We study the behavior of a unimodal map in two parameters, one of the parameters varies the sign of the Schwarzian derivative the second the value of the maximum. We characterize the behavior of the different dynamics in the parameter space.

**Keywords:** Iterated maps; Schwarzian derivative; Feigenbaum constant; nonpolynomial maps; topological entropy.

1. Introduction

The families of one-dimensional maps have been the object of many studies since [Sharkovsky, 1964], for instance in [Sharkovsky et al., 1997] or in [de Melo & van Strien, 1993], whose authors also studied this subject, we have two large surveys of the enormous effort made in the last decades in this field of work. We adopt the usual definitions as for instance symbolic dynamics or topological equivalence from those works. As a rule when we say that some definition is usual or well known the reader can easily find those subjects in any one of the cited books.

Our goal is the study of \( \ell \)-modal families of maps on several parameters with the possibility of changing the sign of the Schwarzian derivative when we change one or more of these parameters. Here we begin this study by one family of unimodal maps.

In this section we present our object of study, the unimodal family of maps \( f_{a,b} \) on the interval \( I = [-1, 1] \), depending on two real parameters, \( a \) and \( b \):

\[
f_{a,b}(x) = b(1 - |x|^a) - 1.
\]

Our aim is to characterize some properties of the topological behavior of the iterates of this kind of maps, namely when we change both the parameters \( a \) and \( b \).

Because of the condition \( f_{a,b}(I) \subset I \), the parameters \( a \) and \( b \) satisfy:

\[
0 < b \leq 2, \quad a > 0.
\]

The dynamics of the iterates of \( f \) are trivial when \( b < 1 \). This family is always continuous for the two parameters in the region considered.

When \( a = 1 \) we have a piecewise linear model, the well-known tent map. When \( k < a \leq k + 1 \), for \( k = 1, 2, 3, \ldots \), this map is \( C^k(I) \) except in the particular cases when \( a = 2n, n = 1, 2, 3, \ldots \), where \( f_{a,b} \) is \( C^\infty(I) \). When \( 0 < a < 1, f'_{a,b} \) is discontinuous at the origin, but \( f_{a,b} \) is \( C^\infty(I \setminus \{0\}) \).

Usually, in the context of one-dimensional dynamics we study maps with negative Schwarzian derivative, mostly because of Singer’s theorem [Singer, 1978] regarding bifurcations and ordering, and the very important property of the nonexistence of wandering intervals proved by Guckenheimer [1979] which is also related to the Denjoy theorem.

More recently, Yoccoz [1984], Sullivan [1985], Lyubich [1989], Blokh and Lyubich [1989], de Melo and van Strien [1988, 1989], the two authors of this
paper with Martens [Martens et al., 1992], found substitutes of the condition on the negativity of the Schwarzian derivative by related conditions, such as conditions on distortions, the Kőebe conditions, or the Zygmund conditions, conditions strongly related with the nonexistence of wandering intervals. The purpose of these studies was to reduce the conditions on the strong differentiability of the maps and to maintain the most important features of the negative Schwarzian derivative finding the weakest possible conditions.

There are two important properties related to the continuity of the topological entropy $h_t$ of unimodal families. The first states the continuity of $h_t$ in the space of unimodal maps with topology $C^1$. The second states the continuity of $h_t$ in the space of unimodal maps with topology $C^0$, topology of uniform convergence, if $h_t > 0$ [Misiurewicz & Szlenk, 1980] and [Misiurewicz & Shlyachkov, 1989].

Another important feature of negative Schwarzian maps is the monotone dependence of the topological entropy on the parameter of the family [Milnor & Thurston, 1988] as for example in the topological entropy on the parameter of the 1980 and [Misiurewicz & Shlyachkov, 1989].

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The Schwarzian derivative of $f_{a,b}$ is:

$$S(f_{a,b}(x)) = \frac{f'''_{a,b}(x)}{f'_{a,b}(x)} - \frac{1}{2} \left( \frac{f''_{a,b}(x)}{f'_{a,b}(x)} \right)^2 = \frac{1 - a^2}{2x^2},$$

thus we have positive Schwarzian derivative when the exponent $a < 1$, and, conversely, we have negative Schwarzian derivative when $a > 1$.

It is an easy task to prove that the functions of a $C^3$ family with positive Schwarzian derivative on the interval can have no critical point, thus the first derivative of any of these functions must be discontinuous at least at the critical point (viewed as a local maximum or minimum), as happens with $f_{a,b}$, with $0 < a < 1$. Additionally, in our family, we define the Schwarzian derivative for every function at any point of $I$, except at $x = 0$.

To study the dynamics of this family when $a > 1$, we can fix this exponent and study the behavior of the iterates and orbits varying the parameter $b$, in that case we have all the nice features of the negative Schwarzian derivative, we have the continuity of the topological entropy on the parameter $b$, the result which states that the topological entropy is a nondecreasing function of the parameter $b$ and the cascades of period doubling.

2. Parameter Space Geometry when $a < 1$

In this section we characterize the different regions in the parameter space where the Schwarzian derivative of $f_{a,b}$ is positive. As we know this happens for our map when $0 < a < 1$. We consider in this section the region:

$$Q = \{(a, b) \in \mathbb{R}^2 : 0 < a < 1, 1 < b < 2\}.$$

The first result concerns a curve in the parameter space which separates the two kinds of behavior of the dynamics of the iterates of $f_{a,b}$ in $Q$, consequently we are going to have two subregions of $Q$ each one with a particular kind of dynamics.

We use here the well-known concepts of symbolic dynamics for unimodal maps, with the usual symbols, $L$ the left half interval and $R$ the right half interval, and the usual concepts of topological entropy and topological equivalence for dynamics of dimension one.

We remember that when we have the symbolic itinerary $RL^\infty$ of the critical point of a unimodal map the dynamics of the iterates is a full shift of two symbols and the topological entropy is one. In our case we are going to see this behavior in a subregion of $Q$, which we shall call $D$.

As we see in Fig. 1 we have a fixed point $x_L$, solution of

$$f_{a,b}(x_L) = x_L,$$

in the interval $] - 1, 0[$, for some parameters in the region $Q$. We will return to condition (4) in Proposition 2.3. When $b > 1$ the first iterate of the critical point $0$ is positive, but the second iterate can be negative. When this second iterate falls in the left fixed point

$$f_{a,b}^2(0) = x_L,$$

the symbolic itinerary of the critical point is precisely $RL^\infty$. When we combine the two conditions (4) and (5) we get the system

$$\begin{cases}
    b(1 - |x_L|^a) - 1 = x_L \\
    b(1 - |b - 1|^a) - 1 = x_L
\end{cases},$$

obtaining the solution $x_L = 1 - b$, eliminating $x_L$ from the equations of the system we get

$$a = \frac{\log \left( \frac{2(b - 1)}{b} \right)}{\log(b - 1)} = \Gamma(b), \quad 1 < b < 2.$$
We can define a curve $\Omega$ parametrized by $\Gamma(b)$

$$\Omega = \{(a, b) \in Q : a = \Gamma(b) , \ 1 < b < 2\} .$$

When a pair $(a, b) \in \Omega$ the second iterate of the critical point of $f_{a,b}$ falls on the left fixed point.

The characterization of the dynamics of the iterates of $f_{a,b}$ below the curve $\Omega$, which is the first subregion considered of $Q$, is given in the two next propositions. We call this first subregion

$$D = \{(a, b) \in \mathbb{R}^2 : 0 < a < \Gamma(b), \ 1 < b < 2\} .$$

**Proposition 2.1.** When $(a, b) \in \Omega \cup D$ the dynamics of the map $f_{a,b}(x)$ is topologically equivalent to a full shift of two symbols with topological entropy 1.

**Proof.** When $(a, b) \in \Omega$ we have a direct consequence of the definition of $\Omega$, the itinerary of the critical point is $RL^\infty$.

It is easy to see that in the region $D$ the critical point 0, has also the symbolic orbit $RL^\infty$. When we fix $b$ and let $a < \Gamma(b)$, the second iterate of 0 is less than the left fixed point, and then is attracted to $-1$. The symbolic itinerary is $RL^\infty$ and we can associate with it a full shift in the alphabet $\{L, R\}$. Finally, the topological entropy of a full shift is 1.

**Proposition 2.2.** The topological entropy $h_t(f_{a,b})$ has a discontinuity when we cross transversally the segment

$$S = \{(a, b) \in \mathbb{R}^2 : b = 1, \ 0 < a < 1\} .$$

When $0 < a < 1$ with $0 < b \leq 1$ the topological entropy is 0, in region $D$ the topological entropy is 1.

**Proof.** Also obvious, for $b < 1$, the critical point lies below the diagonal in the square $I \times I$, so, for that region, the topological entropy is zero. When $b = 1$, in the stated segment, the topological entropy is also 0 because the critical point is a fixed point. When $b > 1$ the previous result assures that the topological entropy is 1, the entropy of a full shift.

Above the curve we do not have so simple statements. Now we consider the second subregion on the space of parameters above the curve

$$C = \{(a, b) \in \mathbb{R}^2 : \Gamma(b) < a < 1, \ 1 < b < 2\} .$$

In $C$ there are, typically, two kinds of geometrical behavior of $f_{a,b}$. As we can see in Fig. 1 $f_{a,b}$ can intersect the diagonal in the left half of the interval $I$ at a point other than $-1$, in Fig. 2 we see that $f_{a,b}$ does not intersect the diagonal in the left half of the interval $I$ at any point other than $-1$. We now study the two subregions of $C$ where we split the geometrical behavior of $f_{a,b}$. We are going to show in the next proposition that in these two subregions of $C$ there is only an invariant set of $I$ where the dynamics are interesting. In the case of double intersection, in the left half interval, this invariant set is a proper
subinterval of \( I \), where the iterates of \( f_{a,b} \) are confined when the initial condition is in that invariant set.

**Proposition 2.3.** When the parameters satisfy
\[
0 < a < \frac{1}{b} \text{ and } 1 < b < 2,
\]
there exists a solution \( x_L \), of
\[
f_{a,b}(x_L) = x_L,
\]
in the interval \([-1, 0[\) for every pair \((a, b)\). Moreover, given \( a \) and \( b \) satisfying
\[
\Gamma(b) < a < \frac{1}{b},
\]
the interval \( J_{a,b} = [x_L, -x_L] \) is invariant for the dynamics of \( f_{a,b} \) and the dynamics of the points in \( I \setminus J_{a,b} \) are trivial with all the iterates converging to the fixed point \(-1\).

**Proof.** In the first part we just have to remember that
\[
f_{a,b}'(x) = ab|x|^{a-1}, \text{ if } x < 0,
\]
therefore the slope of \( f_{a,b} \) is increasing with \( x \) in the left half interval, \( f_{a,b}(0) = b - 1 > 0, f_{a,b}(-1) = -1 \) and \( f_{a,b}'(-1) = ab \). If \( ab > 1 \) we cannot have any intersection of the graph of the map and the diagonal in \([-1, 0[\), but if \( ab < 1 \), we have one and only one solution \( x_L \) of (9) in \([-1, 0[\) because of the intermediate value theorem. It is easy to see that \( \Gamma(b) < 1/b \) for every \( b : 1 < b < 2 \).

The proof of the invariance is also simple. We fix \( b \), consider \( \Gamma(b) = a_0 \) and \( 1/b = a_1 \), we vary \( a \) such that \( 0 < a < a_1 \). Now, with \( b \) fixed, we use the notation \( x_L(a) \) for the solution of (9) which depends only on \( a \). When \( a = a_0 \) there is the solution \( x_L(a_0) = 1 - b \) of (9) that we know from the system (6). The interval
\[
J_{a_0,b} = [1 - b, b - 1]
\]
is clearly an invariant interval. Now we increase \( a \) from \( a_0 \), remembering that \( f_{a,b}(0) \equiv b - 1 \). It is easy to see, using the slope arguments used at the beginning of this proof, that the point \( x_L(a) \) approaches \(-1\) going to the left of \( 1 - b \) when \( a \) increases, but the second iterate of 0, \( f_{a,b}^2(0) \) (which is also negative), approaches the origin. As we know, the itinerary of the critical point is maximal, consequently all the orbits remain in \( J_{a,b} \).

Finally when we are in \( I \setminus J_{a,b} \), the point \(-1\) is an attracting fixed point. ■

In a nutshell, we can say that in the region where \( f_{a,b} \) has positive Schwarzian derivative we have basically two distinct kinds of dynamics, in region \( D \) we have full shifts, in region \( C \) subshifts in the interval \( I \) or in an invariant subinterval of \( I \), what we called \( J_{a,b} \).

**Remark 2.4.** Henceforth we denote the intervals \( J_{a,b} \) or \( I \) by \( J \) when there is no possibility of confusion, meaning the invariant interval related to the map \( f_{a,b} \), i.e. in case \( a \geq 1/b \) denoting \( I \), in case \( a < 1/b \) denoting \( J_{a,b} \).

We are interested in the details of the dynamics in region \( C \) and the bifurcations which can occur in this region are obviously in our line of interest.

The stability coefficient in region \( C \) is:
\[
|f_{a,b}'(x)| > 1,
\]
where \( x \in J \setminus \{0\} \), this coefficient has no meaning for \( x = 0 \), therefore the usual theory of bifurcations does not apply here.

To study the bifurcations we must work on the successive intersections of the critical points of \( f_{a,b}^n \) with the diagonal. As we know from relation (10) the stability coefficient is greater than 1, so there are no tangencies when those intersections occur, and the usual saddle-node and period doubling are overridden. The only possible bifurcation conditions are:
\[
\begin{aligned}
&f_{a,b}^n(x) = x, \\
&\text{sign}(f_{a,b}^n(x^+)) = -\text{sign}(f_{a,b}^n(x^-)).
\end{aligned}
\]

We remember that \( f_{a,b} \) is three times differentiable in \( I \setminus \{0\} \) and \( f_{a,b}'(0^-) = -f_{a,b}'(0^+) = +\infty \).

**Proposition 2.5.** Let \( f \) be a continuous even function on the interval \( I \), with \( f(I) \subset I \) and only one critical point at the origin where it is not differentiable. If sign \( f'''(x^-) = -\text{sign} f'''(x^+) \), then \( f'''(x^-) = -f'''(x^+) \).

**Proof.** Let \( f'(x_0^+) = \lim_{\varepsilon \to 0} f'(x_0 \pm \varepsilon) \), where \( \varepsilon \) is a small positive real parameter such that \( f \circ f \) is differentiable and monotone in \( I_\varepsilon = [-\varepsilon, 0] \) and \( f(x_0) = x_1 \). First we assume that in \( I_\varepsilon \) the function \( f \) is increasing, then we compute the left lateral derivative of \( f \circ f \):
\[
(f \circ f)'(x^-_0) = \lim_{\varepsilon \to 0} f'(f(x_0 - \varepsilon)) f'(x_0 - \varepsilon),
\]
and:
\[
\lim_{\varepsilon \to 0} f(x_0 - \varepsilon) = x_1^-.
\]
Because \( f \) is continuous and increasing we get:

\[
(f \circ f)'(x_n^-) = f'(x_1^-) f'(x_0^-).
\]

Conversely, if \( f \) is decreasing in \( I_\varepsilon \):

\[
(f \circ f)'(x^-) = f'(x_1^+) f'(x_0^-).
\]

For the right derivative, by similar reasonings:

\[
(f \circ f)'(x_0^+) = f'(x_1^+) f'(x_0^+)
\]

and by successive repetition of the same arguments:

\[
(f \circ f)'(x_0^n) = f'(x_1^+) f'(x_0^+) \cdots f'(x_n^+),
\]

the signs \( \pm \) are not necessarily the same in both lateral derivatives, with this notation, we only mean that the \( f^n \) lateral derivatives are products of lateral derivatives (right or/and left) of \( f \). As we know \( f' \) is continuous in every point of \( I \) except in \( 0 \), where \( f(0^-) = -f(0^+) \), so \( |f''(x_0^+)| = |f''(x_0^-)| \) with different signs.

**Corollary 2.6.** If

\[
\text{sign}(f''(x_0^+)) = -\text{sign}(f''(x_0^-)),
\]

then \( x_0 \) is a preimage of 0 or is 0.

**Proof.** Remembering the previous proof, 0 can be present in the sequence \( x_0, x_1, \ldots, x_{n-1} \) an odd number of times. The same result holds even if \( |f''(0^+)| \neq |f''(0^-)| \).

It is quite simple to generalize these last results even when \( f'_{a,b}(0^-) = -f'_{a,b}(0^+) = +\infty \), as in our case. Bearing in mind that we adopt the convention: \( \text{sign}(\pm\infty) = \pm 1 \), when \( \text{sign}(f''(x_0^+)) = -\text{sign}(f''(x_0^-)) \) we have \( f''(x_0^+) = \pm\infty, f''(x_0^-) = \mp\infty \) and \( x_0 \) is either 0 or a preimage of 0.

The study of the creation of fixed points of \( f_{a,b}^n \) is quite easy when \( 0 < a < 1 \), and they are only generated by transversal crossings of the preimages of 0 with the diagonal when we change the parameters, with the consequent creation or destruction of unstable orbits.

### 3. Parameter Space Ordering

In this section we give a result about the ordering of the dynamics of the map \( f_{a,b} \) depending on the region of the parameter space where \( a \) and \( b \) are taken.

In Fig. 3 we see in the parameter space the curves \( f_{a,b}^n(0) = 0 \), for small periods \( (n \leq 5) \) the topological entropy remains constant over each curve. That figure was one of the motivations of the results stated in this section.

We define four regions in our parameter space.

**Region A:**

\[
A = \{(a, b) \in \mathbb{R}^2 : 0 < a < 1, 0 < b < 1, \}
\]

\[
\cup \{(a, b) \in \mathbb{R}^2 : 1 < a, 0 < b < \phi(a)\},
\]

where \( b = \phi(a) \) is a parametrization of the curve \( \Phi \) in the parameter space where lie the Feigenbaum accumulation points, i.e. the pairs \((a, b)\) where the map \( f_{a,b} \) has the stable aperiodic orbit resulting in the limit process of period doubling of the periods \( 2^n \) for \( n \) arbitrarily large.

**Region B:**

\[
B = \{(a, b) \in \mathbb{R}^2 : a > 1, \phi(a) < b < 2\}.
\]

**Region C,** defined previously:

\[
C = \{(a, b) \in \mathbb{R}^2 : \Gamma(b) < a < 1, 1 < b < 2\},
\]

\( \Gamma(b) \) is defined in relation (6).

**Region D,** also defined previously:

\[
D = \{(a, b) \in \mathbb{R}^2 : 0 < a < \Gamma(b), 1 < b < 2\}.
\]

In Fig. 4, we see these regions in the parameter space.

**Theorem 3.1.** The topological entropy in the different regions of \((a, b)\)-parameter space.

1. In the region \( A \), the topological entropy is 0.
2. Each locus of constant topological entropy in \((a, b)\)-parameter space in the region \( B \) is a connected set. When we fix \( a \) and change \( b \) the topological entropy is a continuous and monotonous Cantor function of \( b \).
3. In the region \( C \), the loci of constant topological entropy are connected curves indexed continuously by \( s \), the growth number of \( f_{a,b} \), whose union is the region \( C \).
4. In the region \( D \) the topological entropy \( h_t \) is constant and equal to 1.

**Proof.** Result 1 is obvious, there are no fixed points in that region if \( a < 1 \) and \( b < 1 \). When \( a > 1 \) and \( b < \phi(a) \) it is well known that the topological entropy is 0.

Result 2 is a consequence of the well-known properties of unimodal maps with negative Schwarzian derivative like the map (2).

Result 4 is only a consequence of Proposition 2.1.

Let us prove the result 3. We are now confined to region \( C \). We use the definition of growth number.
Fig. 3. Curves in the parameter space corresponding to periodic orbits of the critical point (periods $n \leq 5$), the labels are the respective periods.

Fig. 4. Regions in the parameter space.

$s$ and topological entropy of an unimodal map of the interval from [Misiurewicz & Szlenk, 1980]. We use the concept of lap number $\ell(f_{a,b}^n)$, i.e. the number of maximal intervals of monotonicity of $f_{a,b}^n$:

$$s(f_{a,b}) = \lim_{n \to \infty} \left( \frac{\ell(f_{a,b}^n)}{n} \right),$$

the growth number $s$ is related to the topological entropy $h_t$ by the relation: $h_t = \log_2 s$.

When given a real constant $k$, such that:

$$1 < k < 2,$$

we claim that the condition on the growth number of $f_{a,b}$:

$$s(f_{a,b}) = k$$

(11)

gives a well defined and continuous curve $\gamma_k$, in the region $C$ of the parameter space and when we change continuously the growth number $s$ varying $k$, we can enumerate uniquely a family of curves $\gamma_k$ by $k$ which union is the region $C$. 
First we fix \( b = \beta \), \( f_{a,b} \) is continuous and all its orbits are unstable so this map is expanding, there exists a homeomorphism \( h_{a,\beta} \) [Parry, 1966], which conjugates topologically \( f_{a,\beta} \) and the tent map \( t_a = s(1 - |x|) - 1 \), if

\[
a \in [\Gamma(\beta), 1],
\]

there exists a reverse orientation homeomorphism \( \varphi_\beta \) between \( a \) and \( s \), such that:

\[
s(f_{a,\beta}) = \varphi_\beta.
\]

We strictly increase the growth number \( s \) when we decrease \( a \).

Observation: We never have bifurcations of period doubling which preserve the growth number and the kneading invariant when we change \( a \) in the region \( C \).

If \( a = 1 \) we have the tent map \( f_{1,\beta}(x) = \beta(1 - |x|) - 1 \), in that case the topological conjugacy is the identity map and \( s(f_{1,\beta}(x)) = \varphi_\beta(1) = \beta \) is the minimum value of the growth number for \( a \) in the stated interval. The maximum \( \varphi_\beta(a) = 2 \) is obtained when \( a = \Gamma(b) \) as we showed before. Therefore the condition (11) defines a unique solution \( a \) for each \( b \) fixed, if \( b < k < 2 \).

We remember that in region \( C \) the topological entropy \( h_t \) is positive: \( h_t = \log_2 s > \log_2 b > 0 \), for each \( b > 1 \) fixed, and \( f_{a,b} \) is a continuous bounded unimodal family, from the results stated in [Misiurewicz & Shlyachkov, 1989] we conclude that the topological entropy \( h_t \) is continuous and the growth number \( s \) is also continuous on every map \( f_{a,b} \) in \( C \) and, obviously, continuous in every point of region \( C \).

Finally, because Eq. (11) has a unique solution \( a \) for each \( b \) fixed, and \( s \) is continuous on every point \( (a, b) \in C \), we have an implicit function \( a_k(b) \) for each \( 1 < k < 2 \), it is easy to see that this function is continuous on \( b \). Using \( a_k(b) \) we can parametrize the solutions of (11):

\[
\gamma_k = \{(a, b) \in C : a = a_k(b), \ 1 < b < 2\}. \quad \blacksquare
\]

**Remark 3.2.** Each one of the curves \( \gamma_k \) intersects the segment:

\[
\{(a, b) \in \mathbb{R}^2 : a = 1, 1 < b < 2\}
\]

exactly at the point \((1, k)\).

**Remark 3.3.** Given any point \((a, b)\) of the region \( C \) there exists one and only one \( k : 1 < k < 2 \), and a curve \( \gamma_k \) solution of (11) such that \((a, b) \in \gamma_k\).

**Corollary 3.4.** All the curves \( \gamma_s \) have an accumulating point in \( a = 1, b = 1 \), where a generalized and degenerated bifurcation occurs.

**Proof.** The result about the degenerated bifurcation when \( a = 1, b = 1 \), follows from the study of the curve \( a = \Gamma(b) \) the \( \lim_{b \to 1^+} \frac{\log(2(b - 1)/b)}{\log(b)} = 1^- \), all the solutions of \( f_{a,b}^n(0) = 0 \) in region \( C \) accumulate at this point. \( \blacksquare \)

**Remark 3.5.** In Fig. 4 we see the curves separating some global properties of the dynamics in each region. We summarize here what happens when the parameters belong to those curves. We have seen in Sec. 2 that over the segment \( S \) the topological entropy is 0 and over the curve \( \Omega \) the topological entropy is 1. When \( a = 1 \) and \( 1 < b < 2 \) we have the tent map with slope \( b \) with well-known behavior. Finally over the curve \( \Phi \) it is well known that the topological entropy is 0, the final stage of the Feigenbaum period doubling process.

4. Conclusions

In our study of \( f_{a,b} \) we find that one parameter is not enough to a full understanding of positive Schwarzian derivative maps. The usual ordering, which occurs in region \( B \), relative to the parameter \( b \), fixed \( a \), does not exist in regions \( C \) and \( D \). There is another ordering of the dynamics of \( f_{a,b} \) relative to \( a \), fixed \( b \), when \( S(f_{a,b}) > 0 \). More precisely: for \( 0 < a < 1 \) when we fix \( a \) and change \( b \) as usual in unimodal families with negative Schwarzian derivative we do not have the ordering of the bifurcations with the parameter \( b \), nor the monotone dependence of the topological entropy with the parameter, but when we fix \( b \) and change \( a \) we have an ordering of the bifurcations and a monotone dependence of the topological entropy with \( a \). This is illustrated in Fig. 3 and is a consequence of the result of Sec. 3.

Some important quantities like the Feigenbaum “constant” \( \delta(a) \), change with the parameter \( a \), the exponent in our map \( f_{a,b} \), when \( a > 1 \), showing the essential role of the two parameters even in the region with negative Schwarzian derivative.

We recall the definition of Feigenbaum constant: for \( a \) fixed, let \( b_n \) be the value of the parameter \( b \) where occurs a period doubling bifurcation with period \( 2^n \). The Feigenbaum constant for that \( a \) is:

\[
\delta(a) = \lim_{n \to +\infty} \frac{b_n - b_{n-1}}{b_{n+1} - b_n}.
\]
When we increase the exponent $a$ this number increases, we obtain the Feigenbaum “constant” as a function of $a$ which we call Feigenbaum function $\delta(a)$.

In Fig. 5 we see an experimental curve of the variation of the Feigenbaum function $\delta(a)$, curve $c_1$, obtained by varying the parameter $a$.

In our numerical computations we used:

$$\delta(a) \simeq \frac{b_4 - b_3}{b_5 - b_4}$$

with an error $\Delta b_k < 10^{-7}$ in the determination of $b_3$, $b_4$ and $b_5$.

Near $a = 1$, the convergence of our numerical computations is difficult, but we can complete the curve in Fig. 5 with the result of Collet et al. [1980] that states:

$$\lim_{a \to 1} \delta(a) = 2.$$ 

Several numeric tests and Fig. 5, are the basis for the experimental fact that follows.

The Feigenbaum function $\delta(a)$ is a monotone increasing function of $a$, such that:

$$3\log(a) + 2 < \delta(a) < 5\sqrt{a} - 1 + 2.$$ 

This result must be viewed as an asymptotic behavior of $\delta(a)$, and for us is a hint to further study this subject. To see more about the Feigenbaum constant, in the case of asymmetric unimodal maps see for instance the works of de Melo and van Strien [1998, 2000].

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