Proper Cartan Groupoids

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Basic concepts

- A pseudo-representation of a Lie groupoid $\Gamma \overset{s}{\underset{t}{\rightrightarrows}} M$ on a vector bundle $E \to M$ is an arbitrary vector-bundle morphism $\lambda : s^*E \to t^*E$.
- A connection on a Lie groupoid $\Gamma \overset{s}{\underset{t}{\rightrightarrows}} M$ is an Ehresmann connection $H \subset T\Gamma$ along the source fibration $\Gamma \xrightarrow{s} M$. Let η^H denote the corresponding horizontal lift

$$\begin{tikzcd} 0 \arrow[r] & \ker ds \arrow[r, "\subset"] & T\Gamma \arrow[r, "ds"] & s^*TM \arrow[r] \arrow[l, bend left, "\eta^H"] & 0 \end{tikzcd}$$

- The effect λ^H of a groupoid connection $H \subset T\Gamma$ is the pseudo-representation of $\Gamma \rightrightarrows M$ on TM given by $dt \circ \eta^H$:

$$s^*TM \xrightarrow{\eta^H} T\Gamma \xrightarrow{dt} t^*TM$$

Basic concepts

- The tangent groupoid of a Lie groupoid $\Gamma \underset{t}{\overset{s}{\rightrightarrows}} M$ is the Lie groupoid $T\Gamma \underset{Tt}{\overset{Ts}{\rightrightarrows}} TM$ whose structure maps are obtained by differentiating those of $\Gamma \rightrightarrows M$.
- A connection $H \subset T\Gamma$ on $\Gamma \rightrightarrows M$ is multiplicative if it constitutes a subgroupoid of the tangent groupoid $T\Gamma \rightrightarrows TM$. Equivalently, H is multiplicative iff

$$\begin{aligned} \eta^H_{1x} &= T_x 1 && \forall x \in M \\ \eta^H_{g'g} v &= (\eta^H_{g'} \lambda^H_g v)\eta^H_g v && \forall (g', g) \in \Gamma\,{}_s\!\times_t \Gamma, v \in T_{sg} M \end{aligned}$$

- Multiplicative connections are effective, that is to say, their effects are *representations*.
- A Cartan groupoid is a Lie groupoid endowed with a multiplicative connection.

Relation to classical Cartan geometries

- The subgroupoid of $GL(TM) \rightrightarrows M$ consisting of all tangent space isometries for a given Riemannian metric carries a canonical multiplicative connection. This Cartan groupoid is always proper.
- G-structures on a manifold M, that is, reductions $P \to Fr(M)$ of the general linear frame bundle from $GL(\mathbb{R}^n)$ to a subgroup G, correspond to tangent representations $\lambda : \Gamma \to GL(TM)$ of transitive Lie groupoids $\Gamma = (P \times P)/G \rightrightarrows M$. The case Γ proper generalizes the case G compact.
- Cartan geometries $(P \to M, \omega)$ modeled on G/H correspond to transitive Cartan groupoids $(P \times P)/H$ over M. These groupoids are proper whenever the closed subgroup $H \subset G$ is compact.

Motivating questions

1. When does a given (proper) Lie groupoid admit Cartan connections? Better, given a (proper) Lie groupoid, decide whether it admits Cartan connections or not.
2. When can two Cartan connections on a given (proper) Lie groupoid be deformed into each other through Cartan connections?

Examples

1) Action groupoids

- Connections on $G \ltimes M \rightrightarrows M$ are in one-to-one correspondence with Lie algebra valued maps $pr_M^* TM \to \mathfrak{g}$.
- A connection H on $G \ltimes M \rightrightarrows M$ is *multiplicative* iff the corresponding Lie algebra valued map $X^H : pr_M^* TM \to \mathfrak{g}$ satisfies the following *cocycle conditions:*

$$
\begin{aligned}
X^H_{1,x} &= 0 \\
\mathrm{Ad}_G(g) \circ X^H_{h,x} - X^H_{gh,x} + X^H_{g,hx} \circ \lambda^H_{h,x} &= 0
\end{aligned}
$$

In particular the unique H for which $X^H = 0$ is multiplicative.

2) Compact abelian Lie group bundles

- Any bundle of compact abelian Lie groups $\varGamma \xrightarrow{s=t} M$ admits a unique multiplicative connection (necessarily flat).

Examples

3) Pair groupoids

- The pair groupoid $M \times M \rightrightarrows M$ over a manifold M admits multiplicative connections iff M is parallelizable (multiplicative connections on $M \times M \rightrightarrows M$ corresponding to trivializations of the tangent bundle TM).

4) Regular Lie groupoids

- Any longitudinal tangent representation $\alpha : \Gamma \to GL(\Lambda)$ of a regular Lie groupoid $\Gamma \rightrightarrows M$ (representation on the regular distribution $\Lambda \subset TM$ tangent to the orbits) is the restriction to Λ of the effect of some *effective* connection on $\Gamma \rightrightarrows M$.
- Any regular Lie groupoid whose longitudinal tangent bundle is trivializable admits effective connections. In particular, any transitive Lie groupoid whose base is parallelizable admits effective connections.

Examples

5) Proper regular groupoids of rank one

Let $\Gamma \rightrightarrows M$ be a proper regular Lie groupoid of rank $(= \mathrm{rk}\, \Lambda =$ *orbit dimension*$) = 1$. Suppose that $\Gamma \rightrightarrows M$ is *source connected*. Then the space of all multiplicative connections on $\Gamma \rightrightarrows M$ is non-empty and smoothly contractible.

6) Proper regular groupoids of rank two

Let $\Gamma \rightrightarrows M$ be a regular Lie groupoid of rank $= 2$. Suppose that $\Gamma \rightrightarrows M$ is *source proper* and *source simply connected*. Then the space of all multiplicative connections on $\Gamma \rightrightarrows M$ is smoothly contractible, and it is non-empty iff the two conditions below are satisfied at every base point $x \in M$:

- the second homotopy group $\pi_2(\Gamma^x)$ is trivial, where Γ^x denotes the source fiber at x
- $\pi_1(c_h) = -id \in Aut(\pi_1(\Gamma_x^x))$ for any $h \in \Gamma_x^x \smallsetminus \mathring{\Gamma}_x^x$, where $\mathring{\Gamma}_x^x$ denotes the identity component of the isotropy group Γ_x^x, and $c_h \in Aut(\Gamma_x^x)$ denotes conjugation by h.

The averaging operator

Remark

A unital connection H on $\Gamma \rightrightarrows M$ is multiplicative iff for every pair of arrows g, h with $sg = sh$ and for every $v \in T_{sg=sh}M$

$$\eta^H_{gh^{-1}} \lambda^H_h v = \eta^H_g v (\eta^H_h v)^{-1}$$

Let $\eta^H_g \div \eta^H_h$ denote the linear map $v \mapsto \eta^H_g v (\eta^H_h v)^{-1}$.

Definition

Under the hypotheses that H be non-degenerate, i.e., that its effect λ^H be an isomorphism, and that $\Gamma \rightrightarrows M$ be proper, we set

$$\hat{\eta}^H_g := \int_{tk=sg} (\eta^H_{gk} \div \eta^H_k) \circ (\lambda^H_k)^{-1} \, dk$$

using an arbitrary normalized left Haar system on $\Gamma \rightrightarrows M$. We refer to the global cross-section $\hat{\eta}^H$ of the vector bundle $L(s^* TM, T\Gamma)$ as the multiplicative average of H.

The averaging operator

The multiplicative average $\hat{\eta}^H$ *of* H *is itself the horizontal lift for a unique connection* $\hat{H}$ *on* $\Gamma \rightrightarrows M$*, which is always unital with effect*

$$\lambda_g^{\hat{H}} = \int_{tk=sg} \lambda_{gk}^H \circ (\lambda_k^H)^{-1}\, dk$$

Proposition

The multiplicative average $\hat{\Phi}$ *(relative to any choice of normalized Haar systems) of an effective (hence non-degenerate) connection* Φ *is a multiplicative connection. When* Φ *is multiplicative,* $\hat{\Phi} = \Phi$*.*

Corollary

Any proper Lie group bundle (more in general any proper Lie groupoid whose associated Lie algebroid has zero anchor map) admits multiplicative connections.
Every proper regular Lie groupoid whose longitudinal tangent bundle is trivializable (in particular every transitive such groupoid over a parallelizable base) admits multiplicative connections.

Averaging pseudo-representations

For a generic non-degenerate connection Φ, we want to consider the sequence of connections $\hat{\Phi}, \hat{\hat{\Phi}}, \ldots$ obtained by repeatedly averaging Φ (provided this is defined) and understand its limiting behavior.

Definition

For an arbitrary invertible pseudo-representation $\lambda : s^*E \xrightarrow{\sim} t^*E$ of a proper Lie groupoid $\Gamma \rightrightarrows M$ on a vector bundle $E \to M$, we set

$$\hat{\lambda}_g = \int_{tk=sg} \lambda_{gk} \circ (\lambda_k)^{-1} \, dk$$

Averaging pseudo-representations

We endow E with some metric, and set

$$b(\lambda) = \sup_g \|\lambda(g)\|$$
$$c(\lambda) = \sup_{(g',g)} \|\lambda(g'g) - \lambda(g') \circ \lambda(g)\|$$

Estimates

Suppose that λ is *unital* and that $c(\lambda) < 1$. Then

- λ must be invertible
- $\|\hat{\lambda}(g)\| \leq \dfrac{b(\lambda)}{1 - c(\lambda)}$
- $\|\hat{\lambda}(g'g) - \hat{\lambda}(g') \circ \hat{\lambda}(g)\| \leq 2\dfrac{b(\lambda)^2}{(1 - c(\lambda))^2} c(\lambda)^2$

Averaging pseudo-representations

Fast Convergence Lemma

Let $\{b_0, b_1, \dots, b_l\}$ *and* $\{c_0, c_1, \dots, c_l\}$ *be finite sequences of non-negative real numbers. Suppose that*

$$c_i < 1 \Rightarrow \begin{cases} b_{i+1} \le \dfrac{b_i}{1 - c_i} & \textit{and} \\ c_{i+1} \le 2\left(\dfrac{b_i}{1 - c_i}\right)^2 c_i^2 \end{cases}$$

Also suppose that $b_0 \ge 1$ *and that* $\varepsilon := 6b_0^2 c_0 \le \frac{2}{3}$. *Then, the following inequalities hold for every index* i.

$$b_i \le \sqrt{3} b_0$$

$$c_i \le \frac{\varepsilon^{2^i}}{6b_0^2}$$

Averaging pseudo-representations

Definition

We call a *unital* pseudo-representation λ of $\Gamma \rightrightarrows M$ on $E \to M$ a near representation if for each point in M it is possible to find an invariant open neighborhood $U = \Gamma U$ with the property that the inequality below holds for some choice of metrics on $E \mid U$

$$c(\lambda \mid U) \leq \frac{1}{9}\, b(\lambda \mid U)^{-2}$$

We call a connection Φ on $\Gamma \rightrightarrows M$ nearly effective if its effect λ^{Φ} is a near representation.

Remark

Near representations λ are always invertible. Their multiplicative averages $\hat{\lambda}$ are themselves near representations.

Fast Convergence Theorem

Any near representation λ gives rise to a whole sequence of averaging iterates $\hat{\lambda}^{(i)}$ constructed recursively by setting $\hat{\lambda}^{(0)} = \lambda$ and $\hat{\lambda}^{(i+1)} = (\hat{\lambda}^{(i)})^\wedge$.

Theorem

Let $\Gamma \rightrightarrows M$ be a proper Lie groupoid. Let λ be a unital pseudo-representation of $\Gamma \rightrightarrows M$ on some vector bundle $E \to M$. Suppose that λ is nearly multiplicative. Then the sequence of averaging iterates of λ

$$\hat{\lambda}^{(0)} = \lambda,\ \hat{\lambda}^{(1)} = \hat{\lambda},\ \ldots,\ \hat{\lambda}^{(i+1)} = (\hat{\lambda}^{(i)})^\wedge,\ \ldots$$

converges uniformly on compact sets, together with its derivatives of any order, towards a unique smooth representation $\hat{\lambda}^{(\infty)}$ of $\Gamma \rightrightarrows M$ on E.

Fast Convergence Theorem

For any non-degenerate connection Φ, we have

$$\lambda^{\hat{\Phi}} = (\lambda^{\Phi})^{\wedge}$$

Thus, any nearly effective connection Φ gives rise by recursive averaging to a sequence of nearly effective connections $\hat{\Phi}^{(i)}$.

Theorem

Let $\Gamma \rightrightarrows M$ be a proper Lie groupoid. Let Ψ be any nearly effective connection on $\Gamma \rightrightarrows M$. The sequence of averaging iterates of Ψ

$$\hat{\Psi}^{(0)} = \Psi,\ \hat{\Psi}^{(1)} = \hat{\Psi},\ \ldots,\ \hat{\Psi}^{(i+1)} = (\hat{\Psi}^{(i)})^{\wedge},\ \ldots$$

converges uniformly on compact sets, together with its derivatives of any order, towards a unique multiplicative connection $\hat{\Psi}^{(\infty)}$ on $\Gamma \rightrightarrows M$.

Reduction to the regular case

Theorem (Corollary of the Fast Convergence Theorem)

Suppose that $\Gamma \rightrightarrows M$ is a Lie groupoid which is proper. Let C be an invariant closed subset of M, let U be an open neighborhood of C, and let Z be an invariant differentiable submanifold of M. Let Φ be a multiplicative connection on $\Gamma \mid U \rightrightarrows U$, and let Θ be a multiplicative connection on $\Gamma \mid Z \rightrightarrows Z$ whose restriction over $Z \cap U$ coincides with the connection induced by Φ on $\Gamma \mid U \cap Z \rightrightarrows U \cap Z$. Then, there exists some open neighborhood V of $C \cup Z$ and some multiplicative connection Ψ on $\Gamma \mid V \rightrightarrows V$ that induces Θ along Z and that agrees with Φ over some open neighborhood of C within $U \cap V$.

Reduction to the regular case

Corollary

Let $\Gamma \rightrightarrows M$, C, U and Z be as in the statement of the preceding theorem, and let Φ_0 and Φ_1 be multiplicative connections on $\Gamma \rightrightarrows M$. Let $\{\Phi_t\}$ be an isotopy through multiplicative connections on $\Gamma \,|\, U \rightrightarrows U$ that connects the restrictions over U of Φ_0 and Φ_1. Let $\{\Theta_t\}$ be a similar isotopy on $\Gamma \,|\, Z \rightrightarrows Z$ joining $\Theta_0 = \Phi_0 \,|\, Z$ and $\Theta_1 = \Phi_1 \,|\, Z$ with the property that for every t the restriction of Θ_t over $Z \cap U$ coincides with $\Phi_t \,|\, U \cap Z$. Then, over some open neighborhood V of $C \cup Z$, the restrictions of Φ_0 and Φ_1 can be connected by means of an isotopy through multiplicative connections on $\Gamma \,|\, V \rightrightarrows V$.

Main application

C = union of all orbits of dimension strictly smaller than q
Z = union of all orbits of dimension equal to q