Reduced smooth stacks?

Giorgio Trentinaglia

CAMGSD, Instituto Superior Técnico, Lisbon

PSSL 97
Louvain-la-Neuve, Belgium
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Groupoid Small category in which all morphisms are invertible

$\mathit{\Gamma} \overset{s}{\underset{t}{\rightrightarrows}} M$ $M =$ set of *objects*/*base points*

$\mathit{\Gamma} =$ set of *morphisms*/*arrows*

$s,\ t =$ *domain*/*source* map, *codomain*/*target* map

For $x, y \in M$

$$\mathit{\Gamma}^x := \mathit{\Gamma}(x, -) := s^{-1}(x) = \{g \in \mathit{\Gamma} \mid s(g) = x\}$$

$$\mathit{\Gamma}_y := \mathit{\Gamma}(-, y) := t^{-1}(y) = \{g \in \mathit{\Gamma} \mid t(g) = y\}$$

$$\mathit{\Gamma}^x_y := \mathit{\Gamma}(x, y) := \mathit{\Gamma}^x \cap \mathit{\Gamma}_y$$

Composition	$g'g$	[whenever $s(g') = t(g)$]
Unit/identity at x	1_x or just x	$[s(1_x) = t(1_x) = x]$
Inverse	g^{-1}	$[s(g^{-1}) = t(g),\ t(g^{-1}) = s(g)]$

Smooth manifold Locally compact manifold of class C^∞ of constant dimension whose topology is Hausdorff and possesses a countable basis of open sets

Lie groupoid Groupoid $\Gamma \overset{s}{\underset{t}{\rightrightarrows}} M$ whose set of base points M and set of arrows Γ are each equipped with a smooth manifold structure so that

- s and t are surjective submersions (in particular C^∞ maps)
  [$\Rightarrow$ the fiber product $\Gamma\, {}_s\!\times_t \Gamma = \{(g', g) \in \Gamma \times \Gamma \mid s(g') = t(g)\}$ is a smooth manifold]
- composition $\Gamma\, {}_s\!\times_t \Gamma \to \Gamma$
  unit $M \to \Gamma$
  inversion $\Gamma \to \Gamma$
  are maps of class C^∞

Fundamental structure theorem

Let $\mathit{\Gamma} \rightrightarrows M$ be a Lie groupoid. Then $\forall x, y \in M$

- The source fiber $\mathit{\Gamma}^x = s^{-1}(x)$ is a smooth submanifold of $\mathit{\Gamma}$
- $\mathit{\Gamma}(x, y)$ is a smooth submanifold of $\mathit{\Gamma}$ [obviously closed since it equals $s^{-1}(x) \cap t^{-1}(y)$]
- The *isotropy/vertex* group $G_x = \mathit{\Gamma}^x_x = \mathit{\Gamma}(x, x)$ has a canonical Lie group structure
- Composition of arrows restricts to a free smooth action of G_x on $\mathit{\Gamma}^x$ from the right
- The quotient set $\mathit{\Gamma}^x/G_x$ admits a unique smooth structure making the canonical projection $pr_x : \mathit{\Gamma}^x \twoheadrightarrow \mathit{\Gamma}^x/G_x$ into a submersion

We set $O_x = \mathit{\Gamma}^x/G_x$ and call this the *orbit* through x. The data

$$pr_x : \mathit{\Gamma}^x \twoheadrightarrow O_x \qquad \mathit{\Gamma}^x \circlearrowleft G_x$$

constitute a principal right G_x-bundle over O_x

There is a unique map $in_x : O_x \to M$ such that the following diagram commutes ($t^x =$ restriction of the target map to $\mathit{\Gamma}^x$)

$$\begin{tikzcd}
\mathit{\Gamma}^x \arrow[r, "t^x"] \arrow[d, two heads, "pr_x"'] & M \\
O_x \arrow[ur, tail, dashed, "in_x"'] &
\end{tikzcd}$$

This map is necessarily of class C^∞, injective, and immersive

We define

$$T_x^{\overset{\leftrightarrow}{\frown}} \mathit{\Gamma}_{(0)} := \mathrm{im}(T_{[x]} in_x) \subset T_x M$$

(*longitudinal tangent space* at x)

$$T_x^{\overset{\updownarrow}{\frown}} \mathit{\Gamma}_{(0)} := T_x M \; / \; T_x^{\overset{\leftrightarrow}{\frown}} \mathit{\Gamma}_{(0)}$$

(*transversal tangent space* at x)

Effect of an arrow

1. Let an arrow $g \in \Gamma(x, x')$ be given
2. Consider an arbitrary C^∞ local section $\gamma : U \hookrightarrow \Gamma$ to the source map $s : \Gamma \twoheadrightarrow M$ through $g = \gamma(x)$
3. The tangent linear map $T_x(t \circ \gamma) : T_x M \to T_{x'} M$ must carry $T_x^{\leftrightarrow} \Gamma_{(0)} \subset T_x M$ into $T_{x'}^{\leftrightarrow} \Gamma_{(0)} \subset T_{x'} M$
4. Hence $T_x(t \circ \gamma)$ must induce a well-defined linear map between the transversal tangent spaces $T_x^{\updownarrow} \Gamma_{(0)}$ and $T_{x'}^{\updownarrow} \Gamma_{(0)}$
5. Let this last map be indicated by ε_x^γ (provisionally)

Lemma

The linear map $\varepsilon_x^\gamma : T_x^{\updownarrow} \Gamma_{(0)} \to T_{x'}^{\updownarrow} \Gamma_{(0)}$ *does not depend on the choice of a local source section* γ *through* $g \in \Gamma(x, x')$

Definition

We set $\varepsilon(g) := \varepsilon_x^\gamma$ and call this the (*infinitesimal*) *effect* of g
Note that $\varepsilon(g'g) = \varepsilon(g') \circ \varepsilon(g)$ and $\varepsilon(1_x) = id$

Ineffective isotropy

Lemma

The correspondence that to each arrow $g \in \Gamma_x^x$ associates its effect $\varepsilon(g)$ gives rise to a Lie group homomorphism

$$\varepsilon_x : \Gamma_x^x \to GL(T_x^{\pitchfork}\Gamma_{(0)})$$

Definition

We shall refer to the closed subgroup

$$\dot{\Gamma}_x^x := \ker \varepsilon_x \subset \Gamma_x^x$$

as the *ineffective* isotropy group of Γ at x

A *homomorphism* of Lie groupoids is a C^∞ functor
$$\begin{tikzcd} \Gamma \arrow[r, "\phi"] \arrow[d, shift left] \arrow[d, shift right] & \Delta \arrow[d, shift left] \arrow[d, shift right] \\ M \arrow[r, "f"] & N \end{tikzcd}$$

For each $x \in M$ there is a unique map $O_x^\phi : O_x^\Gamma \to O_{f(x)}^\Delta$ such that

$$\begin{tikzcd} \Gamma^x \arrow[rrr, "\phi^x"] \arrow[dr, "t^x"] \arrow[dd, two heads, "pr_x^\Gamma"'] & & & \Delta^{f(x)} \arrow[dr, "t^{f(x)}"] \arrow[dd, two heads, "pr_{f(x)}^\Delta"'] & \\ & M \arrow[rrr, "f"] & & & N \\ O_x^\Gamma \arrow[ur, hook, "in_x^\Gamma"'] \arrow[rrr, dashed, "O_x^\phi"] & & & O_{f(x)}^\Delta \arrow[ur, hook, "in_{f(x)}^\Delta"'] & \end{tikzcd}$$

1. This map is necessarily C^∞
2. The tangent map of f at x, $T_x f : T_x M \to T_{f(x)} N$, carries
   $$T_x^{\leftrightarrow} \Gamma_{(0)} \subset T_x M \qquad \text{into} \qquad T_{f(x)}^{\leftrightarrow} \Delta_{(0)} \subset T_{f(x)} N$$
3. Therefore $T_x f$ yields a well-defined linear map
   $$T_x^{\updownarrow} \phi_{(0)} : T_x^{\updownarrow} \Gamma_{(0)} \to T_{f(x)}^{\updownarrow} \Delta_{(0)}$$

Lemma

The following diagram commutes for every isotropic arrow $g \in \Gamma_x^x$

$$\begin{tikzcd}
T_x^{\updownarrow}\Gamma_{(0)} \arrow[r, "\varepsilon_x^{\Gamma}(g)"] \arrow[d, "T_x^{\updownarrow}\phi_{(0)}"] & T_x^{\updownarrow}\Gamma_{(0)} \arrow[d, "T_x^{\updownarrow}\phi_{(0)}"] \\
T_{f(x)}^{\updownarrow}\Delta_{(0)} \arrow[r, "\varepsilon_{f(x)}^{\Delta}(\phi(g))"] & T_{f(x)}^{\updownarrow}\Delta_{(0)}
\end{tikzcd}$$

Proposition

The following implications hold for any homomorphism of Lie groupoids $\phi : \Gamma \to \Delta$ *for every base point* x *of* Γ

1. *If the map* $T_x^{\updownarrow}\phi_{(0)}$ *is* **surjective** *then* $\phi(\dot{\Gamma}_x^x) \subset \dot{\Delta}_{\phi x}^{\phi x}$
2. *If the map* $T_x^{\updownarrow}\phi_{(0)}$ *is* **injective** *then*
$$\phi^{-1}(\dot{\Delta}_{\phi x}^{\phi x}) \cap \Gamma_x^x \subset \dot{\Gamma}_x^x$$
3. *If the map* $T_x^{\updownarrow}\phi_{(0)}$ *is* **bijective** *then* $\forall g \in \Gamma_x^x$
$$g \in \dot{\Gamma}_x^x \Leftrightarrow \phi(g) \in \dot{\Delta}_{\phi x}^{\phi x}$$

Weak equivalences

A homomorphism of Lie groupoids
$$\begin{tikzcd} \Gamma \arrow[r, "\phi"] \arrow[d, shift left] \arrow[d, shift right] & \Delta \arrow[d, shift left] \arrow[d, shift right] \\ M \arrow[r, "f"] & N \end{tikzcd}$$
is (*completely*) *transversal* if the following map is a (surjective) submersion

$$s^{\Delta} \circ pr_2 : M\ {}_f\!\times_t \Delta \to N$$

ϕ is a *weak equivalence* if, in addition, the square diagram

$$\begin{tikzcd} \Gamma \arrow[r, "\phi"] \arrow[d, "(s,t)"] & \Delta \arrow[d, "(s,t)"] \\ M \times M \arrow[r, "f \times f"] & N \times N \end{tikzcd}$$

is a pullback within the category of smooth manifolds

Proposition

Let $\phi : \Gamma \to \Delta$ *be a transversal homomorphism of Lie groupoids. Then for each base point* x *of* Γ *the linear map* $T^{\ddagger}_{x}\phi_{(0)} : T^{\ddagger}_{x}\Gamma_{(0)} \to T^{\ddagger}_{\phi x}\Delta_{(0)}$ *is surjective.*

Proposition

Let $\phi : \Gamma \to \Delta$ *be a weak equivalence of Lie groupoids. Then for each base point* x *of* Γ *the linear map* $T^{\ddagger}_{x}\phi_{(0)} : T^{\ddagger}_{x}\Gamma_{(0)} \to T^{\ddagger}_{\phi x}\Delta_{(0)}$ *is bijective.*

Corollary

For any weak equivalence of Lie groupoids $\phi : \Gamma \to \Delta$ *and for any base point* x *of* Γ*, the Lie group isomorphism* $\phi^{x}_{x} : \Gamma^{x}_{x} \xrightarrow{\sim} \Delta^{\phi x}_{\phi x}$ *establishes a bijection between the ineffective subgroup* $\dot{\Gamma}^{x}_{x}$ *of* Γ^{x}_{x} *and the ineffective subgroup* $\dot{\Delta}^{\phi x}_{\phi x}$ *of* $\Delta^{\phi x}_{\phi x}$*.*

Definition

We say that a Lie groupoid $\Gamma \rightrightarrows M$ is **effective** if $\dot{\Gamma} = 1$ (that is to say $\dot{\Gamma}_x^x = \{1_x\}$ for every $x \in M$)

Ideally we would like to be able to associate an effective Lie groupoid $\Gamma/\dot{\Gamma} \rightrightarrows M$ to each Lie groupoid $\Gamma \rightrightarrows M$ (in a functorial way)

Problem: The quotient $\Gamma/\dot{\Gamma}$ is not smooth in general

Proposition

Let
$$\begin{tikzcd} \Gamma \arrow[r, "\phi"] \arrow[d, shift left] \arrow[d, shift right] & \Delta \arrow[d, shift left] \arrow[d, shift right] \\ M \arrow[r, "f"] & N \end{tikzcd}$$
*be a homomorphism of Lie groupoids which is transversal and full (as an abstract functor). Then ϕ is automatically C^∞-**full**:*

$$\begin{tikzcd} U' \arrow[r, dashed, two heads, "c"] \arrow[drr, dashed, "g"'] & U \arrow[drr, bend left, "h"] \arrow[ddr, bend right, "{(x,x')}"'] & & \\ & & \Gamma \arrow[r, "\phi"] \arrow[d, "{(s,t)}"] & \Delta \arrow[d, "{(s,t)}"] \\ & & M \times M \arrow[r, "f \times f"] & N \times N \end{tikzcd}$$

Corollary

Let $\phi : \Gamma \to \Delta$ be a completely transversal homomorphism of Lie groupoids which is fully faithful (as an abstract functor). Then ϕ is a weak equivalence.

Recall that a Lie groupoid $\Gamma \rightrightarrows M$ gives rise to an associated **orbit space** M/Γ

Lemma

Any full, completely transversal homomorphism of Lie groupoids induces a homeomorphism between the associated orbit spaces.

We say that a homomorphism of Lie groupoids $\phi : \Gamma \to \Delta$ is **faithfully** transversal if (it is transversal and) for every point x in the base of Γ the linear map $T_x^{\ddagger}\phi_{(0)} : T_x^{\ddagger}\Gamma_{(0)} \to T_{\phi x}^{\ddagger}\Delta_{(0)}$ is injective (hence bijective)

Lemma

Any full and transversal homomorphism of Lie groupoids is faithfully transversal.

For each base point x of $\Gamma \rightrightarrows M$ let us put

$$\overset{\circlearrowright}{\Gamma}{}^x_x := \Gamma^x_x \,/\, \dot{\Gamma}^x_x$$

We shall refer to the effective $\overset{\circlearrowright}{\Gamma}{}^x_x$ space $T^{⇞}_x \Gamma_{(0)}$ as the **effective infinitesimal model** for Γ at x $\qquad (\overset{\circlearrowright}{\Gamma}{}^x_x, T^{⇞}_x \Gamma_{(0)})$

A transversal $\phi : \Gamma \to \Delta$ must induce a Lie group homomorphism $\overset{\circlearrowright}{\phi}{}^x_x : \overset{\circlearrowright}{\Gamma}{}^x_x \to \overset{\circlearrowright}{\Delta}{}^{\phi x}_{\phi x}$

which is injective whenever ϕ is *faithfully* transversal

A C^∞-full $\phi : \Gamma \to \Delta$ must induce an epimorphism of Lie groups $\phi^x_x : \Gamma^x_x \twoheadrightarrow \Delta^{\phi x}_{\phi x}$

Thus when ϕ is (C^∞-)full and transversal we have an isomorphism

$$(\overset{\circlearrowright}{\phi}{}^x_x, T^{⇞}_x \phi_{(0)}) : (\overset{\circlearrowright}{\Gamma}{}^x_x, T^{⇞}_x \Gamma_{(0)}) \xrightarrow{\ \sim\ } (\overset{\circlearrowright}{\Delta}{}^{\phi x}_{\phi x}, T^{⇞}_{\phi x} \Delta_{(0)})$$

Natural congruences

For any Lie groupoid $\mathit{\Delta} \rightrightarrows N$ the notation

'$h_1 \equiv h_2 \pmod{\dot{\mathit{\Delta}}}$' will mean

$$\text{'}sh_1 = sh_2 = y,\ th_1 = th_2 \text{ and } h_2^{-1}h_1 \in \dot{\mathit{\Delta}}_y^y\text{'}$$

The binary relation $\equiv \pmod{\dot{\mathit{\Delta}}}$ thus defined on the arrows of $\mathit{\Delta}$ is a categorical congruence on $\mathit{\Delta} \rightrightarrows N$

Definition

Let $\mathit{\Gamma} \underset{\psi}{\overset{\phi}{\rightrightarrows}} \mathit{\Delta}$ be a pair of homomorphisms between two given Lie groupoids $\mathit{\Gamma} \rightrightarrows M$ and $\mathit{\Delta} \rightrightarrows N$. By a *natural congruence* τ between ϕ and ψ, in symbols '$\tau : \phi \dot{\Rightarrow} \psi$', we shall mean a map $\tau : M \to \mathit{\Delta}$ of class C^∞ from the base manifold M of $\mathit{\Gamma}$ into the manifold of arrows of $\mathit{\Delta}$ such that $\tau(x) \in \mathit{\Delta}(\phi x, \psi x)$ for all $x \in M$ and such that for all $g \in \mathit{\Gamma}$

$$\tau(tg)\phi(g) \equiv \psi(g)\tau(sg) \pmod{\dot{\mathit{\Delta}}}$$

Main construction

LGpd Category of Lie groupoids

LGpd$^{\boldsymbol{\cdot}}$ Subcategory of **LGpd** with

- same objects (Lie groupoids)
- morphisms $\phi \in \mathbf{LGpd}^{\boldsymbol{\cdot}}(\varGamma, \varDelta)$ all those Lie groupoid homomorphisms $\phi : \varGamma \to \varDelta$ such that $\phi^x_x(\dot{\varGamma}^x_x) \subset \dot{\varDelta}^{\phi x}_{\phi x}$ for every base point x of $\varGamma$

The binary relation on the collection $\mathrm{Mor}(\mathbf{LGpd}^{\boldsymbol{\cdot}})$

$$\phi \doteqdot \psi \overset{def}{\Leftrightarrow} \exists \textit{ a natural congruence between } \phi \textit{ and } \psi$$

is a categorical congruence on **LGpd**$^{\boldsymbol{\cdot}}$

$\mathbf{LGpd}^{\boldsymbol{\cdot}}{}_{/\doteqdot}$ Quotient category (with morphisms all $\doteqdot$-equivalence classes of **LGpd**$^{\boldsymbol{\cdot}}$-morphisms)

$\mathcal{E}$ Collection of all ($\doteqdot$-classes of) (C^∞-)full and completely transversal Lie groupoid homomorphisms

$\mathbf{LGpd}^{\boldsymbol{\cdot}}{}_{/\doteqdot}[\mathcal{E}^{-1}]$ Localized category (universal construction)

Claim

The localized category $\mathbf{LGpd}^{\cdot}_{/\dot{\equiv}}[\mathcal{E}^{-1}]$ *admits a calculus of right fractions*

Definition

We shall call $\mathbf{LGpd}^{\cdot}_{/\dot{\equiv}}[\mathcal{E}^{-1}]$ the category of *reduced Lie groupoids* and use the shorthand **RedLGpd** for it

Axiom I

The class of morphisms $\mathcal{E} \subset \mathrm{Mor}(\mathbf{LGpd}^{\boldsymbol{\cdot}}_{/\dot{\equiv}})$ *is multiplicative viz. contains the identities and is closed under composition*

Proof.

$\mathcal{E}$ is multiplicative already as a subclass of $\mathrm{Mor}(\mathbf{LGpd}^{\boldsymbol{\cdot}})$. $\square$

Axiom II

Notation $[\ ]^{\cdot}$ = Natural congruence class of a homomorphism

$$\forall \quad \begin{tikzcd} & \Gamma' \arrow[d, squiggly, "{[\phi]^{\cdot}\in\mathcal{E}}"] \\ \Delta \arrow[r, "{[\psi]^{\cdot}}"] & \Gamma \end{tikzcd} \qquad \exists \quad \begin{tikzcd} \Delta' \arrow[r, dashed, "{[\psi']^{\cdot}}"] \arrow[d, squiggly, "{[\phi']^{\cdot}\in\mathcal{E}}"] & \Gamma' \arrow[d, squiggly] \\ \Delta \arrow[r] & \Gamma \end{tikzcd}$$

Proof.

Since ϕ is transversal we can form the **weak pullback**

$$\begin{tikzcd} \Delta \,{}_{\psi}{\sqcap}_{\phi}\, \Gamma' \arrow[r, "pr_{\Gamma'}"] \arrow[d, "pr_{\Delta}"] & \Gamma' \arrow[d, "\phi"] \\ \Delta \arrow[r, "\psi"] \arrow[ur, phantom, "\Swarrow"] & \Gamma \end{tikzcd}$$

where pr_{Δ} must belong to $\mathcal{E} \subset \mathrm{Mor}(\mathbf{LGpd}^{\cdot})$.
Using the faithful transversality of ϕ we also see that $pr_{\Gamma'}$ must belong to $\mathrm{Mor}(\mathbf{LGpd}^{\cdot})$. □

Axiom III

$$\forall \quad \Gamma \underset{[\psi_2]^\cdot}{\overset{[\psi_1]^\cdot}{\rightrightarrows}} \Delta \overset{[\phi]^\cdot \in \mathcal{E}}{\rightsquigarrow} \Delta' \quad \textit{such that} \quad [\phi]^\cdot[\psi_1]^\cdot = [\phi]^\cdot[\psi_2]^\cdot$$

$$\exists \quad \Gamma' \overset{[\pi]^\cdot \in \mathcal{E}}{\rightsquigarrow} \Gamma \underset{[\psi_2]^\cdot}{\overset{[\psi_1]^\cdot}{\rightrightarrows}} \Delta \quad \textit{such that} \quad [\psi_1]^\cdot[\pi]^\cdot = [\psi_2]^\cdot[\pi]^\cdot$$

Proof.

Make use of the C^∞-fullness of ϕ:

$$\begin{tikzcd}
M' \arrow[r, dashed, two heads, "c"] \arrow[drr, dashed, bend right, "\tau"] & M \arrow[drr, bend left, "\tau'"] \arrow[ddr, bend right, "(\psi_1{,}\psi_2)"'] & & \\
& & \Delta \arrow[r, "\phi"] \arrow[d, "(s{,}t)"] & \Delta' \arrow[d, "(s{,}t)"] \\
& & N \times N \arrow[r, "\phi\times\phi"] & N' \times N'
\end{tikzcd}$$

Take Γ' to be: $c^*\Gamma \rightrightarrows M'$ (pullback along c).
Take π to be: $c^*\Gamma \to \Gamma$ (canonical projection). $\square$

Explicit model (category of fractions)

- Objects: Lie groupoids
- Morphisms: equivalence classes of "spans"

$$\Delta \xleftarrow{\alpha} \Gamma' \overset{\varepsilon}{\rightsquigarrow} \Gamma \qquad \text{with } \varepsilon \in \mathcal{E}$$

$(\alpha_1, \varepsilon_1) \sim (\alpha_2, \varepsilon_2)$ meaning: there exists a diagram

$$\begin{tikzcd}
 & \Gamma_1' \arrow[ld, "\alpha_1"'] \arrow[rd, squiggly, "\varepsilon_1"] & \\
\Delta & \Gamma'' \arrow[u] \arrow[d] \arrow[r, squiggly, "\varepsilon'"] & \Gamma \\
 & \Gamma_2' \arrow[lu, "\alpha_2"] \arrow[ru, squiggly, "\varepsilon_2"'] &
\end{tikzcd} \qquad \text{with } \varepsilon' \in \mathcal{E}$$

- Composition of morphisms:

$$\begin{tikzcd}
 & & \bullet \arrow[ld] \arrow[rd, squiggly] & & \\
 & \bullet \arrow[ld] \arrow[rd, squiggly] & & \bullet \arrow[ld] \arrow[rd, squiggly] & \\
\bullet & & \bullet & & \bullet
\end{tikzcd}$$

Effective equivalences

An **effective equivalence** is an element $\phi \in \mathrm{Mor}(\mathbf{LGpd}^{\boldsymbol{\cdot}})$ which becomes invertible under the canonical functor

$$\mathbf{LGpd}^{\boldsymbol{\cdot}} \longrightarrow \mathbf{LGpd}^{\boldsymbol{\cdot}}{}_{/\dot{\equiv}}[\mathcal{E}^{-1}]$$

Proposition

Any effective equivalence of Lie groupoids $\phi : \Gamma \to \Delta$ induces a homeomorphism $M/\Gamma \overset{\approx}{\to} N/\Delta$ between the orbit space of Γ and the orbit space of Δ.

Proposition

For any effective equivalence of Lie groupoids $\phi : \Gamma \to \Delta$, and for each base point x of Γ, the Lie group homomorphism $\overset{\circlearrowright}{\phi}{}^x_x : \overset{\circlearrowright}{\Gamma}{}^x_x \to \overset{\circlearrowright}{\Delta}{}^{\phi x}_{\phi x}$ is an isomorphism and the $\overset{\circlearrowright}{\phi}{}^x_x$-equivariant linear map $T^{\nUparrow}_x \phi_{(0)} : T^{\nUparrow}_x \Gamma_{(0)} \to T^{\nUparrow}_{\phi x} \Delta_{(0)}$ is bijective.

Reduced orbifolds

A Lie groupoid is **étale** if its source and its target are C^∞-étale maps (local diffeomorphisms)

A Lie groupoid is **proper** if for each compact subset K of its base manifold the set $s^{-1}(K) \cap t^{-1}(K)$ is compact

An **orbifold groupoid** is a proper étale Lie groupoid

efforbGpd Category of effective orbifold groupoids

$\mathbf{efforbGpd}_{/\equiv}$ Quotient category with homomorphisms identified modulo natural isomorphism

$\mathcal{W}$ Collection of all [$\equiv$-classes of] weak equivalences between effective orbifold groupoids

RedOrb Category of reduced orbifolds $\stackrel{def}{=} \mathbf{efforbGpd}_{/\equiv}[\mathcal{W}^{-1}]$ (localized category)

effLGpd	Category of effective Lie groupoids
$\mathbf{effLGpd}_{/\equiv}$	Quotient category with homomorphisms identified modulo natural isomorphism
$\mathcal{W}$	Collection of all [$\equiv$-classes of] weak equivalences between effective Lie groupoids

Proposition

The canonical functor

$$\mathbf{effLGpd}_{/\equiv}[\mathcal{W}^{-1}] \longrightarrow \mathbf{LGpd}^{\boldsymbol{\cdot}}_{/\dot{\equiv}}[\mathcal{E}^{-1}] = \mathbf{RedLGpd}$$

is fully faithful and hence an embedding of categories.

Corollary

There is a canonical embedding of categories

$$\mathbf{RedOrb} \longrightarrow \mathbf{RedLGpd}.$$