oscillates around zero after $t = 40$.

In Figure 6.5 we have represented the closed-loop trajectory of w_h over the domain Ω at 4 different times. We can see that w_h is stabilized at $t = 100$ but due to the noise state the solution is not identical to zero. In order to better understand the role of filtering operator in the estimation process, we have represented in Figure 6.6 the evolution of the three measurements with respect to time. We recall that

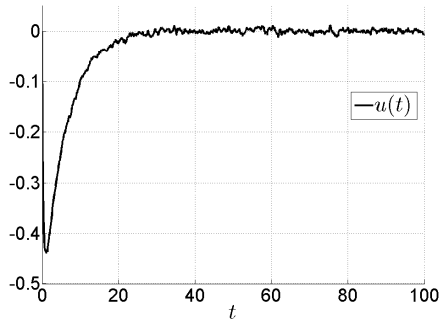


FIG. 6.4. *Evolution of control law for the linear closed-loop system.*

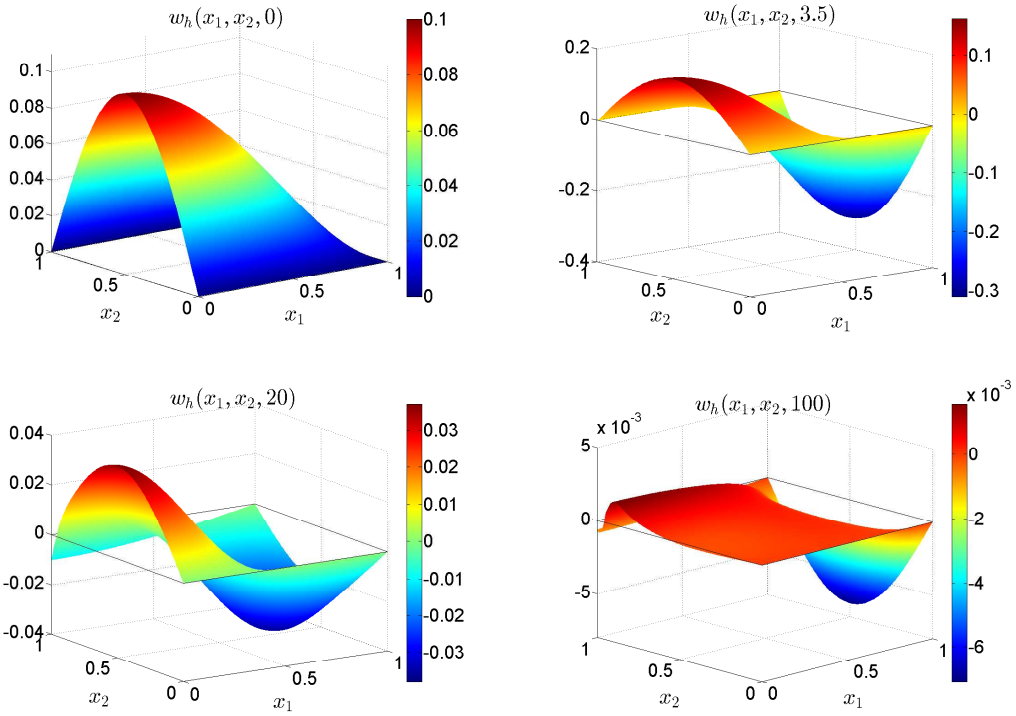


FIG. 6.5. *Evolution of the solution w_h for the linear closed-loop system* (5.16) *at time $t = 0$, $t = 3.5$, $t = 20$ and $t = 100$.*

$y_{obs,1}$, $y_{obs,2}$ and $y_{obs,3}$ correspond respectively to the noisy measurements of w on $\{0\} \times (0.2, 0.25)$, $\{0\} \times (0.5, 0.55)$ and $\{0\} \times (0.8, 0.85)$. As expected, we can see that the estimated measurements are much smoother than the noisy measurements. Due to the large gain of ℓ_3, we notice that the estimated measurement $y_{e,3}$ converges rapidly to the noisy measurement $y_{obs,3}$.

6.3. Numerical tests for the nonlinear system. Next we deal with the fully nonlinear system (5.16). Such a system evolves differently than the linear one. In fact, as we can see in Figure 6.7, the uncontrolled solutions, corresponding to different initial magnitudes δ, have a different behaviour for initial times but they coincide after time $T = 20$ around a single nonzero solution plotted on the right. We remark that although the uncontrolled solution does not blow up, stabilizing to the zero solution is still an issue. As in the linear case, we set δ equal to 0.1.

In Figure 6.8, on the left, we have represented the L^2 norm of the solution $w_h(t)$ for the nonlinear closed-loop system (5.11). As expected, the solution behaves differently than in the linear case. On

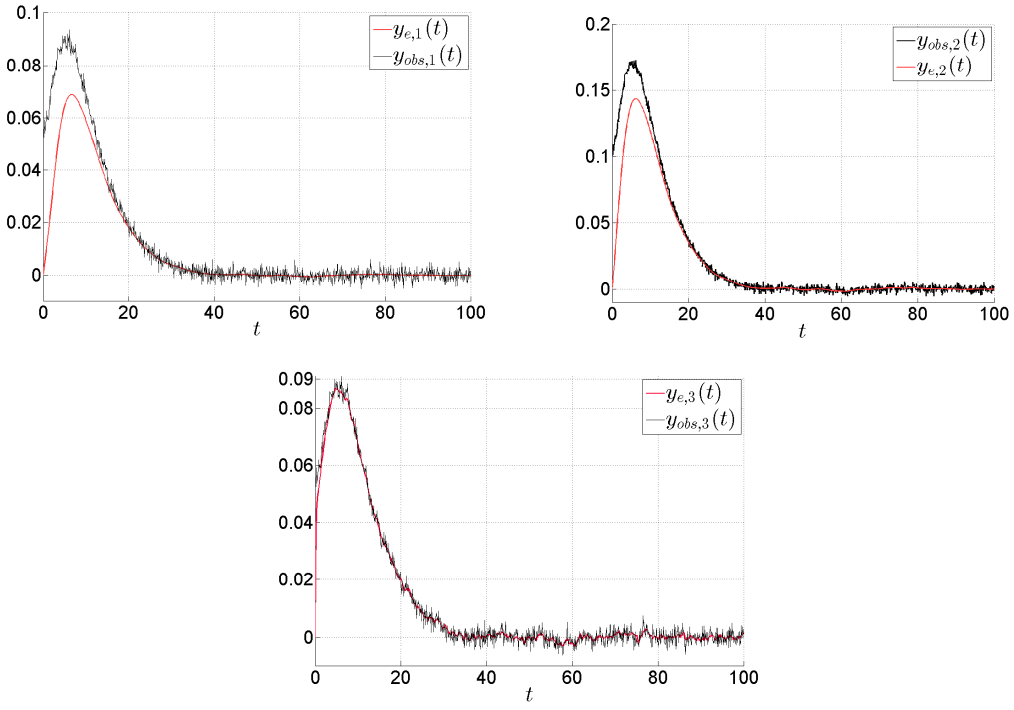


FIG. 6.6. *Evolution of the controlled and estimated measurements for the linear closed-loop system.*

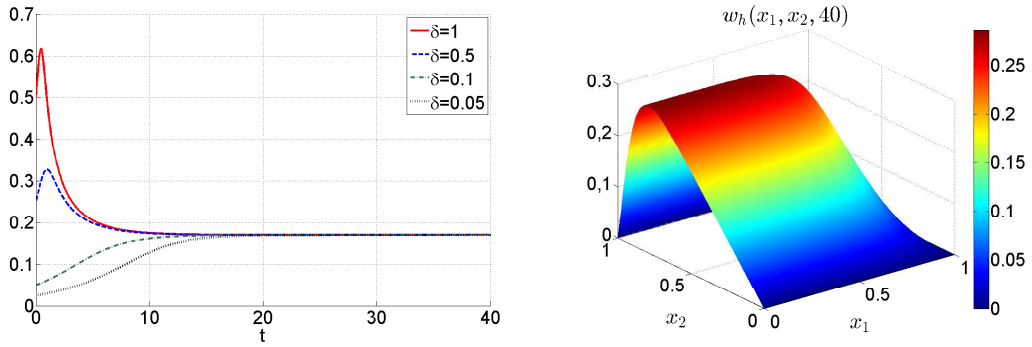


FIG. 6.7. *(Left) L^2 norm of the solution for the uncontrolled nonlinear system* (5.1) *for different initial conditions. (Right) the corresponding solution at $t = 40$.*

the right picture, we can see the L^2 norm of $e_h(t) = w_h(t) - w_{eh}(t)$. We notice that the convergence $||e_h(t)||_{L^2(\Omega)}$ to zero is very slow.

Figure 6.9 shows the evolution of the corresponding control law. Contrary to the classical behaviour of the control law for the linear closed-loop system, here we have clearly a different behaviour. The control law is able to stabilize the nonlinear system but the stabilization is much slower than in the linear case and with higher oscillations. These results illustrate how the linear LQG controller can locally stabilize the nonlinear system when only partial observations are available. In Figure 6.10, the nonlinear closed-loop solution w_h is represented for four different times. We can check that the stabilization is achieved up to small oscillations on the boundary where the control and the noise state are acting.

In order to better understand the role of the filtering on the estimation, we can look at Figure 6.11 the time evolution of the noisy and estimated observations. We can verify that, as in the linear case, the effect of the noise is highly reduced by the estimation process but the convergence of the estimated

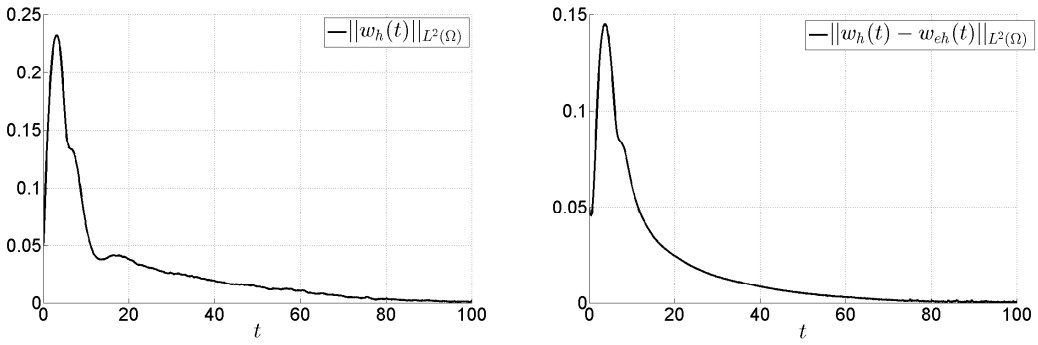


FIG. 6.8. *(Left) Evolution of the L^2 norm of the solution $w_h(t)$ for the nonlinear closed-lop system. (Right) Evolution of the L^2 norm of estimated error $w_h(t) - w_{eh}(t)$.*

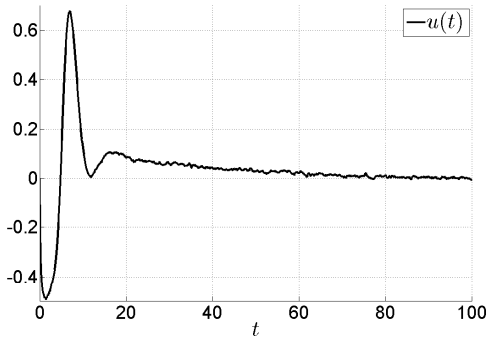


FIG. 6.9. *Evolution of the control law for the nonlinear closed-loop system* (5.16).

observation towards the noisy observation is slower than in the linear case.

Conclusion. In this paper, we studied theoretically and numerically the boundary stabilization problem of a Burger's equation in presence of measurement and model errors. A Luenberger compensator is built from the linearized equation that corresponds to the heat equation. The feedback gains are obtained by solving LQR problems. In section 2 and 3, we give some assumptions on the control and observation operators that allows us to study the well posedness of the infinite dimensional compensator. At the end of section 3, we verify on some particular examples whether these assumptions are satisfied or not. In section 4, we prove that the controlled nonlinear Burgers equation coupled with the linear estimator can be locally stabilized. In section 5, the numerical results illustrate these theoretical results: The solution estimated from noisy measurements may be used into a feedback law to stabilize efficiently the nonlinear unstable system, when the initial data and the boundary perturbation are not too large.

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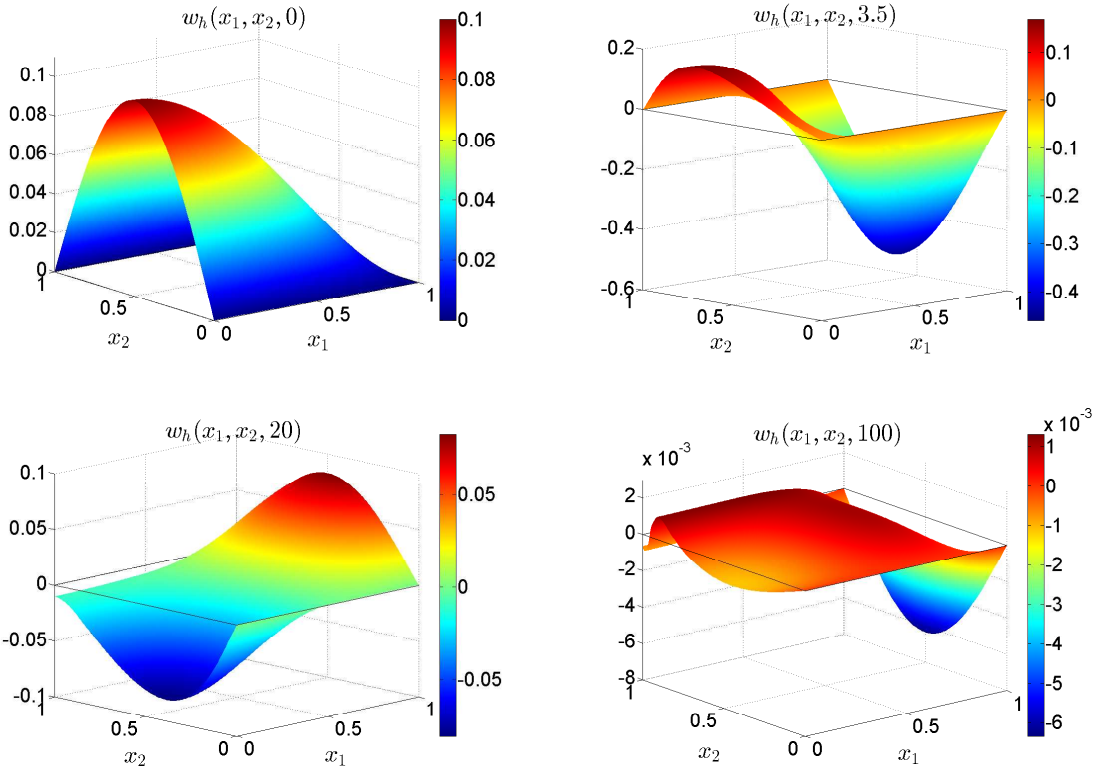


FIG. 6.10. *Evolution of the solution w_h for the nonlinear closed-loop system* (5.16) *at time $t = 0$, $t = 3.5$, $t = 20$ and $t = 100$.*

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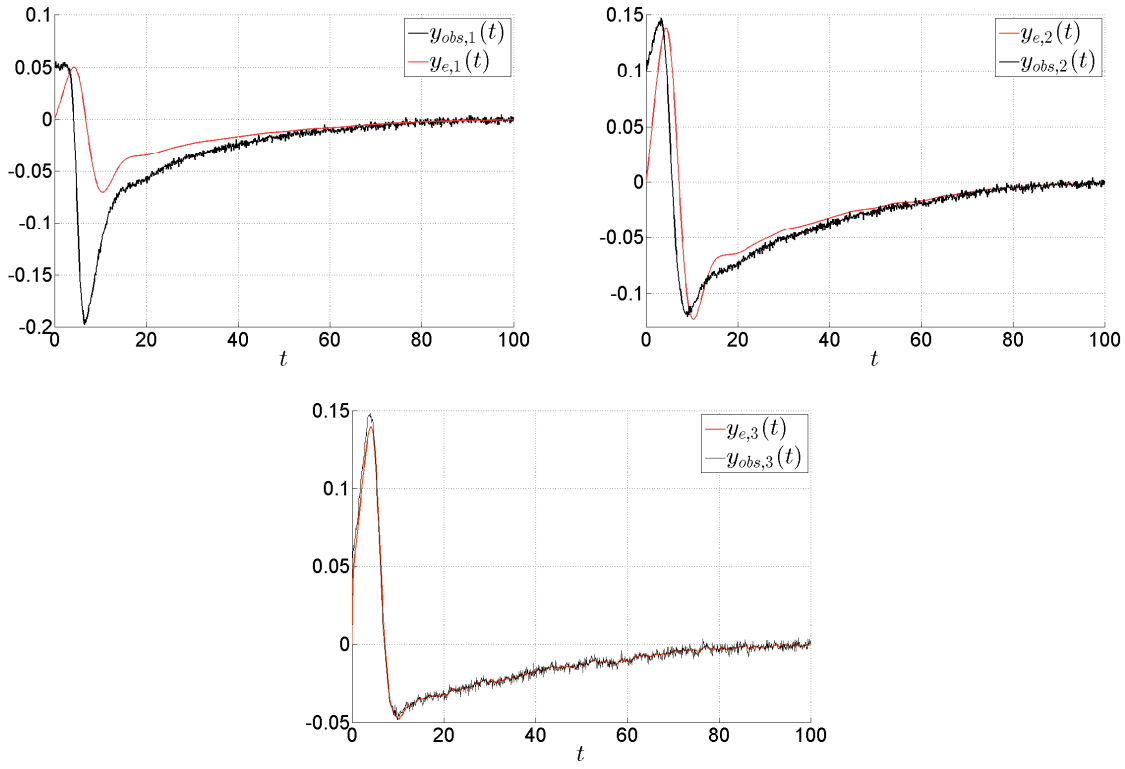


FIG. 6.11. *Evolution of the noisy and estimated observations for the nonlinear closed-loop system* (5.16).

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