Institute of Mathematics of the Academy of Sciences of the Czech Republic

Diffusion limits in a model of radiative flow

Šárka Nečasová

Workshop on PDE's and Biomedical Applications

Basic principle of mathematical modeling

Johann von Neumann [1903-1957]

In mathematics you don't understand things. You just get used to them.

General questions

Compressible vs. incompressible

Is air compressible? Is it important?

Is the physical space bounded or unbounded?

Viscous vs. inviscid

What is turbulence?

Do extremely viscous fluids exhibit turbulent behavior?

Gaseous stars in astrophysics

The effect of coupling between the macroscopic description of the fluid and the statistical character of the motion of the massless photons.

Eulerian description of motion

Physical space

- **time** $t \in [0, \infty)$
- **position** $\vec{x} \in \Omega \subset R^3$

Leonhard Paul Euler [1707-1783]

Thermostatic variable

- **mass density** $\varrho = \varrho(t, x)$

Motion

- **macroscopic velocity** $\vec{u} = \vec{u}(t, x)$

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{X}(t, \vec{x}) = \vec{u}\Big(t, \vec{X}(t, \vec{x})\Big), \ \vec{X}(0, \vec{x}) = \vec{x}$$

Momentum balance

$$\partial_t(\varrho\vec{u}) + \mathrm{div}_x(\varrho\vec{u}\otimes\vec{u}) + \nabla_x p(\varrho) = \mathrm{div}_x\mathbb{S} + \varrho\vec{f}$$

Newton's rheological law

$$\mathbb{S} = \mu\left(\nabla_x\vec{u} + \nabla_x^t\vec{u} - \frac{2}{3}\mathrm{div}_x\vec{u}\mathbb{I}\right) + \eta\mathrm{div}_x\vec{u}\mathbb{I}$$

Mass conservation

$$\partial_t\varrho + \mathrm{div}_x(\varrho\vec{u}) = 0$$

A mathematical model of radiative flow

$$\partial_t \varrho + \mathrm{div}_x(\varrho \vec{u}) = 0 \quad \text{in } (0,T) \times \Omega, \tag{1.1}$$

$$\partial_t(\varrho \vec{u}) + \mathrm{div}_x(\varrho \vec{u} \otimes \vec{u}) + \nabla_x p(\varrho, \vartheta) = \mathrm{div}_x \mathbb{S} + \vec{S}_F \quad \text{in } (0,T) \times \Omega, \tag{1.2}$$

$$\partial_t \left(\varrho \left(\frac{1}{2}|\vec{u}|^2 + e(\varrho, \vartheta) \right) \right) + \mathrm{div}_x \left(\varrho \left(\frac{1}{2}|\vec{u}|^2 + e(\varrho, \vartheta) \right) \vec{u} \right) \tag{1.3}$$

$$+ \mathrm{div}_x \left(p\vec{u} + \vec{q} - \mathbb{S}\vec{u} \right) = -S_E \quad \text{in } (0,T) \times \Omega,$$

$$\frac{1}{c} \partial_t I + \vec{\omega} \cdot \nabla_x I = S \quad \text{in } (0,T) \times \Omega \times (0,\infty) \times \mathcal{S}^2. \tag{1.4}$$

the mass density $\varrho = \varrho(t, x)$,

the velocity field $\vec{u} = \vec{u}(t, x)$,

the temperature $\vartheta = \vartheta(t, x)$

the radiative intensity $I = I(t, x, \vec{\omega}, \nu)$, depending on the direction $\vec{\omega} \in \mathcal{S}^2$,

$\mathcal{S}^2 \subset \mathbb{R}^3$ the unit sphere,

the frequency $\nu \geq 0$.

$p = p(\varrho, \vartheta)$ the thermodynamic pressure

$e = e(\varrho, \vartheta)$ is the specific internal energy

Maxwell's equation

$$\frac{\partial e}{\partial \varrho} = \frac{1}{\varrho^2}\left(p(\varrho, \vartheta) - \vartheta \frac{\partial p}{\partial \vartheta} \right). \tag{1.5}$$

$\mathbb{S}$ is the viscous stress tensor

$$\mathbb{S} = \mu \left(\nabla_x \vec{u} + \nabla_x^t \vec{u} - \frac{2}{3} \mathrm{div}_x \vec{u} \right) + \eta \ \mathrm{div}_x \vec{u} \ \mathbb{I}, \tag{1.6}$$

the shear viscosity coefficient $\mu = \mu(\vartheta) > 0$

the bulk viscosity coefficient $\eta = \eta(\vartheta) \geq 0$ are effective functions of the temperature

$\vec{q}$ is the heat flux given by Fourier's law

$$\vec{q} = -\kappa \nabla_x \vartheta, \tag{1.7}$$

the heat conductivity coefficient $\kappa = \kappa(\vartheta) > 0$.

$$S = S_{a,e} + S_s, \tag{1.8}$$

$$S_{a,e} = \sigma_a \Big(B(\nu, \vartheta) - I \Big), \; S_s = \sigma_s \left(\tilde{I} - I \right). \tag{1.9}$$

$$S_E = \int_{\mathcal{S}^2} \int_0^\infty S(\cdot, \nu, \vec{\omega}) \, \mathrm{d}\nu \, \mathrm{d}\vec{\omega}, \; \vec{S}_F = \frac{1}{c} \int_{\mathcal{S}^2} \int_0^\infty \vec{\omega} S(\cdot, \nu, \vec{\omega}) \, \mathrm{d}\nu \, \mathrm{d}\vec{\omega}, \tag{1.10}$$

the absorption coefficient $\sigma_a = \sigma_s(\nu, \vartheta) \geq 0$,
the scattering coefficient $\sigma_s = \sigma_s(\nu, \vartheta) \geq 0$

$$\tilde{I} := \frac{1}{4\pi} \int_{\mathcal{S}^2} I(\cdot, \vec{\omega}) \, \mathrm{d}\vec{\omega}$$

$$B(\nu, \vartheta) = 2h\nu^3 c^{-2} \left(e^{\frac{h\nu}{k\vartheta}} - 1 \right)^{-1}$$

– the radiative equilibrium function

h and k are the Planck and Boltzmann constants, the boundary conditions

$$\vec{u}|_{\partial\Omega} = 0, \ \vec{q} \cdot \vec{n}|_{\partial\Omega} = 0, \tag{1.11}$$

$$I(t, x, \nu, \vec{\omega}) = 0 \text{ for } x \in \partial\Omega, \ \vec{\omega} \cdot \vec{n} \leq 0, \tag{1.12}$$

$\vec{n}$ the outer normal vector to $\partial\Omega$.

- **Pomraning**
- **Mihalas and Weibel-Mihalas** in the framework of special relativity.
- astrophysics, laser applications (in the relativistic and inviscid case) by **Lowrie, Morel and Hittinger, Buet and Després**
- with a special attention to asymptotic regimes **Dubroca and Feugeas, Lin, Lin, Coulombel and Goudon**
  a simplified version of the system (non conducting fluid at rest) - investigated by **Golse and Perthame** , where global existence was proved by means of the theory of nonlinear semi-groups under very general hypotheses.

Global weak solutions

barotropic case

P. L. Lions (98)

$p(\varrho) = \varrho^\gamma$, $\gamma \geq 9/5$

generalization to a larger class of exponents $\gamma > 3/2$
E. Feireisl, A. Novotný and H. Petzeltová

Full system - the Navier - Stokes - Fourier system
global existence of weak solution E. Feireisl
Singular limits for Navier type of boundary conditions,
Concept of weak- strong uniqueness
E. Feireisl, A. Novotný

Hypotheses

Pressure

$$p(\varrho, \vartheta) = \vartheta^{5/2} P\left(\frac{\varrho}{\vartheta^{3/2}}\right) + \frac{a}{3}\vartheta^4, \ \ a > 0, \tag{2.1}$$

$P : [0, \infty) \to [0, \infty)$

$$P \in C^1[0, \infty), \ \ P(0) = 0, \ \ P'(Z) > 0 \text{ for all } Z \geq 0, \tag{2.2}$$

$$0 < \frac{\frac{5}{3}P(Z) - P'(Z)Z}{Z} < c \text{ for all } Z \geq 0, \tag{2.3}$$

$$\lim_{Z \to \infty} \frac{P(Z)}{Z^{5/3}} = p_\infty > 0. \tag{2.4}$$

$\frac{a}{3}\vartheta^4$- “equilibrium” radiation pressure.

the specific internal energy e is

$$e(\varrho,\vartheta)=\frac{3}{2}\vartheta\left(\frac{\vartheta^{3/2}}{\varrho}\right)P\left(\frac{\varrho}{\vartheta^{3/2}}\right)+a\frac{\vartheta^4}{\varrho}, \tag{2.5}$$

the associated specific entropy reads

$$s(\varrho,\vartheta)=M\left(\frac{\varrho}{\vartheta^{3/2}}\right)+\frac{4a}{3}\frac{\vartheta^3}{\varrho}, \tag{2.6}$$

with

$$M'(Z)=-\frac{3}{2}\frac{\frac{5}{3}P(Z)-P'(Z)Z}{Z^2}<0.$$

$$0 < c_1(1+\vartheta) \le \mu(\vartheta),\ \ \mu'(\vartheta) < c_2,\ \ 0 \le \eta(\vartheta) \le c(1+\vartheta), \tag{2.7}$$

$$0 < c_1(1+\vartheta^3) \le \kappa(\vartheta) \le c_2(1+\vartheta^3) \tag{2.8}$$

for any $\vartheta \ge 0$.

$$0 \le \sigma_a(\nu,\vartheta), \sigma_s(\nu,\vartheta), |\partial_\vartheta \sigma_a(\nu,\vartheta)|, |\partial_\vartheta \sigma_s(\nu,\vartheta)| \le c_1, \tag{2.9}$$

$$0 \le \sigma_a(\nu,\vartheta)B(\nu,\vartheta), |\partial_\vartheta\{\sigma_a(\nu,\vartheta)B(\nu,\vartheta)\}| \le c_2, \tag{2.10}$$

$$\sigma_a(\nu,\vartheta), \sigma_s(\nu,\vartheta), \sigma_a(\nu,\vartheta)B(\nu,\vartheta) \le h(\nu),\ \ h \in L^1(0,\infty). \tag{2.11}$$

for all $\nu \ge 0$, $\vartheta \ge 0$, where $c_{1,2,3}$ are positive constants. **Relations (2.9) - (2.11) represent "cut-off" hypotheses neglecting the effect of radiation at large frequencies ν**

Weak formulation: *(weak) renormalized version represented by the family of integral identities*

$$\int_0^T \int_\Omega \left(b(\varrho)\partial_t\varphi + b(\varrho)\vec{u}\cdot\nabla_x\varphi\right)\ \mathrm{d}x\ \mathrm{d}t \tag{2.12}$$

$$+\int_0^T \int_\Omega \left(\left(b(\varrho) - b'(\varrho)\varrho\right)\mathrm{div}_x\vec{u}\varphi\right)\ \mathrm{d}x\mathrm{d}t$$

$$= -\int_\Omega b(\varrho_0)\varphi(0,\cdot)\ \mathrm{d}x$$

satisfied for any $\varphi \in C_c^\infty([0,T)\times\overline{\Omega})$, and any $b \in C^\infty[0,\infty)$, $b' \in C_c^\infty[0,\infty)$, where (2.12) implicitly includes the initial condition $\varrho(0,\cdot) = \varrho_0$.

The momentum equation (1.2) is replaced by

$$\int_0^T \int_\Omega \left(\varrho \vec{u} \cdot \partial_t \varphi + \varrho \vec{u} \otimes \vec{u} : \nabla_x \varphi + p \operatorname{div}_x \varphi \right) \, \mathrm{d}x \, \mathrm{d}t \tag{2.13}$$

$$= \int_0^T \int_\Omega \mathbb{S} : \nabla_x \varphi + \vec{S}_F \cdot \varphi \, \mathrm{d}x \, \mathrm{d}t - \int_\Omega (\varrho \vec{u})_0 \cdot \varphi(0, \cdot) \, \mathrm{d}x$$

for any $\varphi \in C_c^\infty([0,T) \times \Omega; \mathbb{R}^3)$.

$$\vec{u} \in L^2(0, T; W_0^{1,2}(\Omega; \mathbb{R}^3)), \tag{2.14}$$

where (2.14) already includes the no-slip boundary condition (1.11).

the entropy equation is replaced by the inequality

$$\int_0^T \int_\Omega \left(\varrho s \partial_t \varphi + \varrho \vec{u} \cdot \nabla_x \varphi + \vec{q} \vartheta \cdot \nabla_x \varphi \right) \, \mathrm{d}x \, \mathrm{d}t \tag{2.15}$$

$$\leq - \int_\Omega (\varrho s)_0 \varphi(0, \cdot) \, \mathrm{d}x$$

$$- \int_0^T \int_\Omega \frac{1}{\vartheta} \left(\mathbb{S} : \nabla_x \vec{u} - \frac{\vec{q} \cdot \nabla_x \vartheta}{\vartheta} \right) \varphi \, \mathrm{d}x \, \mathrm{d}t$$

$$- \int_0^T \int_\Omega \frac{1}{\vartheta} \left(\vec{u} \cdot \vec{S}_F - S_E \right) \varphi \, \mathrm{d}x \, \mathrm{d}t$$

for any $\varphi \in C_c^\infty([0, T) \times \overline{\Omega})$, $\varphi \geq 0$.
where not the sign of all the terms in the right hand side may be controlled.

the total energy balance

$$\int_\Omega \left(\frac{1}{2}\varrho|\vec{u}|^2 + \varrho e(\varrho,\vartheta) + E^R \right)(\tau,\cdot)\ \mathrm{d}x \tag{2.16}$$

$$+ \int_0^\tau \int \int_{\partial\Omega\times\mathcal{S}^2,\ \vec{\omega}\cdot\vec{n}\geq 0} \int_0^\infty \vec{\omega}\cdot\vec{n} I(t,x,\vec{\omega},\nu)\ \mathrm{d}\nu\ \mathrm{d}\vec{\omega}\ \mathrm{d}S_x\ \mathrm{d}t$$

$$= \int_\Omega \left(\frac{1}{2\varrho_0}|(\varrho\vec{u})_0|^2 + (\varrho e)_0 + E_{R,0} \right)\ \mathrm{d}x,$$

where E^R is given by

$$E^R(t,x) = \frac{1}{c}\int_{\mathcal{S}^2}\int_0^\infty I(t,x,\vec{\omega},\nu)\ \mathrm{d}\vec{\omega}\ \mathrm{d}\nu. \tag{2.17}$$

and $E_{R,0} = \int_{\mathcal{S}^2}\int_0^\infty I_0(\cdot,\vec{\omega},\nu)\ \mathrm{d}\vec{\omega}\ \mathrm{d}\nu.$

Definition 2.1 *We say that* $\varrho, \vec{u}, \vartheta, I$ *is a weak solution of problem (1.1 - 1.12) if*

$$\varrho \geq 0,\ \vartheta > 0 \text{ for a.a. } (t,x) \times \Omega,\ I \geq 0 \text{ a.a. in } (0,T) \times \Omega \times \mathcal{S}^2 \times (0,\infty),$$

$$\varrho \in L^\infty(0,T; L^{5/3}(\Omega)),\ \vartheta \in L^\infty(0,T; L^4(\Omega)),$$

$$\vec{u} \in L^2(0,T; W_0^{1,2}(\Omega;\mathbb{R}^3)),\ \vartheta \in L^2(0,T; W^{1,2}(\Omega)),$$

$$I \in L^\infty((0,T) \times \Omega \times \mathcal{S}^2 \times (0,\infty)),\ I(t,\cdot) \in L^\infty(0,T; L^1(\Omega \times \mathcal{S}^2 \times (0,\infty)),$$

and if ϱ, $\vec{u}$, ϑ, I *satisfy the integral identities (2.12), (2.13), (2.15), (2.16), together with the transport equation (1.4).*

Theorem

(B. Ducomet, E.Feireisl,Š. Nečasová) Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain. Assume that the thermodynamic functions p, e, s satisfy hypotheses (2.1 - 2.6), and that the transport coefficients μ, λ, κ, σ_a, and σ_s comply with (2.7 - 2.11).
Let $\{\varrho_\varepsilon, \vec{u}_\varepsilon, \vartheta_\varepsilon, I_\varepsilon\}_{\varepsilon>0}$ be a family of weak solutions to problem (1.1 - 1.12) in the sense of Definition 2.1 such that

$$\varrho_\varepsilon(0,\cdot) \equiv \varrho_{\varepsilon,0} \to \varrho_0 \text{ in } L^{5/3}(\Omega), \tag{2.18}$$

$$\int_\Omega \left(\frac{1}{2}\varrho_\varepsilon |\vec{u}_\varepsilon|^2 + \varrho_\varepsilon e(\varrho_\varepsilon, \vartheta_\varepsilon) + E_{R,\varepsilon} \right)(0,\cdot)\, \mathrm{d}x \tag{2.19}$$

$$\equiv \int_\Omega \left(\frac{1}{2\varrho_{0,\varepsilon}} |(\varrho\vec{u})_{0,\varepsilon}|^2 + (\varrho e)_{0,\varepsilon} + E_{R,0,\varepsilon} \right) \mathrm{d}x \leq E_0,$$

Then

$$\varrho_\varepsilon \to \varrho \text{ in } C_{\mathrm{weak}}([0,T]; L^{5/3}(\Omega)),$$

$$\vec{u}_\varepsilon \to \vec{u} \text{ weakly in } L^2(0,T; W_0^{1,2}(\Omega;\mathbb{R}^3)),$$

$$\vartheta_\varepsilon \to \vartheta \text{ weakly in } L^2(0,T; W^{1,2}(\Omega)),$$

and

$$I_\varepsilon \to I \text{ weakly-(*) in } L^\infty((0,T)\times\Omega\times\mathcal{S}^2\times(0,\infty)),$$

at least for suitable subsequences, where $\{\varrho, \vec{u}, \vartheta, I\}$ is a weak solution of problem (1.1 - 1.12).

Asymptotic analysis

Scaled equations

Scaling

$$X \approx \frac{X}{X_{\mathrm{char}}}$$

Mass conservation

$$[\mathrm{Sr}]\partial_t \varrho + \mathrm{div}_x(\varrho \vec{u}) = 0$$

Momentum balance

$$[\mathrm{Sr}]\partial_t(\varrho \vec{u}) + \mathrm{div}_x(\varrho \vec{u} \otimes \vec{u}) + \frac{1}{[\mathrm{Ro}]}\varrho\omega \times \vec{u} + \left[\frac{1}{\mathrm{Ma}^2}\right] \nabla_x p(\varrho)$$

$$= \left[\frac{1}{\mathrm{Re}}\right] (\Delta_x \vec{u} + \lambda \nabla_x \mathrm{div}_x \vec{u}) + (\text{external forces})$$

Characteristic numbers - Strouhal number

Čeněk Strouhal [1850-1922]

Strouhal number

$$[\mathrm{Sr}] = \frac{\mathrm{length}_{\mathrm{char}}}{\mathrm{time}_{\mathrm{char}}\mathrm{velocity}_{\mathrm{char}}}$$

Scaling by means of Strouhal number is used in the study of the long-time behavior of the fluid system, where the characteristic time is large

Mach number

Ernst Mach [1838-1916]

Mach number

$$[\mathrm{Ma}] = \frac{\mathrm{velocity}_{\mathrm{char}}}{\sqrt{\mathrm{pressure}_{\mathrm{char}}/\mathrm{density}_{\mathrm{char}}}}$$

Mach number is the ratio of the characteristic speed to the speed of sound in the fluid. Low Mach number limit, where, formally, the speed of sound is becoming infinite, characterizes incompressibility

Reynolds number

Osborne Reynolds [1842-1912]

Reynolds number

$$[\mathrm{Re}] = \frac{\mathrm{density}_{\mathrm{char}}\mathrm{velocity}_{\mathrm{char}}\mathrm{length}_{\mathrm{char}}}{\mathrm{viscosity}_{\mathrm{char}}}$$

High Reynolds number is attributed to turbulent flows, where the viscosity of the fluid is negligible

Rossby number

Carl Gustav Rossby [1898-1957]

Rossby number

$$[\mathrm{Ro}] = \frac{\mathrm{velocity}_{\mathrm{char}}}{\omega_{\mathrm{char}} \mathrm{length}_{\mathrm{char}}}$$

Rossby number characterizes the speed of rotation of the fluid

1.Simplified model $\vec{S}_F = 0$

Derivation of the entropy inequality

The internal energy equation

$$\partial_t(\varrho e) + \mathrm{div}_x(\varrho e \vec{u}) + \mathrm{div}_x \vec{q} = \mathbb{S} : \nabla_x \vec{u} - p\,\mathrm{div}_x \vec{u} - S_E. \quad (2.20)$$

the entropy equation

$$\partial_t(\varrho s) + \mathrm{div}_x(\varrho s \vec{u}) + \mathrm{div}_x\left(\frac{\vec{q}}{\vartheta}\right) = \varsigma, \quad (2.21)$$

where

$$\varsigma = \frac{1}{\vartheta}\left(\mathbb{S} : \nabla_x \vec{u} - \frac{\vec{q}\cdot\nabla_x \vartheta}{\vartheta}\right) - \frac{S_E}{\vartheta}, \quad (2.22)$$

$\varsigma_m := \frac{1}{\vartheta}\left(\mathbb{S} : \nabla_x \vec{u} - \frac{\vec{q}\cdot\nabla_x\vartheta}{\vartheta}\right)$ is the (positive) matter entropy production.

The formula for the entropy of a photon gas

$$s^R = -\frac{2k}{c^3}\int_0^\infty \int_{\mathcal{S}^2} \nu^2 \left[n \log n - (n+1)\log(n+1)\right] d\vec{\omega} d\nu, \quad (2.23)$$

$n = n(I) = \frac{c^2 I}{2h\nu^3}$ is the occupation number.
Defining the radiative entropy flux

$$\vec{q}^R = -\frac{2k}{c^2}\int_0^\infty \int_{\mathcal{S}^2} \nu^2 \left[n \log n - (n+1)\log(n+1)\right] \vec{\omega}\ d\vec{\omega} d\nu, \quad (2.24)$$

$$\partial_t s^R + \mathrm{div}_x \vec{q}^R = -\frac{k}{h}\int_0^\infty \int_{\mathcal{S}^2} \frac{1}{\nu} \log \frac{n}{n+1}\ S\ d\vec{\omega} d\nu =: \varsigma^R. \quad (2.25)$$

$$\partial_t \left(\varrho s + s^R \right) + \mathrm{div}_x \left(\varrho s \vec{u} + \vec{q}^R \right) + \mathrm{div}_x \left(\frac{\vec{q}}{\vartheta} \right) = \varsigma + \varsigma^R. \quad (2.26)$$

The right-hand side of (2.65) rewrites

$$\varsigma^R =:$$

$$\frac{S_E}{\vartheta} - \frac{k}{h} \int_0^\infty \int_{\mathcal{S}^2} \frac{1}{\nu} \left[\log \frac{n(I)}{n(I)+1} - \log \frac{n(B)}{n(B)+1} \right] \sigma_a(B - I) \; d\vec{\omega} d\nu$$

$$- \frac{k}{h} \int_0^\infty \int_{\mathcal{S}^2} \frac{1}{\nu} \left[\log \frac{n(I)}{n(I)+1} - \log \frac{n(\tilde{I})}{n(\tilde{I})+1} \right] \sigma_s(\tilde{I} - I) \; d\vec{\omega} d\nu,$$

Rescaling

We denote by

$$Sr := \frac{L_{ref}}{T_{ref}\, U_{ref}},\ \ Ma := \frac{U_{ref}}{\sqrt{\rho_{ref}\, p_{ref}}},\ \ Re := \frac{U_{ref}\, \rho_{ref}\, L_{ref}}{\mu_{ref}},$$

$$Pe := \frac{U_{ref}\, p_{ref}\, L_{ref}}{\vartheta_{ref}\, \kappa_{ref}},\ \ \mathcal{C} := \frac{c}{U_{ref}},$$

the Strouhal, Mach, Reynolds, Péclet (dimensionless) and "infrarelativistic" numbers corresponding to hydrodynamics, and by

$$\mathcal{L} := L_{ref}\, \sigma_{a,ref},\ \ \mathcal{L}_s := \frac{\sigma_{s,ref}}{\sigma_{a,ref}},\ \ \mathcal{P} := \frac{2k_B^4 \vartheta_{ref}^4}{h^3 c^3\ \ \rho_{ref}\, e_{ref}},$$

various dimensionless numbers corresponding to radiation.

$$\frac{Sr}{\mathcal{C}}\partial_t I + \vec{\omega}\cdot\nabla_x I = s = \mathcal{L}\sigma_a\left(B - I\right) + \mathcal{L}\mathcal{L}_s\sigma_s\left(\frac{1}{4\pi}\int_{\mathcal{S}^2} I \,\mathrm{d}\vec{\omega} - I\right), \tag{2.27}$$

The continuity equation is now

$$Sr\ \partial_t \varrho + \mathrm{div}_x(\varrho \vec{u}) = 0, \tag{2.28}$$

and the momentum equation

$$Sr\ \partial_t(\varrho \vec{u}) + \mathrm{div}_x(\varrho \vec{u} \otimes \vec{u}) + \frac{1}{Ma^2} \nabla_x p(\varrho, \vartheta) - \frac{1}{Re} \mathrm{div}_x \mathbb{T} = 0. \tag{2.29}$$

The balance of matter (fluid) entropy

$$Sr\ \partial_t (\varrho s) + \mathrm{div}_x \left(\varrho s \vec{u} + \mathcal{P} \vec{q}^R \right) + \frac{1}{Pe}\ \mathrm{div}_x \left(\frac{\vec{q}}{\vartheta} \right) = \varsigma, \qquad (2.30)$$

with

$$\varsigma = \frac{1}{\vartheta} \left(\frac{Ma^2}{Re} \mathbb{S} : \nabla_x \vec{u} - \frac{1}{Pe} \frac{\vec{q} \cdot \nabla_x \vartheta}{\vartheta} \right) - \frac{S_E}{\vartheta},$$

The balance of radiative entropy

$$\frac{Sr}{\mathcal{C}}\,\partial_t s^R + \mathrm{div}_x \vec{q}^R = \varsigma^R, \tag{2.31}$$

with

$$\varsigma^R = \mathcal{L}\int_0^\infty \int_{\mathcal{S}^2} \frac{1}{\nu}\left[\log\frac{n(I)}{n(I)+1} - \log\frac{n(B)}{n(B)+1}\right]\sigma_a(I-B)\; d\vec{\omega}\, d\nu$$

$$+\mathcal{L}\mathcal{L}_s\int_0^\infty \int_{\mathcal{S}^2} \frac{1}{\nu}\left[\log\frac{n(I)}{n(I)+1} - \log\frac{n(\tilde{I})}{n(\tilde{I})+1}\right]\sigma_s(I-\tilde{I})\; d\vec{\omega}\, d\nu + \frac{S_E}{\vartheta}.$$

The scaled equation for total energy gives finally the total energy balance

$$Sr\,\frac{d}{dt}\int_{\Omega}\left(\frac{Ma^2}{2}\,\varrho|\vec{u}|^2+\varrho e+\frac{1}{\mathcal{C}}\,E^R\right)\,dx+ \tag{2.32}$$

$$+\mathcal{P}\int_0^{\infty}\int_{\Gamma_+}\vec{\omega}\cdot\vec{n}I\;d\Gamma_+ d\nu=0.$$

Diffusion limit

equilibrium diffusion regime $\mathcal{P} = O(\varepsilon)$- a small amount of radiation is present

$\mathcal{C} = O(\varepsilon^{-1})$ -the flow is strongly under-relativistic

$Ma = Sr = Pe = Re = 1,\ \mathcal{P} = \varepsilon,\ \mathcal{C} = \varepsilon^{-1},\ \mathcal{L}_s = \varepsilon^2$ and $\mathcal{L} = \varepsilon^{-1}$,

$$\varepsilon\ \partial_t I + \vec{\omega} \cdot \nabla_x I = \frac{1}{\varepsilon}\sigma_a (B - I) + \varepsilon\sigma_s \left(\frac{1}{4\pi} \int_{\mathcal{S}^2} I \ \mathrm{d}\vec{\omega} - I \right), \quad (2.33)$$

$$\partial_t \varrho + \mathrm{div}_x(\varrho\vec{u}) = 0, \quad (2.34)$$

$$\partial_t(\varrho\vec{u}) + \mathrm{div}_x(\varrho\vec{u} \otimes \vec{u}) + \nabla_x p(\varrho, \vartheta) - \ \mathrm{div}_x \mathbb{T} = 0. \quad (2.35)$$

$$\partial_t \left(\varrho s + \varepsilon s_R\right) + \operatorname{div}_x \left(\varrho \vec{u} s + \vec{q}_R\right) + \operatorname{div}_x \left(\frac{\vec{q}}{\vartheta}\right) \geq \frac{1}{\vartheta}\left(\mathbb{S} : \nabla_x \vec{u} - \frac{\vec{q}\cdot\nabla_x\vartheta}{\vartheta}\right)$$

$$+\frac{1}{\varepsilon}\int_0^\infty \int_{\mathcal{S}^2} \frac{1}{\nu}\left[\log\frac{n(I)}{n(I)+1} - \log\frac{n(B)}{n(B)+1}\right]\sigma_a(I-B)\ d\vec{\omega} d\nu$$

$$+\,\varepsilon \int_0^\infty \int_{\mathcal{S}^2} \frac{1}{\nu}\left[\log\frac{n(I)}{n(I)+1} - \log\frac{n(\tilde{I})}{n(\tilde{I})+1}\right]\sigma_s(I-\tilde{I})\ d\vec{\omega} d\nu, \tag{2.36}$$

$$\frac{d}{dt}\int_\Omega \left(\frac{1}{2}\ \varrho|\vec{u}|^2 + \varrho e + E_R\right)\ dx + \frac{1}{\varepsilon}\int_0^\infty \int_{\Gamma_+} \vec{\omega}\cdot\vec{n}\ I\ d\Gamma_+ d\nu = 0. \tag{2.37}$$

the " non-equilibrium diffusion regime"
$Ma = Sr = Pe = Re = 1,\ \mathcal{P} = \varepsilon,\ \mathcal{C} = \varepsilon^{-1},\ \mathcal{L} = \varepsilon^2$ and $\mathcal{L}_s = \varepsilon^{-1}$.

$$\varepsilon\ \partial_t I + \vec{\omega} \cdot \nabla_x I = \varepsilon \sigma_a (B - I) + \frac{1}{\varepsilon} \sigma_s \left(\frac{1}{4\pi} \int_{\mathcal{S}^2} I \ \mathrm{d}\vec{\omega} - I \right), \quad (2.38)$$

$$\partial_t \varrho + \mathrm{div}_x (\varrho \vec{u}) = 0, \quad (2.39)$$

$$\partial_t (\varrho \vec{u}) + \mathrm{div}_x (\varrho \vec{u} \otimes \vec{u}) + \nabla_x p(\varrho, \vartheta) - \ \mathrm{div}_x \mathbb{T} = 0. \quad (2.40)$$

$$\partial_t (\varrho s + \varepsilon s_R) + \mathrm{div}_x (\varrho \vec{u} s + \vec{q}_R) + \mathrm{div}_x \left(\frac{\vec{q}}{\vartheta} \right) \geq \frac{1}{\vartheta} \left(\mathbb{S} : \nabla_x \vec{u} - \frac{\vec{q} \cdot \nabla_x \vartheta}{\vartheta} \right)$$

$$+ \varepsilon \int_0^\infty \int_{\mathcal{S}^2} \frac{1}{\nu} \left[\log \frac{n(I)}{n(I)+1} - \log \frac{n(B)}{n(B)+1} \right] \sigma_a (I - B) \, d\vec{\omega} d\nu$$

$$+ \frac{1}{\varepsilon} \int_0^\infty \int_{\mathcal{S}^2} \frac{1}{\nu} \left[\log \frac{n(I)}{n(I)+1} - \log \frac{n(\tilde{I})}{n(\tilde{I})+1} \right] \sigma_s (I - \tilde{I}) \, d\vec{\omega} d\nu. \tag{2.41}$$

The limit system–equilibrium system

$$\partial_t \varrho + \mathrm{div}_x(\varrho \vec{u}) = 0, \tag{2.42}$$

$$\partial_t(\varrho \vec{u}) + \mathrm{div}_x(\varrho \vec{u} \otimes \vec{u}) = \mathrm{div}_x \mathbb{T}(\varrho, \vartheta), \tag{2.43}$$

$$\begin{aligned} \partial_t \left(\varrho \mathcal{E}(\varrho, \vartheta)\right) + \mathrm{div}_x \left(\varrho e(\varrho, \vartheta)\vec{u}\right) + \mathrm{div}_x \left(\mathcal{K}(\varrho, \vartheta)\nabla_x \vartheta\right) \\ = \mathbb{S}(\varrho, \vartheta) : \nabla_x \vec{u} - p(\varrho, \vartheta)\mathrm{div}_x \vec{u}, \end{aligned} \tag{2.44}$$

$$I = B(\nu, \vartheta). \tag{2.45}$$

$$\vec{u}|_{\partial\Omega} = 0, \ \ \nabla\vartheta \cdot \vec{n}|_{\partial\Omega} = 0, \tag{2.46}$$

$$\left(\varrho(x,t), \ \vec{u}(x,t), \ \vartheta(x,t)\right)|_{t=0} = \left(\varrho^0(x), \ \vec{u}^0(x), \ \vartheta^0(x)\right), \tag{2.47}$$

with the following compatibility conditions

$$\vec{u}^0(x)|_{\partial\Omega} = 0, \ \ \nabla\vartheta^0 \cdot \vec{n}|_{\partial\Omega} = 0. \tag{2.48}$$

$$\mathcal{E}(\varrho,\vartheta) = e(\varrho,\vartheta) + \frac{B(\vartheta)}{\varrho}, \text{ and } \mathcal{K}(\vartheta) = \kappa(\vartheta) - \frac{1}{3\sigma_a(\vartheta)}\,\partial_\vartheta B(\vartheta).$$

- existence of a global solution for the small data - Matsumura and Nishida

Remark: When one considers the formal "nonconducting at rest" situation where $\kappa = 0$ and $\vec{u} = 0$ and in the no-scattering case ($\sigma_s \equiv 0$), one obtains the simplified system introduced by Bardos, Golse and Perthame for which they proved global existence and diffusion limit (called "Rosseland approximation") under assumptions much more general than ours.

The non-equilibrium diffusion regime

$$\partial_t \varrho + \mathrm{div}_x(\varrho \vec{u}) = 0, \tag{2.49}$$

$$\partial_t(\varrho \vec{u}) + \mathrm{div}_x(\varrho \vec{u} \otimes \vec{u}) = \mathrm{div}_x \mathbb{T}(\varrho, \vartheta), \tag{2.50}$$

$$\partial_t \left(\varrho e(\varrho, \vartheta)\right) + \mathrm{div}_x \left(\varrho e(\varrho, \vartheta) \vec{u}\right) + \mathrm{div}_x \left(\kappa(\vartheta) \nabla_x \vartheta\right)$$

$$= \mathbb{S}(\varrho, \vartheta) : \nabla_x \vec{u} - p(\varrho, \vartheta) \mathrm{div}_x \vec{u} - \sigma_a(\vartheta)[B(\vartheta) - N], \tag{2.51}$$

$$\partial_t N - \frac{1}{3} \mathrm{div}_x \left(\frac{1}{\sigma_s(\vartheta)} \nabla_x N \right) = \sigma_a(\vartheta) \left(B(\vartheta) - N\right). \tag{2.52}$$

$$\vec{u}|_{\partial\Omega} = 0, \ \ \nabla \vartheta \cdot \vec{n}|_{\partial\Omega} = 0, \tag{2.53}$$

$$N := \int_0^\infty I_0 \, d\nu$$

$$N|_{\partial\Omega} = 0. \tag{2.54}$$

$$\left(\varrho(x,t),\ \vec{u}(x,t),\ \vartheta(x,t), N(x,t)\right)\Big|_{t=0} = \left(\varrho^0(x),\ \vec{u}^0(x),\ \vartheta^0(x), N^0(x)\right), \tag{2.55}$$

and the compatibility conditions

$$\vec{u}^2|_{\partial\Omega} = 0,\ \nabla\vartheta^0 \cdot \vec{n}|_{\partial\Omega} = 0,\ \ N^0\Big|_{\partial\Omega} = 0. \tag{2.56}$$

$N^0(x) = \int_0^\infty \int_{\mathcal{S}^2} I^0(x,\nu,\vec{\omega})\ d\vec{\omega}\ d\nu.$

- Strong solution of the limit system for small data

Given three numbers $\overline{\varrho} \in \mathbb{R}_+$, $\overline{\vartheta} \in \mathbb{R}_+$ and $\overline{E} \in \mathbb{R}_+$ we define $\mathcal{O}^H_{ess}$ the set of hydrodynamical essential values

$$\mathcal{O}^H_{ess} := \left\{ (\varrho, \vartheta) \in \mathbb{R}^2 \ : \ \frac{\overline{\varrho}}{2} < \varrho < 2\overline{\varrho}, \ \frac{\overline{\vartheta}}{2} < \vartheta < 2\overline{\vartheta} \right\}, \qquad (2.57)$$

$\mathcal{O}^R_{ess}$ the set of radiative essential values

$$\mathcal{O}^R_{ess} := \left\{ E^R \in \mathbb{R} \ : \ \frac{\overline{E}}{2} < E^R < 2\overline{E} \right\}, \qquad (2.58)$$

with $\mathcal{O}_{ess} := \mathcal{O}^H_{ess} \cup \mathcal{O}^R_{ess}$, and their residual counterparts

$$\mathcal{O}^H_{res} := (\mathbb{R}_+)^2 \backslash \mathcal{O}^H_{ess}, \quad \mathcal{O}^R_{res} := \mathbb{R}_+ \backslash \mathcal{O}^R_{ess}, \quad \mathcal{O}_{res} := (\mathbb{R}_+)^3 \backslash \mathcal{O}_{ess}. \qquad (2.59)$$

Theorem

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain of class $C^{2,\nu}$. Assume that the thermodynamic functions p, e, s satisfy hypotheses (2.1 - 2.4) with $P \in C^1[0,\infty) \cap C^2(0,\infty)$, and that the transport coefficients μ, λ, κ, σ_a, σ_s and the equilibrium function B comply with (2.7 - 2.11), $B \in C^1$.
Let $(\varrho_\varepsilon, \vec{u}_\varepsilon, \vartheta_\varepsilon, I_\varepsilon)$ be a weak solution to the scaled radiative Navier-Stokes system for $(t, x, \vec{\omega}, \nu) \in [0, T] \times \Omega \times \mathcal{S}^2 \times \mathbb{R}_+$, supplemented with the boundary conditions (1.11 - 1.12) and the initial conditions $(\varrho_{0,\varepsilon}, \vec{u}_{0,\varepsilon}, \vartheta_{0,\varepsilon}, I_{0,\varepsilon})$

such that

$$\varrho_\varepsilon(0,\cdot) = \varrho_0 + \varepsilon \varrho_{0,\varepsilon}^{(1)}, \quad \vec{u}_\varepsilon(0,\cdot) = \vec{u}_{0,\varepsilon}, \quad \vartheta_\varepsilon(0,\cdot) = \vartheta_0 + \varepsilon \vartheta_{0,\varepsilon}^{(1)},$$

where $(\varrho_0, \vec{u}, \vartheta_0) \in H^3(\Omega)$ are smooth functions such that (ϱ_0, ϑ_0) belong to the set $\mathcal{O}_{ess}^H$ defined in (2.59) where $\overline{\varrho} > 0$, $\overline{\varrho} > 0$, are two constants and $\int_\Omega \varrho_{0,\varepsilon}^{(1)} \, dx = 0$, $\int_\Omega \vartheta_{0,\varepsilon}^{(1)} \, dx = 0$.

Suppose also that

$$\vec{u}_{0,\varepsilon} \to \vec{u}_0 \ \ strongly \ in \ L^\infty(\Omega;\mathbb{R}^3),$$

$$\varrho_{0,\varepsilon}^{(1)} \to \varrho_0^{(1)} \ strongly \ in \ L^2(\Omega),$$

$$\vartheta_{0,\varepsilon}^{(1)} \to \vartheta_0^{(1)} \ strongly \ in \ L^2(\Omega),$$

$$I_{0,\varepsilon}^{(1)} \to I_0^{(1)} \ strongly \ in \ L^\infty((0,T)\times\Omega\times(0,\infty)).$$

Then up to subsequences

$$\varrho_\varepsilon \to \varrho \ \ strongly \ in \ L^\infty(0,T;L^{\frac{5}{3}}(\Omega)),$$

$$\vec{u}_\varepsilon \to \vec{u} \ strongly \ in \ L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^3)),$$

$$\vartheta_\varepsilon \to \vartheta \ \ strongly \ in \ L^\infty(0,T;L^4(\Omega)),$$

where $(\varrho, \vec{u}, \vartheta)$ is the smooth solution of the equilibrium decoupled system (2.42)-(2.45) on $[0,T]\times\Omega$ and $I(t,x,\nu,\vec{\omega}) = B(\nu,\vartheta(t,x))$, with initial data $(\varrho_0, \vec{u}_0, \vartheta_0)$.

Theorem

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain of class $C^{2,\nu}$. Assume that the thermodynamic functions p, e, s satisfy hypotheses (2.1 - 2.4) with $P \in C^1[0,\infty) \cap C^2(0,\infty)$, and that the transport coefficients μ, λ, κ, σ_a, σ_s and the equilibrium function B comply with (2.7 - 2.11), $B \in C^1$.
Let $(\varrho_\varepsilon, \vec{u}_\varepsilon, \vartheta_\varepsilon, I_\varepsilon)$ be a weak solution to the scaled radiative Navier-Stokes system for $(t, x, \vec{\omega}, \nu) \in [0, T] \times \Omega \times \mathcal{S}^2 \times \mathbb{R}_+$, supplemented with the boundary conditions (1.11 - 1.12) and the initial conditions $(\varrho_{0,\varepsilon}, \vec{u}_{0,\varepsilon}, \vartheta_{0,\varepsilon}, I_{0,\varepsilon})$ such that

$$\varrho_\varepsilon(0,\cdot) = \varrho_0 + \varepsilon \varrho^{(1)}_{0,\varepsilon}, \quad \vec{u}_\varepsilon(0,\cdot) = \vec{u}_{0,\varepsilon}, \quad \vartheta_\varepsilon(0,\cdot) = \vartheta_0 + \varepsilon \vartheta^{(1)}_{0,\varepsilon},$$

$$I_\varepsilon(0,\cdot) = I_0 + \varepsilon I^{(1)}_{0,\varepsilon},$$

where the functions $(\varrho_0, \vec{u}, \vartheta_0)$ and $x \to I_0(x, \vec{\omega}, \nu)$ belong to $H^3(\Omega)$ and are such that $(\varrho_0, \vartheta_0, E_R(I_0))$ belong to the set $\mathcal{O}_{ess}$ defined in (2.59) where $\overline{\varrho} > 0$, $\overline{\varrho} > 0$, $\overline{E_R} > 0$ are three constants and $\int_\Omega \varrho^{(1)}_{0,\varepsilon}\ dx = 0$, $\int_\Omega \vartheta^{(1)}_{0,\varepsilon}\ dx = 0$, $\int_\Omega I^{(1)}_{0,\varepsilon}\ dx = 0$.

Suppose also that

$$\vec{u}_{0,\varepsilon} \to \vec{u}_0 \ \ \textit{strongly in } L^\infty(\Omega;\mathbb{R}^3),$$

$$\varrho_{0,\varepsilon}^{(1)} \to \varrho_0^{(1)} \ \textit{strongly in } L^2(\Omega),$$

$$\vartheta_{0,\varepsilon}^{(1)} \to \vartheta_0^{(1)} \ \textit{strongly in } L^2(\Omega),$$

$$I_{0,\varepsilon}^{(1)} \to I_0^{(1)} \ \textit{strongly in } L^\infty((0,T)\times\Omega\times(0,\infty)).$$

Then up to subsequences

$$\varrho_\varepsilon \to \varrho \ \ \textit{strongly in } L^\infty(0,T;L^{\frac{5}{3}}(\Omega)),$$

$$\vec{u}_\varepsilon \to \vec{u} \ \textit{strongly in } L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^3)),$$

$$\vartheta_\varepsilon \to \vartheta \ \ \textit{strongly in } L^\infty(0,T;L^4(\Omega)),$$

and

$$N_\varepsilon \to N \quad strongly \ in \ L^\infty((0,T)\times\Omega),$$

where $N_\varepsilon = \int_0^\infty \int_{\mathcal{S}^2} I_\varepsilon \ d\vec{\omega} \ \ d\nu$ and $(\varrho, \vec{u}, \vartheta, N)$ is the smooth solution of the Navier-Stokes-Rosseland system (2.49)-(2.52) on $[0,T]\times\Omega$ with initial data $(\varrho_0, \vec{u}_0, \vartheta_0, N_0)$.

We establish a relative entropy inequality satisfied by any weak solution $(\varrho, \vec{u}, \vartheta, I)$ of the radiative Navier-Stokes system

Let us consider a set $\{r, \Theta, \vec{U}\}$ of smooth functions such that r and Θ are bounded below away from zero and $\vec{U}\Big|_{\partial\Omega} = 0$.
We call *ballistic free energy* the thermodynamical potential given by

$$H_\Theta(\varrho, \vartheta) = \varrho e(\varrho, \vartheta) - \Theta \varrho s(\varrho, \vartheta),$$

and *radiative ballistic free energy* the potential

$$H^R_\Theta(I) = E^R(I) - \Theta s^R(I).$$

The *relative entropy* is then defined by

$$\mathcal{E}(\varrho, \vartheta | r, \Theta) := H_\Theta(\varrho, \vartheta) - \partial_\varrho H_\Theta(r, \Theta)(\varrho - \Theta) - H_\Theta(r, \Theta).$$

$$\int_\Omega \left(\frac{1}{2}\ \varrho_\varepsilon |\vec{u}_\varepsilon - \vec{U}|^2 + \mathcal{E}\left(\varrho_\varepsilon, \vartheta_\varepsilon | r, \Theta\right) + \varepsilon H^R(I_\varepsilon) \right) (\tau, \cdot) dx$$

$$+ \int_0^\tau \int_{\Gamma_+} \vec{\omega} \cdot \vec{n}_x I_\varepsilon(t, x, \vec{\omega}, \nu)\ d\Gamma d\nu dt$$

$$+ \int_0^\tau \int_\Omega \frac{\Theta}{\vartheta_\varepsilon} \left(\mathbb{S}_\varepsilon : \nabla_x \vec{u}_\varepsilon - \frac{\vec{q}_\varepsilon \cdot \nabla_x \vartheta_\varepsilon}{\vartheta_\varepsilon} \right) dxdt$$

$$+ \int_0^\tau \int_\Omega \int_0^\infty \int_{\mathcal{S}^2} \frac{\Theta}{\nu} \left[\log \frac{n(I_\varepsilon)}{n(I_\varepsilon) + 1} - \log \frac{n(B_\varepsilon)}{n(B_\varepsilon) + 1} \right] \sigma_{a_\varepsilon}^{(j)} (B_\varepsilon - I_\varepsilon) d\vec{\omega} d\nu d$$

$$+ \int_0^\tau \int_\Omega \int_0^\infty \int_{\mathcal{S}^2} \frac{\Theta}{\nu} \left[\log \frac{n(I_\varepsilon)}{n(I_\varepsilon) + 1} - \log \frac{n(\tilde{I}_\varepsilon)}{n(\tilde{I}_\varepsilon) + 1} \right] \sigma_{s_\varepsilon}^{(j)} (\tilde{I}_\varepsilon - I_\varepsilon) d\vec{\omega} d\nu dx$$

$$\leq - \int_\Omega \frac{1}{2} \mathbf{E}(0) dx + \int_0^\tau \int_\Omega \mathcal{R}(x,t)$$

where

$\mathbf{E}(0) =$
$\frac{1}{2}(\varrho_{0,\varepsilon}|\vec{u}_{0,\varepsilon} - \vec{U}(0,\cdot)|^2 + \mathcal{E}\left(\varrho_{0,\varepsilon}, \vartheta_{0,\varepsilon}|r(0,\cdot), \Theta(0,\cdot)\right) + \varepsilon H^R(I_{0,\varepsilon})\right)\, dx$

$$\int_0^\tau \int_\Omega \mathcal{R}(x,t) =:$$

$$\int_0^\tau \int_\Omega \varrho_\varepsilon (\vec{u}_\varepsilon - \vec{U}) \cdot \nabla_x \vec{U} \cdot \left(\vec{U} - \vec{u}_\varepsilon \right) \ dx \ dt$$

$$+ \int_0^\tau \int_\Omega \varrho_\varepsilon \left(s_\varepsilon - s(r,\Theta) \right) \left(\vec{U} - \vec{u}_\varepsilon \right) \cdot \nabla_x \Theta \ dx \ dt$$

$$+ \int_0^\tau \int_\Omega \left(\varrho_\varepsilon \left(\partial_t \vec{U} + \vec{U} \cdot \nabla_x \vec{U} \right) \cdot \left(\vec{U} - \vec{u}_\varepsilon \right) \right) \ dx \ dt$$

$$- \int_0^\tau \int_\Omega \left(p_\varepsilon \ \mathrm{div}_x \vec{U} - \mathbb{S}_\varepsilon : \nabla_x \vec{U} \right) \ dx \ dt$$

$$- \int_0^\tau \int_\Omega \left(\varepsilon s_\varepsilon^R \partial_t \Theta + \vec{q}_\varepsilon^R \cdot \nabla_x \Theta \right) \ dx \ dt$$

$$-\int_0^\tau \int_\Omega \Big(\varrho_\varepsilon \left(s_\varepsilon - s(r,\Theta) \right) \partial_t \Theta \Big) \ dx \ dt$$

$$-\int_0^\tau \int_\Omega \varrho_\varepsilon \left(s_\varepsilon - s(r,\Theta) \right) \vec{U} \cdot \nabla_x \Theta \ dx \ dt$$

$$-\int_0^\tau \int_\Omega \frac{\vec{q}_\varepsilon}{\vartheta_\varepsilon} \cdot \nabla_x \Theta \ dx \ dt$$

$$+\int_0^\tau \int_\Omega \left(\left(1 - \frac{\varrho_\varepsilon}{r} \right) \partial_t p(r,\Theta) - \frac{\varrho_\varepsilon}{r} \ \vec{u}_\varepsilon \nabla_x p(r,\Theta) \right) \ dx \ dt$$

$(r = \rho, \vec{U} = \vec{u}, \Theta = \vartheta)$, where $(\varrho, \vec{u}, \vartheta)$ is a classical solution of the target system (in the equilibrium case or in the non equilibrium case)

Semi-relativistic model

$$B(\nu, \vec{\omega}, \vec{u}, \vartheta) = \frac{2h}{c^2} \frac{\nu^3}{e^{\frac{h\nu}{k\vartheta}\left(1 - \alpha \frac{\vec{\omega}\cdot\vec{u}}{c}\right)} - 1}, \qquad (2.60)$$

$0 \leq \alpha(\vartheta) \leq 1$

If $\frac{|\vec{u}|}{c} << 1$ one recovers the standard equilibrium Planck's function $B(\nu, \vartheta) = \frac{2h}{c^2} \frac{\nu^3}{e^{\frac{h\nu}{k\vartheta}} - 1}$.

Berthon, Buet, Coulombel, Depres, Dubois, Goudon, Morel, Turpault

$M1$ Levermore model

$$\alpha = \frac{\sigma_a + \sigma_s}{\sigma_a + 2\sigma_s}, \qquad (2.61)$$

$\sigma_a(\vartheta, \vec{u}) = \chi(|\vec{u}|)\tilde{\sigma}_a(\vartheta) \geq 0$ and $\sigma_s(\vartheta) \geq 0$

$$\chi(s) = \begin{cases} 1 & \text{if } \; s \leq c, \\ 0 & \text{if } \; s \geq c + \beta, \end{cases}$$

for an arbitrary $\beta > 0$.

The role of this cut-off is to deal with the singularity of B

In the "over-relativistic" regime ($|\vec{u}| \geq c$) we decide to decouple matter and radiation.

$$\partial_t (\varrho s) + \mathrm{div}_x (\varrho s \vec{u}) + \mathrm{div}_x \left(\frac{\vec{q}}{\vartheta} \right) =$$

$$\frac{1}{\vartheta} \left(\mathbb{S} : \nabla_x \vec{u} - \frac{\vec{q} \cdot \nabla_x \vartheta}{\vartheta} \right) - \frac{S_E}{\vartheta} + \frac{\vec{S}_F \cdot \vec{u}}{\vartheta} \tag{2.62}$$

$$=: \varsigma,$$

where the first term of the right hand side $\varsigma_m := \frac{1}{\vartheta} \left(\mathbb{S} : \nabla_x \vec{u} - \frac{\vec{q} \cdot \nabla_x \vartheta}{\vartheta} \right)$ is the (positive) matter entropy production.

the formula for the entropy of a photon gas

$$s^R = -\frac{2k}{c^3} \int_0^\infty \int_{\mathcal{S}^2} \nu^2 \left[n \log n - (n+1) \log(n+1) \right] d\vec{\omega} d\nu, \quad (2.63)$$

where $n = n(I) = \frac{c^2 I}{2h\alpha^3 \nu^3}$ is the occupation number. Defining the radiative entropy flux

$$\vec{q}^R = -\frac{2k}{c^2} \int_0^\infty \int_{\mathcal{S}^2} \nu^2 \left[n \log n - (n+1) \log(n+1) \right] \vec{\omega} \; d\vec{\omega} d\nu, \quad (2.64)$$

and using the radiative transfer equation, we get the equation

$$\partial_t s^R + \mathrm{div}_x \vec{q}^R = -\frac{k}{h} \int_0^\infty \int_{\mathcal{S}^2} \frac{1}{\nu} \log \frac{n}{n+1} \; S \; d\vec{\omega} d\nu =: \varsigma^R. \quad (2.65)$$

$$\varsigma^R = -\frac{k}{h}\int_0^\infty \int_{\mathcal{S}^2} \frac{1}{\nu}\Big\{ \Big[\log \frac{n(I)}{n(I)+1} - \log \frac{n(B)}{n(B)+1}\Big] \sigma_a(B - I)$$

$$+ \Big[\log \frac{n(I)}{n(I)+1} - \log \frac{n(\tilde{I})}{n(\tilde{I})+1}\Big] \sigma_s(\tilde{I} - I)\Big\}\ d\vec{\omega} d\nu,$$

$$+\frac{1}{\vartheta}\int_0^\infty \int_{\mathcal{S}^2} \Big(1 - \alpha\frac{\vec{\omega}\cdot\vec{u}}{c}\Big) S\ d\vec{\omega} d\nu - \frac{\alpha\sigma_s}{\sigma_a+\sigma_s}\frac{\vec{S}_F\cdot\vec{u}}{\vartheta}.$$

Choosing now

$$\alpha = \frac{\sigma_a+\sigma_s}{\sigma_a+2\sigma_s}, \qquad (2.66)$$

We get

$$\varsigma^R = -\frac{k}{h}\int_0^\infty \int_{\mathcal{S}^2} \frac{1}{\nu}\left[\log\frac{n(I)}{n(I)+1} - \log\frac{n(B)}{n(B)+1}\right]\sigma_a(B-I)\, d\vec{\omega} d\nu$$

$$-\frac{k}{h}\int_0^\infty \int_{\mathcal{S}^2} \frac{1}{\nu}\left[\log\frac{n(I)}{n(I)+1} - \log\frac{n(\tilde{I})}{n(\tilde{I})+1}\right]\sigma_s(\tilde{I} - I)\, d\vec{\omega} d\nu \tag{2.67}$$

$$+\frac{1}{\vartheta}S_E - \frac{\vec{S}_F\cdot\vec{u}}{\vartheta}.$$

we obtain finally

$$\partial_t\left(\varrho s + s^R\right) + \mathrm{div}_x\left(\varrho s\vec{u} + \vec{q}^R\right) + \mathrm{div}_x\left(\frac{\vec{q}}{\vartheta}\right)$$

$$= \frac{1}{\vartheta}\left(\mathbb{S} : \nabla_x\vec{u} - \frac{\vec{q}\cdot\nabla_x\vartheta}{\vartheta}\right) - \frac{k}{h}\int_0^\infty\int_{\mathcal{S}^2}\frac{1}{\nu}\left[\log\frac{n(I)}{n(I)+1} - \log\frac{n(B)}{n(B)+1}\right]$$

$$-\frac{k}{h}\int_0^\infty\int_{\mathcal{S}^2}\frac{1}{\nu}\left[\log\frac{n(I)}{n(I)+1} - \log\frac{n(\tilde{I})}{n(\tilde{I})+1}\right]\sigma_s(\tilde{I} - I)\ d\vec{\omega}d\nu, \tag{2.68}$$

Diffusion limits- non-relativistic limits

The equilibrium-diffusion regime

$$Ma = Sr = Pe = Re = \mathcal{P} = 1,$$

$$\mathcal{C} = \varepsilon^{-1}, \ \mathcal{L}_s = \varepsilon^2 \text{ and } \mathcal{L} = \varepsilon^{-1},$$

$$\partial_t \varrho + \mathrm{div}_x(\varrho \vec{u}) = 0, \tag{2.69}$$

$$\partial_t(\varrho \vec{u}) + \mathrm{div}_x(\varrho \vec{u} \otimes \vec{u}) + \nabla_x \mathbf{p} = \mathrm{div}_x \mathbb{S}, \tag{2.70}$$

$$\partial_t \left(\frac{1}{2} \varrho |\vec{u}_0|^2 + \varrho \mathbf{e} \right) + \mathrm{div}_x \left(\left(\frac{1}{2} \varrho |\vec{u}|^2 + \varrho \mathbf{e} + \mathbf{p} \right) \vec{u} + \vec{\mathbf{q}} - \mathbb{S} \vec{u} \right) = 0, \tag{2.71}$$

$$\partial_t (\varrho \mathbf{s}) + \operatorname{div}_x (\varrho \mathbf{s} \vec{u}) + \operatorname{div}_x \left(\frac{\vec{\mathbf{q}}}{\vartheta} \right) = \frac{1}{\vartheta} \left(\mathbb{S} : \nabla_x \vec{u} - \frac{\vec{\mathbf{q}} \cdot \nabla_x \vartheta}{\vartheta} \right), \tag{2.72}$$

$$I = B(\nu, \vartheta), \tag{2.73}$$

where $\mathbf{p}(\varrho, \vartheta) = p(\varrho, \vartheta) + \frac{a}{3}\vartheta^4$, $\mathbf{e}(\varrho, \vartheta) = e(\varrho, \vartheta) + \frac{a}{\varrho}\vartheta^4$, $\mathbf{k}(\vartheta_0) = \kappa(\vartheta) + \frac{4a}{3\sigma_a}\vartheta^3$, $\vec{\mathbf{q}} = -\mathbf{k}(\vartheta)\nabla_x \vartheta$ and $\varrho \mathbf{s} = \varrho s + \frac{4}{3}a\vartheta^3$.

We also get boundary conditions

$$\vec{u}|_{\partial\Omega} = 0, \ \nabla\vartheta \cdot \vec{n}|_{\partial\Omega} = 0, \tag{2.74}$$

initial conditions

$$(\varrho(x,t), \ \vec{u}(x,t), \ \vartheta(x,t))|_{t=0} = \left(\varrho^0(x), \ \vec{u}^0(x), \ \vartheta^0(x)\right), \tag{2.75}$$

for any $x \in \Omega$, with the following compatibility conditions

$$\vec{u}^0(x)|_{\partial\Omega} = 0, \ \nabla\vartheta^0 \cdot \vec{n}|_{\partial\Omega} = 0. \tag{2.76}$$

- existence of unique solution for strong solution (**local** (small time) or **global** for (small data))
- rigorous proof of singular limit using relative entropy inequality

The non-equilibrium diffusion regime

$$Ma = Sr = Pe = Re = \mathcal{P} = 1,$$

$$\mathcal{C} = \varepsilon^{-1},\ \mathcal{L} = \varepsilon^{2} \text{ and } \mathcal{L}_s = \varepsilon^{-1}.$$

$$\partial_t \varrho + \mathrm{div}_x(\varrho \vec{u}) = 0, \tag{2.77}$$

$$\partial_t(\varrho \vec{u}) + \mathrm{div}_x(\varrho \vec{u} \otimes \vec{u}) + \nabla_x \mathbf{p} = \mathrm{div}_x \mathbb{S}, \tag{2.78}$$

$$\partial_t \left(\frac{1}{2}\ \varrho |\vec{u}_0|^2 + \varrho \mathbf{e} \right) + \mathrm{div}_x \left(\left(\frac{1}{2}\ \varrho |\vec{u}|^2 + \varrho \mathbf{e} + \mathbf{p} \right) \vec{u} + \vec{\mathbf{q}} - \mathbb{S} \vec{u} \right) = 0, \tag{2.79}$$

$$\partial_t \left(\varrho s \right) + \mathrm{div}_x \left(\varrho s \vec{u} \right) + \mathrm{div}_x \left(\frac{\vec{q}}{\vartheta} \right) =$$

$$\frac{1}{\vartheta} \left(\mathbb{S} : \nabla_x \vec{u} - \frac{\vec{q} \cdot \nabla_x \vartheta}{\vartheta} \right) + \frac{1}{3} \frac{\nabla_x N \cdot \vec{u}}{\vartheta} - \frac{\sigma_a(\vartheta)}{\vartheta} \left(a \vartheta^4 - N \right), \tag{2.80}$$

$$\partial_t N - \frac{1}{3}\, \mathrm{div}_x \left(\frac{1}{\sigma_s(\vartheta)} \nabla_x N \right) = \sigma_a(\vartheta) \left(a\vartheta^4 - N \right), \qquad (2.81)$$

where $\mathbf{p} = p + \frac{1}{3}N$, $\mathbf{e} = e + \frac{N}{\varrho}$ and $\vec{\mathbf{q}} = \kappa \nabla_x \vartheta + \frac{1}{3\sigma_s} \nabla_x N$ with boundary conditions

$$\vec{u}|_{\partial\Omega} = 0, \ \nabla\vartheta \cdot \vec{n}|_{\partial\Omega} = 0, \ \ N|_{\partial\Omega} = 0, \qquad (2.82)$$

initial conditions

$$\left(\varrho(x,t), \ \vec{u}(x,t), \ \vartheta(x,t), N(x,t) \right)|_{t=0} = \left(\varrho^0(x), \ \vec{u}^0(x), \ \vartheta^0(x), N^0(x) \right), \qquad (2.83)$$

for any $x \in \Omega$, with $N^0(x) = \int_0^\infty \int_{\mathcal{S}^2} I^0(x,\nu,\vec{\omega}) \, d\vec{\omega} \, d\nu$ and and the compatibility conditions

$$\vec{u}^2|_{\partial\Omega} = 0, \ \nabla\vartheta^0 \cdot \vec{n}|_{\partial\Omega} = 0, \ \ N^0\big|_{\partial\Omega} = 0. \qquad (2.84)$$

$$N = a\theta_r^4. \qquad (2.85)$$

- existence of unique solution for strong solution (**local** (small time) or **global** for (small data))
- rigorous proof of singular limit using relative entropy inequality

"Non-relativistic" limit

$\mathcal{C} = O(\varepsilon^{-1})$
A small amount of radiation is present so $\mathcal{P} = \varepsilon$. Finally we put $Ma = 1$, $Sr = 1$, $Pe = 1$, $Re = 1$, $\mathcal{L} = \mathcal{L}_s = 1$