A Wasserstein gradient flow approach to Poisson-Nernst-Planck equations

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Abstract

In this talk I will discuss some recent results obtained with D. Kinderlehrer (Carnegie Mellon Univ.) and X. Xiang (Purdue Univ.) for the Poisson-Nernst-Planck equations

$$t \geq 0,\ x \in \mathbb{R}d,\ d \geq 3, \qquad \begin{cases} \partial_t u = \Delta u^m + \operatorname{div}\left(u \nabla \left(U + \psi\right)\right) \\ \partial_t v = \Delta v^m + \operatorname{div}\left(v \nabla \left(V - \psi\right)\right) \\ -\Delta \psi = u - v \end{cases} \tag{PNP}$$

The unknowns $u, v \geq 0$ represent the density of some positively and negatively charged particles, U, V are prescribed confining potentials, $\Psi = (-\Delta)^{-1}$ is the self-induced electrostatic potential, and $m \geq 1$ a fixed diffusion exponent. We show that (PNP) is the gradient flow of a certain energy functional in the metric space $\left(\mathcal{P}_2(\mathbb{R}^d), \mathcal{W}_2\right)$ of Borel probability measures endowed with the quadratic Wasserstein distance $\mathcal{W}_2$. The gradient flow approach in $\left(\mathcal{P}_2(\mathbb{R}^d), \mathcal{W}_2\right)$ is closely related to the theory of optimal mass transport and has already been successfully applied for several scalar PDE's (Porous Media, Keller-Segel, thin film...) We exploit this variational structure in order to semi-discretize the system in time and construct approximate solutions by means of the DeGiorgi minimizing movement. Sending the time step $h \downarrow 0$ we retrieve global weak solutions for initial data with very low regularity. In addition to energy monotonicity we also recover some regularity and new L^p estimates. The proof deals with linear and nonlinear diffusion diffusion ($m = 1$, $m > 1$) in a unified energetic framework.