Decay of viscoelastic waves with memory

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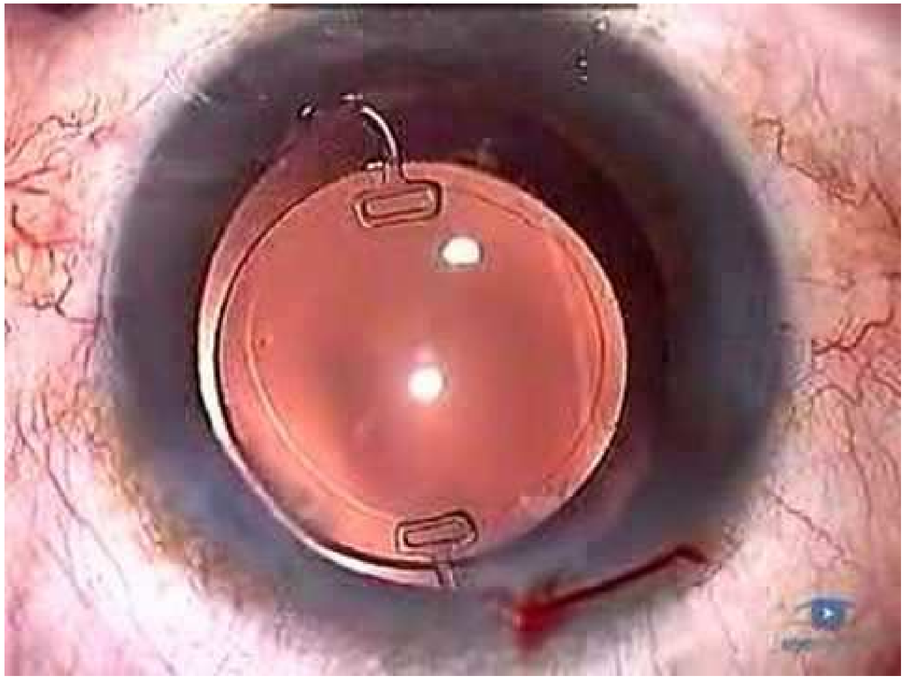


Summary

- Modelling the displacement
- The energy behavior of the wave equation with memory
- Numerical waves with memory
- Conclusions

Modelling the displacement

- Newton's second law -u displacement of the solid; force; stress

$$\rho \frac{d^2u}{dt^2} = \nabla \cdot \sigma + f$$

- Stress and strain: Boltzmann representation

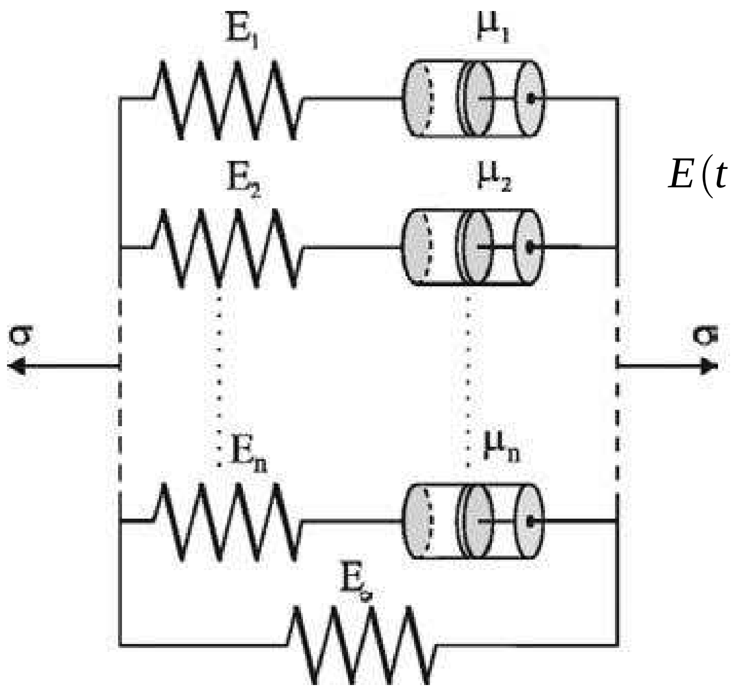
$$\sigma(t) = \underbrace{E(0)\mathbf{D}\varepsilon(t)}_{\text{Hooke's law}} - \underbrace{\int_0^t \frac{\partial}{\partial s} E(t-s)\mathbf{D}\varepsilon(s)\,ds}_{\text{hereditary integral}}$$

- Strain and displacement:

$$\varepsilon(t) = \frac{1}{2}\left(\nabla u(t) + \nabla u(t)^t\right)$$

$$\rho \frac{d^2u}{dt^2} - \nabla \cdot \left(\frac{1}{2}E(0)\mathbf{D}(\nabla u(t) + \nabla u(t)^t)\right) = -\int_0^t \frac{\partial}{\partial s} E(t-s)\nabla \cdot \left(\mathbf{D}(\nabla u(s) + \nabla u(s)^t)\right) ds + f$$

Maxwell-Wiechert model:



$$E(t) = E_0 + \sum_{i=1}^{n} E_i e^{-\frac{E_i}{\mu_i} t}$$

$$\rho \frac{d^2 u}{dt^2} - \hat{E} \nabla \cdot \left(\frac{1}{2} \mathbf{D} (\nabla u(t) + \nabla u(t)^t) \right) = - \int_0^t K_{er}(t-s) \nabla \cdot \left(\mathbf{D} (\nabla u(s) + \nabla u(s)^t) \right) ds + f$$

$$\hat{E} = \sum_{j=0}^{n} E_j \qquad K_{er}(s) = \sum_{i=1}^{n} \frac{E_i^2}{\mu_i} e^{-\frac{E_i}{\mu_i} s}, \qquad u(t) = 0 \,\text{on}\, \partial\Omega$$

General wave equations with memory

$$\frac{d^2u}{dt^2}(t) + c\frac{du}{dt}(t) + \mathcal{A}u(t) = \int_0^t K_{er}(t-s)\mathcal{B}u(s)ds$$

$$\mathcal{A}v = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\big(a_{ij}\frac{\partial v}{\partial x_j}\big) + \sum_{i=1}^{n}\frac{\partial}{\partial x_i}(a_i v) + a_0 v,$$

$$\mathcal{B}v = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\big(b_{ij}\frac{\partial v}{\partial x_j}\big) + \sum_{i=1}^{n}\frac{\partial}{\partial x_i}(b_i v) + b_0 v,$$

Let $u \in H^1(\mathbb{R}^+, H_0^1(\Omega))$ be such that $\frac{d^2u}{dt^2} \in L^2(\mathbb{R}^+, L^2(\Omega))$ and, for all $T > 0$, holds the following

$$\begin{cases} (\frac{d^2u}{dt^2} + c\frac{du}{dt}, w) + a(u,w) = \int_0^t K_{er}(t-s)b(u(s),w)ds \text{ in } (0,T), \forall w \in H_0^1(\Omega), \\ \frac{du}{dt}(0) = u_1, \\ u(0) = u_0, \end{cases}$$

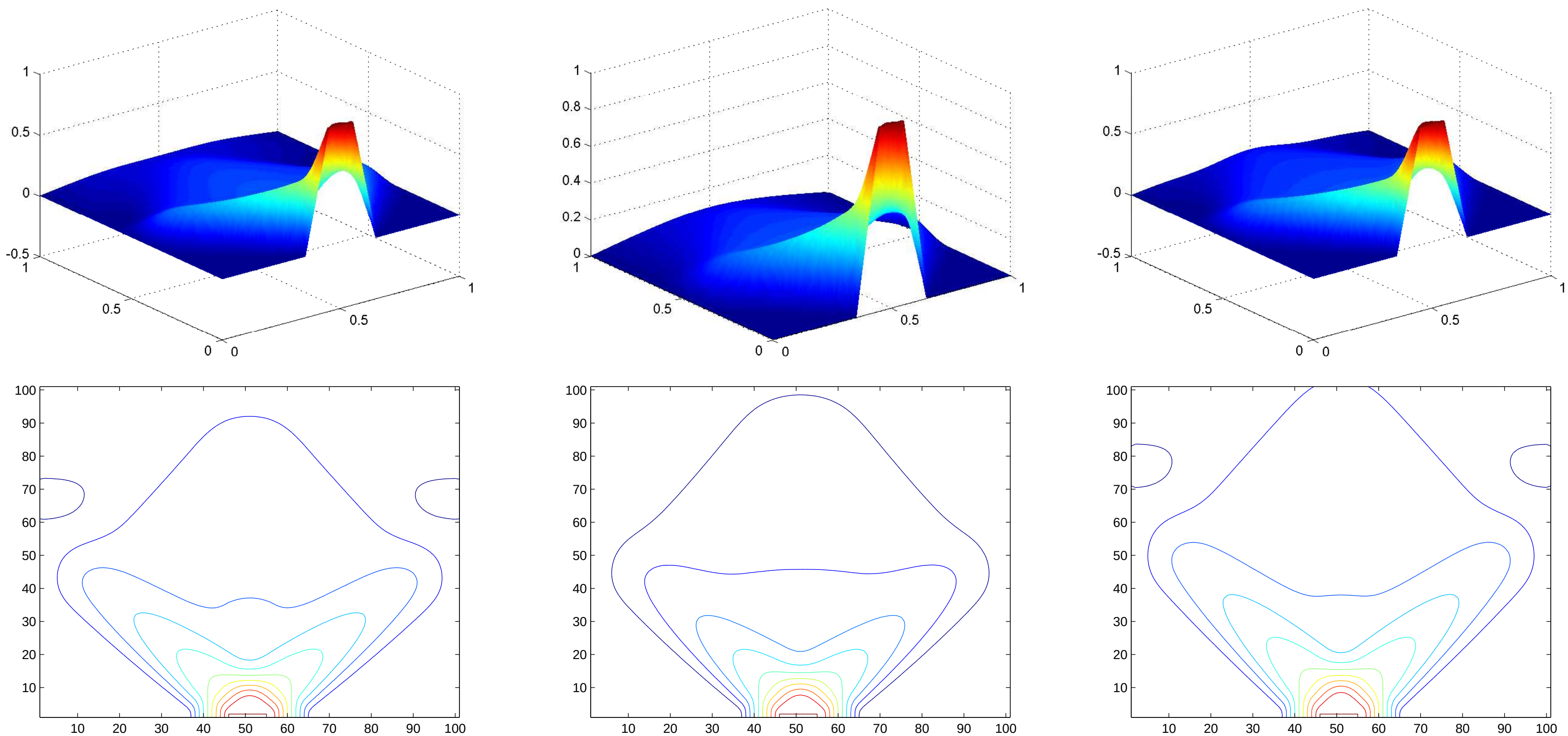
$$u_0 \in H_0^1(\Omega), u_1 \in L^2\Omega)$$

Wave equation with memory- relaxation of a wave equation

Wave equation with memory (WEM): $\rho \dfrac{d^2u}{dt^2}(t) + c\dfrac{du}{dt}(t) - D_1\Delta u(t) = -\displaystyle\int_0^t K_{er}(t-s)D_2\Delta u(s)ds$

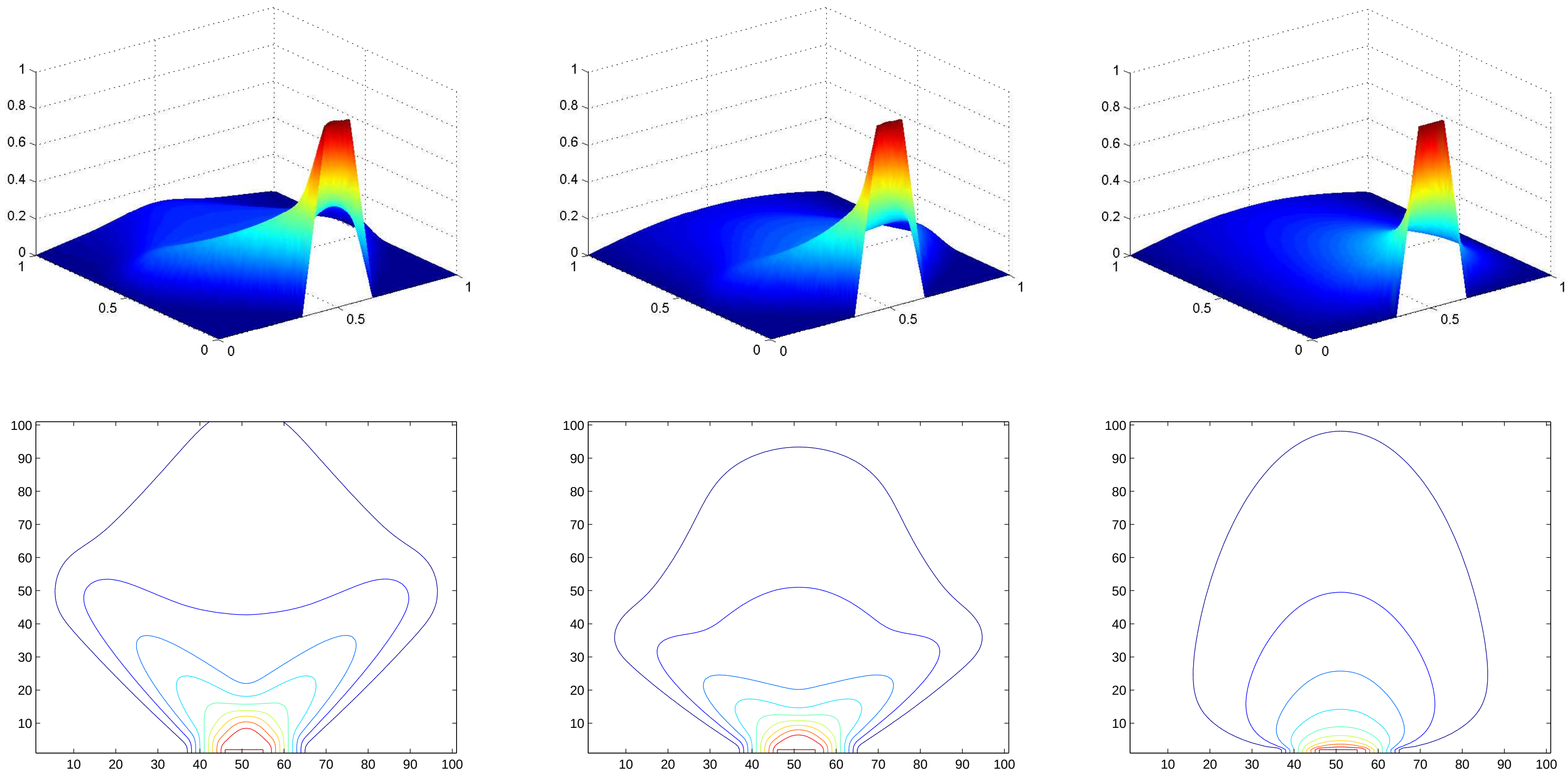
$\tau = \dfrac{\mu_1}{E_1},\ K_{er}(s) = \dfrac{1}{\tau}e^{-\frac{s}{\tau}}$

- $\tau \to 0$ $\qquad \rho \dfrac{d^2u}{dt^2}(t) + c\dfrac{du}{dt}(t) = (D_1 - D_2)\Delta u(t)$



Wave equation with memory- singular perturbation of a diffusion equation with memory

- $\rho \to 0$ $$\frac{du}{dt}(t) - D_1 \Delta u(t) = -\int_0^t K_{er}(t-s) D_2 \Delta u(s)\, ds$$



Wave equation

$$\text{Energy} \rightarrow E(t) = \frac{\rho}{2}\|\frac{du}{dt}(t)\|^2 + \frac{D}{2}\|\nabla u(t)\|^2, t \geq 0$$

- Without damping effect ($c = 0$) - Energy conservation principle

$$E(t) = E(0), t \geq 0$$

- With damping effect ($c > 0$) - $E(t)$ is a decreasing function and

$$E(t) \rightarrow 0, t \rightarrow \infty, \text{exponentially}$$

Diffusion equation with memory

$$\text{Energy} \to E(t) = \|u(t)\|^2 + \int_0^t \|\nabla u(s)\|^2 \, ds, t \geq 0$$

- $E(t)$ is bounded in bounded time intervals
- There exists $\gamma > 0$ such that

$$\|u(t)\|^2 + \int_0^t e^{-2\gamma(t-s)} \|\nabla u(s)\|^2 \, ds \to 0, t \to \infty, \text{ exponentially}$$

Wave equation with memory - damping effect

$$((\int_0^t K_{er}(t-s)\nabla u(s), \nabla \frac{du}{dt}(t))\,ds$$

Using the representation

$$((\int_0^t K_{er}(t-s)\nabla u(s), \nabla \frac{du}{dt}(t))\,ds = -\frac{1}{2}\frac{d}{dt}\|\int_0^t K_{er}(t-s)\nabla u(s)\,ds - \nabla u(t)\|^2$$
$$+\|\int_0^t K'_{er}(t-s)\nabla u(s)\,ds - \nabla u(t)\|^2 + \frac{1}{2}\frac{d}{dt}\|\nabla u(t)\|^2$$

then with $K_{er}(s) = \frac{1}{\tau}e^{-\frac{s}{\tau}}$, we have

$$E(t) + \|\int_0^t K_{er}(t-s)\nabla u(s)\,ds - \nabla u(t)\|^2$$

is a decreasing function in time, and consequently the memory term has a damping effect.

Moreover

$$\|\int_0^t K_{er}(t-s)\nabla u(s)\,ds - \nabla u(t)\|^2$$

decreases exponentially in time.

The wave equation with memory has an exponential decreasing energy?

Using the representation

$$((\int_0^t K_{er}(t-s)\nabla u(s), \nabla \frac{du}{dt}(t))\, ds = -\frac{1}{2}\frac{d}{dt}\int_0^t K_{er}(t-s)\|\nabla u(s)-\nabla u(t)\|^2 ds$$
$$+\frac{1}{2}\int_0^t K'_{er}(t-s)\|\nabla u(s)-\nabla u(t)\|^2\, ds + \frac{1}{2}\int_0^t K_{er}(t-s)\frac{d}{dt}\|\nabla u(t)\|^2$$

then with $K_{er}(s) = \frac{1}{\tau}e^{-\frac{s}{\tau}}$, we have

$$E(t) - \int_0^t K_{er}(t-s)\, ds \|\nabla u(t)\|^2 + \int_0^t K_{er}(t-s)\|\nabla u(s)-\nabla u(t)\|^2\, ds$$

is a decreasing function in time, and

$$\|\nabla u(t)\|^2 + \int_0^t K_{er}(t-s)\|\nabla u(s)-\nabla u(t)\|^2\, ds$$

decreases exponentially in time.

The kinetic energy of the wave equation with memory decreases exponentially in time?

Kinetic energy $\|\frac{du}{dt}(t)\|^2$

Wave equation with damping effect:

$$\frac{d^2u}{dt^2} + c\frac{du}{dt} + \beta u(t) = D\Delta u(t)$$

Then

$$E(t) + \|u(s)\|^2 \to 0, t \to \infty, \text{ exponentially.}$$

Wave equation with memory and damping effect:

$$\frac{d^2u}{dt^2} + c\frac{du}{dt} + \beta u(t) - D_1\Delta u(t) = -\int_0^t K_{er}(t-s)D_2\Delta u(s)\,ds$$

Then, for $K_{er}(s) = \dfrac{1}{\tau}e^{-\frac{s}{\tau}}$, $\exists\gamma > 0$ such that

$$E(t) + \|u(t)\|^2 + \|\int_0^t K_{er}(t-s)e^{-\gamma(t-s)}\nabla u(s)\,ds - \nabla u(t)\|^2 \to 0, t \to \infty, \text{ exponentially.}$$

General wave equations with memory

$$
\begin{aligned}
\int_0^t K_{er}(t-s)b(u(s),\frac{du}{dt}(t))\,ds &= \frac{d}{dt}\int_0^t K_{er}(t-s)b(u(s),u(t))\,ds \\
&-K_{er}(0)b(u(t),u(t)) - \int_0^t K'_{er}(t-s)b(u(s),u(t))\,ds \\
&-\int_0^t K_{er}(t-s)b(u(s),\frac{du}{dt}(t))\,ds
\end{aligned}
$$

$a(.,.)$, $b(.,.)$are H^1- elliptic, $b(.,.)$is bounded, $K_{er} \in H^1(\mathbb{R}^+)$

- If $c=0$, then $E(t)$ is bounded in bounded time intervals

- If $c,\beta>0$, then there exists a family of wave problems such that

$$E(t)+\|u(t)\|^2 \to 0, t\to\infty, \text{ exponentially.}$$

Numerical waves and their qualitative behavior

- $\mathcal{T}_h$ triangulation of $\Omega \subset \mathbb{R}^2$ with diameter h
- finite element space

$$\mathcal{V}_h = \{v \in C^0(\overline{\Omega}) : v = 0 \text{ on } \partial\Omega, v \in \mathbb{P}(K), K \in \mathcal{T}_h\}$$

$$\begin{cases} (\dfrac{d^2 u_h}{dt^2} + c\dfrac{du_h}{dt}, w_h) + a(u_h, w_h) = \displaystyle\int_0^t K_{er}(t-s) b(u_h(s), w_h)\, ds \text{ in } (0,T), \forall w_h \in \mathcal{V}_h, \\ \dfrac{du_h}{dt}(0) = u_{1,h}, \\ u_h(0) = u_{0,h}. \end{cases}$$

Energy estimates: numerical waves mimic their continuous counterpart

$$\mathcal{A}u = -\nabla.(D_1 \nabla u),\ \mathcal{B}u = -\nabla.(D_2 \nabla u)$$

- Discretize in space (FD+trapz)

$$
\begin{aligned}
&\Big(\frac{u_h^{n+1}-2u_h^n+u_h^{n-1}}{\Delta t^2},v\Big)+\Big(c\frac{u_h^{n+1}-u_h^{n-1}}{2\Delta t},v\Big)+\big(D_1\nabla u_h^{n+1},\nabla v\big)\\
&=\underbrace{\frac{\Delta t}{2}\sum_{j=0}^{n}\big(K_{er}(t_{n+1}-t_{j+1})D_2\nabla u_h^{j+1}+K_{er}(t_{n+1}-t_j)D_2\nabla u_h^{j},\nabla v\big)}_{I_{n+1}}
\end{aligned}
$$

- $K_{er}(s)=\dfrac{1}{\tau}e^{-\frac{s}{\tau}}$

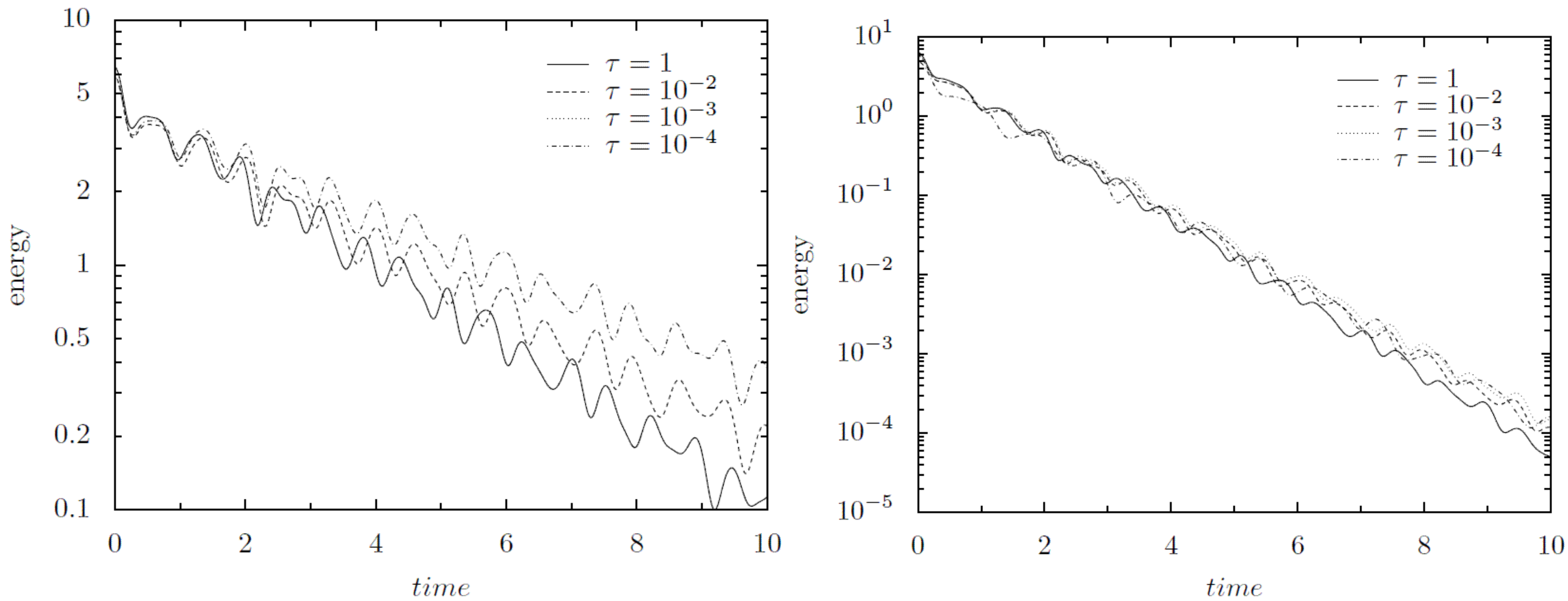
$$
\begin{aligned}
&\Big(\big(\tfrac{1}{\Delta t^2}+\tfrac{c}{2\Delta t}\big)u_h^{n+1},v\Big)+\Big(\big(D_1-\tfrac{\Delta t}{2\tau}D_2\big)\nabla u_h^{n+1},\nabla v\Big)\\
&=\Big(\big(\tfrac{2}{\Delta t^2}+\tfrac{\Delta t}{2\tau}e^{-\frac{-\Delta t}{\tau}}D_2\nabla\big)u_h^{n},v\Big)+\Big(\big(\tfrac{c}{2\Delta t}-\tfrac{1}{\Delta t^2}\big)u_h^{n-1},v\Big)+e^{-\frac{\Delta t}{\tau}}\big(I_n,\nabla v\big)
\end{aligned}
$$

  - Known data: u_h^0, u_h^1 and I_1
  - Given u_h^n and I_n, do
    - Solve for u_h^{n+1}
    - Update $I_{n+1}: I_{n+1}=e^{-\frac{\Delta t}{\tau}}I_n+\dfrac{D_2\Delta t}{2\tau}\big(e^{-\frac{\Delta t}{\tau}}\nabla u_h^n+\nabla u_h^{n+1}\big)$

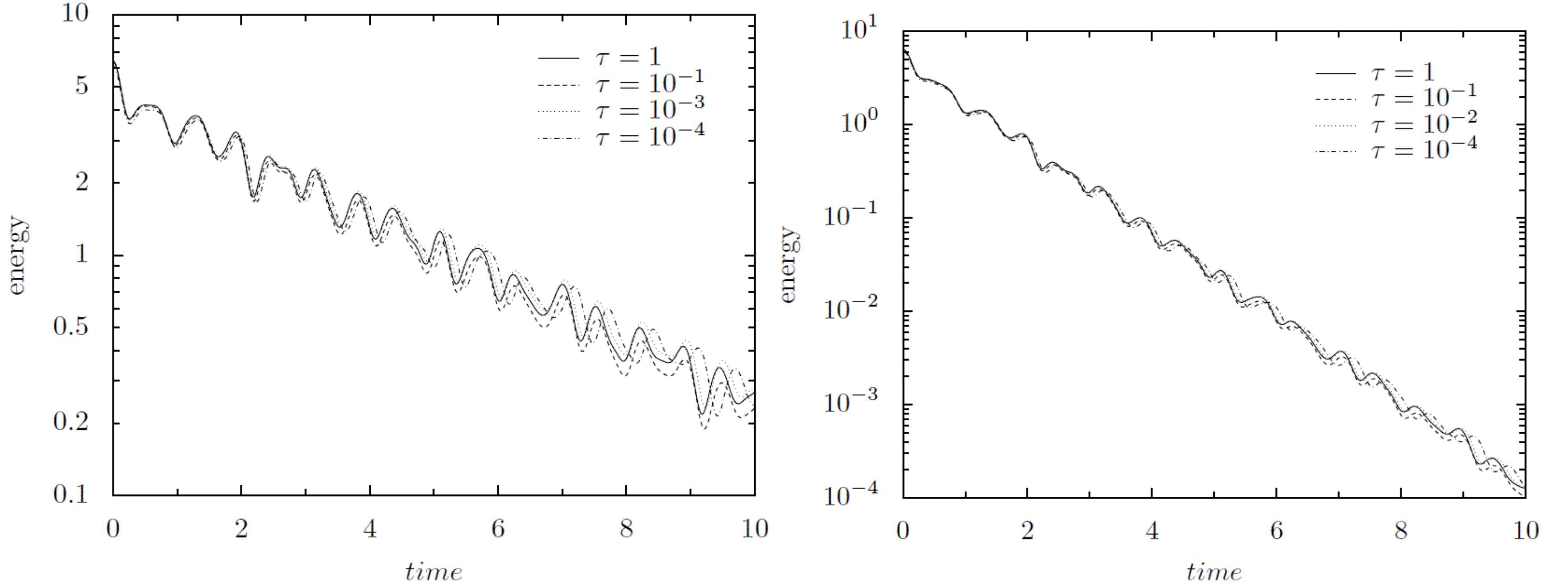
Behavior of the (discrete) energy

$$E_{h,n}=\|\frac{u_h^n-u_h^{n-2}}{2\Delta t}\|^2+\|u_h^n\|_1^2+\|I_n-\nabla u_n^n\|^2,\ n\geq 2.$$

$\Omega = (-1,1)^2, u_0(x,y) = e^{-\frac{x^2+y^2}{0.1}}$



Effect of memory coefficient τ for different damping effects ($D_2 = 0.1, c = 0.25, 1$)



Effect of memory coefficient τ for different damping effects ($D_2 = 0.01, c = 0.25, 1$)

The discrete energy decays exponentially to zero.

Conclusions:

- The energy of the wave equation satisfies the energy conservation principle. If a damping effect is presented ($c > 0$) then $E(t)$ decreases exponentially in time.
- The memory term in the wave equation with memory (WEM) ($c = 0$) has a damping effect - the potential and kinetic energies decrease in time and the potential energy decreases exponentially in time.
- The energy of the WEM with $c, \beta > 0$, decreases exponentially in time.
- The last result can be generalized for a general class of wave equations with memory.