Diagrammatic categorification of the extended affine Hecke and the affine q-Schur algebras

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January 30, 2014

Aim and open questions

Aim Explain the diagrammatic categorification of the extended affine Hecke algebra and the affine q-Schur algebra, extending work by Elias-Khovanov, Elias-Williamson, Härterich, Khovanov-Lauda, Soergel, Williamson,...

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- Categorification of Kirillov-Reshetikhin modules?

Plan

1. Extended affine Weyl group and extended affine Hecke algebra.

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5. Affine Schur-Weyl duality and affine q-Schur algebra.
6. y-Deformed and extended affine Khovanov-Lauda calculus.

The extended affine Weyl group

Definition

The *extended affine Weyl group* $\widehat{\mathcal{W}}_{\widehat{A}_{r-1}}$ is generated by

$$\sigma_1, \ldots, \sigma_r, \rho,$$

subject to the relations

$$\begin{aligned}
\sigma_i^2 &= 1 && \text{for } i = 1, \ldots, r \\
\sigma_i \sigma_j &= \sigma_j \sigma_i && \text{for distant } i, j = 1, \ldots, r \\
\sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1} && \text{for } i = 1, \ldots, r \\
\rho \sigma_i \rho^{-1} &= \sigma_{i+1} && \text{for } i = 1, \ldots, r
\end{aligned}$$

The indices are understood modulo r. We say that i and j are *distant* if $j \not\equiv i \pm 1 \mod r$.

The (non-extended) affine Weyl group

Definition

The (non-extended) affine Weyl group $\mathcal{W}_{\widehat{A}_{r-1}} \subset \widehat{\mathcal{W}}_{\widehat{A}_{r-1}}$ is the subgroup generated by the σ_i.

Lemma

Any $w \in \widehat{\mathcal{W}}_{\widehat{A}_{r-1}}$ can be written as

$$w = \rho^k w' = \rho^k \sigma_{i_1} \cdots \sigma_{i_l}$$

where $k \in \mathbb{Z}$ is unique and $\sigma_{i_1} \cdots \sigma_{i_l}$ is a reduced expression of an element $w' \in \mathcal{W}_{\widehat{A}_{r-1}}$.

The extended affine Hecke algebra

Definition

Similarly, the *extended affine Hecke algebra* $\widehat{\mathcal{H}}_{\widehat{A}_{r-1}}$ is the $\mathbb{Q}(q)$-algebra generated by

$$\{T_\rho, T_\rho^{-1}, T_{\sigma_i}, i = 1, \ldots r\}$$

satisfying all the relations as in $\widehat{\mathcal{W}}_{\widehat{A}_{r-1}}$, except that

$$T_{\sigma_i}^2 = (q^2 - 1)T_{\sigma_i} + q^2 \quad \text{for all } i = 1, \ldots, r$$

with q being a formal parameter.

A $\mathbb{Q}(q)$–basis of $\widehat{\mathcal{H}}_{\widehat{A}_{r-1}}$ is given by the set $\{T_w, w \in \widehat{\mathcal{W}}_{\widehat{A}_{r-1}}\}$, with

$$T_w = T_\rho^k T_{w'} = T_\rho^k T_{\sigma_{i_1}} \cdots T_{\sigma_{i_l}}.$$

Kazhdan-Lusztig generators

Alternatively, $\widehat{\mathcal{H}}_{\widehat{A}_{r-1}}$ is generated by

$$\{T_\rho, T_{\rho^{-1}}, b_i, i = 1, \ldots, r\}$$

where the $b_i := C'_{\sigma_i} = q^{-1}(1 + T_{\sigma_i})$ are the *Kazhdan-Lusztig generators*.

Lemma

We have:

$$\begin{aligned}
b_i^2 &= (q + q^{-1})b_i && \text{for } i = 1, \ldots, r \\
b_i b_j &= b_j b_i && \text{for distant } i, j = 1, \ldots, r \\
b_i b_{i+1} b_i + b_{i+1} &= b_{i+1} b_i b_{i+1} + b_i && \text{for } i = 1, \ldots, r \\
T_\rho b_i T_\rho^{-1} &= b_{i+1} && \text{for } i = 1, \ldots, r.
\end{aligned}$$

Kazhdan-Lusztig basis

Theorem (Grojnowski-Haiman)

$\widehat{\mathcal{H}}_{\widehat{A}_{r-1}}$ *has the following* Kazhdan-Lusztig basis

$$\{T_\rho^k C'_w, k \in \mathbb{Z} \textit{ and } w \in \mathcal{W}_{\widehat{A}_{r-1}}\},$$

with the usual positive integrality property.

Extended reflection faithful representation

Definition

$\widehat{\mathcal{W}}_{\widehat{A}_{r-1}}$ acts faithfully on $R = \mathbb{Q}[y][x_1, \ldots, x_r]$:

$$\rho(x_i) = \begin{cases} x_{i+1} & \text{for } i = 1, \ldots, r-1 \\ x_1 - y & \text{for } i = r \end{cases}$$

$$\rho^{-1}(x_i) = \begin{cases} x_{i-1} & \text{for } i = 2, \ldots, r \\ x_r + y & \text{for } i = 1 \end{cases}$$

$$\sigma_j(x_i) = \begin{cases} x_{j+1} & \text{for } i = j \\ x_j & \text{for } i = j+1 \\ x_i & \text{otherwise} \end{cases} \qquad \text{for } j = 1, \ldots, r-1$$

$$\sigma_r(x_i) = \begin{cases} x_r + y & \text{for } i = 1 \\ x_1 - y & \text{for } i = r \\ x_i & \text{otherwise} \end{cases}$$

Extended affine Soergel bimodules

Definition

For any $i = 1, \dots, r$, define the graded bimodule

$$B_i := R \otimes_{R^{\sigma_i}} R$$

with $\deg(y) = \deg(x_k) = 2$ and

$$\begin{aligned} R^{\sigma_i} &:= \mathbb{Q}[y][x_1, \dots, x_i + x_{i+1}, x_i x_{i+1}, \dots, x_r] \quad (i = 1, \dots, r-1) \\ R^{\sigma_r} &:= \mathbb{Q}[y]\,[x_2, \dots, x_{r-1}, x_r + x_1, (x_r + y/2)\,(x_1 - y/2)] \end{aligned}$$

Define also the *twisted bimodule* $B_{\rho^{\pm 1}}$. As a left R-module, we have $B_{\rho^{\pm 1}} := R$. The right R-module structure is defined by

$$x \triangleleft a := \rho(a)^{\pm 1} x$$

for all $x \in B_{\rho^{\pm 1}}$ and $a \in R$.

Härterich's categorification theorem

Definition

The *category of extended Soergel bimodules* $\mathrm{Kar}\mathcal{E}\mathcal{B}im_{\widehat{A}_{r-1}}$, is the idempotent completion of the $\mathbb{Q}$–linear graded additive monoidal category with translation generated by the bimodules above.

Let $\mathrm{Kar}\mathcal{B}im_{\widehat{A}_{r-1}}$ be the idempotent completion of the monoidal subcategory generated by the B_i.

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Let $\mathrm{Kar}\mathcal{B}im_{\widehat{A}_{r-1}}$ be the idempotent completion of the monoidal subcategory generated by the B_i.

Theorem (Härterich)

We have

$$\mathcal{H}_{\widehat{A}_{r-1}} \cong K_0^{\mathbb{Q}(q)}(\mathrm{Kar}\mathcal{B}im_{\widehat{A}_{r-1}})$$

such that

$$C'_w \mapsto [B_w\{-1\}],$$

for any $w \in \mathcal{W}_{\widehat{A}_{r-1}}$. Here B_w (e.g. B_i) is an indecomposable bimodule uniquely determined by w (e.g. σ_i).

Lemma

For any $i = 1, \ldots, r$, there exist R–bimodule isomorphisms

$$B_\rho^{\otimes k} \cong B_{\rho^k}$$

$$B_\rho \otimes_R B_i \cong B_{i+1} \otimes_R B_\rho$$

Corollary (M.M.-Thiel)

We have

$$\widehat{\mathcal{H}}_{\widehat{A}_{r-1}} \cong K_0^{\mathbb{Q}(q)}(\mathrm{Kar}\mathcal{EB}im_{\widehat{A}_{r-1}})$$

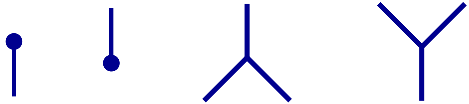
such that

$$T_\rho^k C'_w \mapsto [B_{\rho^k} B_w\{-1\}].$$

Note that $B_\rho^{\otimes k} \ncong R$ for any $k \in \mathbb{Z}$, because the action of ρ is faithful. Putting $y = 0$ gives $B_\rho^{\otimes r} \cong R$.

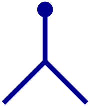
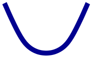
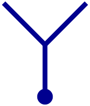
Elias-Khovanov type generators:

- involving only one color:



Degree	1	1	-1	-1

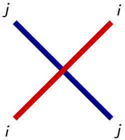
For simplicity, define the degree-zero morphisms:

 $:=$     $:=$ 

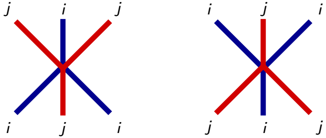
- Boxes (degree 2): $\boxed{i}$ $(i = 1, \dots, r)$ and $\boxed{y}$

The diagrammatic Soergel 2-category $\mathcal{DEB}im_{\widehat{A}_{r-1}}$

- The 4–valent vertex with distant colors, of degree 0:



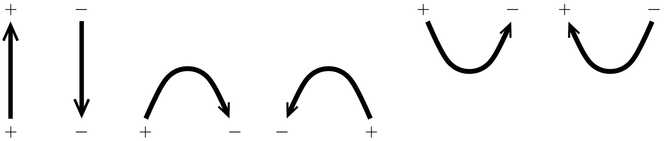
- The 6–valent vertices with adjacent colors i and j, of degree 0:



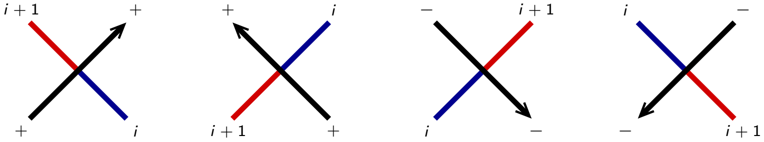
The diagrammatic Soergel 2-category $\mathcal{DEB}im_{\widehat{A}_{r-1}}$

New generators:

- Generators involving only oriented strands (degree 0):

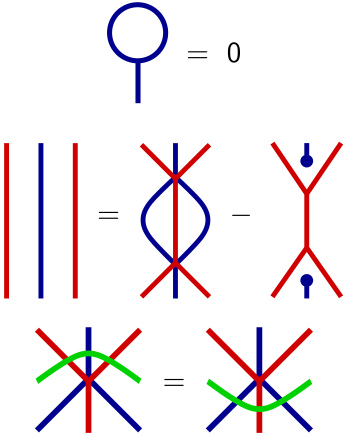


- The mixed 4–valent vertex (degree 0):

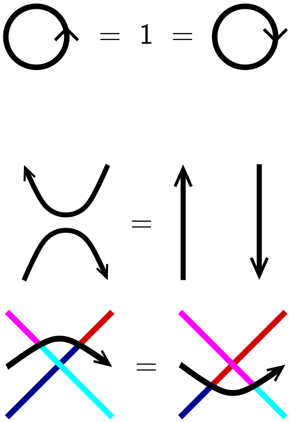


The diagrammatic Soergel 2-category $\mathcal{DEB}im_{\widehat{A}_{r-1}}$

The usual Elias-Khovanov relations, but with colors modulo r, e.g.:



New relations, e.g.:



The equivalence

Theorem (Elias-Williamson, M.M.-Thiel)

There exists an equivalence of graded 2-categories

$$\mathcal{DEB}im_{\widehat{A}_{r-1}} \to \mathcal{EB}im_{\widehat{A}_{r-1}}$$

for any $r \geq 3$*.*

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Corollary

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$$\widehat{\mathcal{H}}_{\widehat{A}_{r-1}} \cong K_0^{\mathbb{Q}(q)}(\mathrm{Kar}\mathcal{DEB}im_{\widehat{A}_{r-1}})$$

Remark

The ideal generated by y is virtually nilpotent, so $K_0^{\mathbb{Q}(q)}(\mathrm{Kar}\mathcal{DEB}im_{\widehat{A}_{r-1}}/[y]) \cong K_0^{\mathbb{Q}(q)}(\mathrm{Kar}\mathcal{DEB}im_{\widehat{A}_{r-1}})$.

The extended affine quantum algebras

Assume $n \geq 3$ from now on.

Definition (Green)

The *extended quantum general linear algebra* $\widehat{\mathbf{U}}_q(\widehat{\mathfrak{gl}}_n)$ is the associative unital $\mathbb{Q}(q)$-algebra generated by $R^{\pm 1}$, $K_i^{\pm 1}$ and $E_{\pm i}$, for $i = 1, \ldots, n$, subject to the usual relations together with

$$RR^{-1} = R^{-1}R = 1 \tag{0.1}$$

$$RX_iR^{-1} = X_{i+1} \qquad \text{for } X_i \in \{E_{\pm i}, K_i^{-1}\}. \tag{0.2}$$

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Definition

The *affine quantum general linear algebra* $\mathbf{U}_q(\widehat{\mathfrak{gl}}_n) \subseteq \widehat{\mathbf{U}}_q(\widehat{\mathfrak{gl}}_n)$ is generated by $E_{\pm i}$ and $K_i^{\pm 1}$, for $i = 1, \ldots, n$.

The *affine quantum special linear algebra* $\mathbf{U}_q(\widehat{\mathfrak{sl}}_n) \subseteq \mathbf{U}_q(\widehat{\mathfrak{gl}}_n)$ is generated by $E_{\pm i}$ and $K_iK_{i+1}^{-1}$, for $i = 1, \ldots, n$.

These algebras are all Hopf algebras.

The idempotented version

The degenerate level-zero $\widehat{\mathbf{U}}_q(\widehat{\mathfrak{gl}}_n)$-weight lattice can be identified with $\mathbb{Z}^n$. Note that

$$R1_{(\lambda_1,\dots,\lambda_n)} = 1_{(\lambda_n,\lambda_1,\dots,\lambda_{n-1})}R.$$

This relation plus the usual ones give

Definition

The *idempotented extended affine quantum general linear algebra* is defined by

$$\widehat{\dot{\mathbf{U}}}(\widehat{\mathfrak{gl}}_n) = \bigoplus_{\lambda,\mu\in\mathbb{Z}^n} 1_\lambda \widehat{\mathbf{U}}_q(\widehat{\mathfrak{gl}}_n) 1_\mu.$$

The idempotented version

Definition

Define $\dot{\mathbf{U}}(\widehat{\mathfrak{gl}}_n) \subset \widehat{\dot{\mathbf{U}}}(\widehat{\mathfrak{gl}}_n)$ as the idempotented subalgebra generated by 1_λ and $E_{\pm i}1_\lambda$, for $i = 1, \ldots, n$ and $\lambda \in \mathbb{Z}^n$.

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To define $\dot{\mathbf{U}}(\mathfrak{sl}_n)$ adjoin an idempotent 1_μ for each $\mu \in \mathbb{Z}^{n-1}$.

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Remark: $\mathfrak{gl}_n$-weights versus $\mathfrak{sl}_n$-weights

Recall the map $\mathbb{Z}^n \to \mathbb{Z}^{n-1}$ given by $\lambda \mapsto \overline{\lambda}$ with

$$\overline{\lambda} := (\lambda_1 - \lambda_2, \ldots, \lambda_{n-1} - \lambda_n).$$

Affine tensor space

Let V be the $\mathbb{Q}(q)$-vector space freely generated by $\{e_t \mid t \in \mathbb{Z}\}$.

Definition (Green)

The following defines an action of $\widehat{\mathbf{U}}_q(\widehat{\mathfrak{gl}}_n)$ on V

$$E_i e_{t+1} = e_t \quad \text{if } i \equiv t \mod n \tag{0.3}$$
$$E_i e_{t+1} = 0 \quad \text{if } i \not\equiv t \mod n \tag{0.4}$$
$$E_{-i} e_t = e_{t+1} \quad \text{if } i \equiv t \mod n \tag{0.5}$$
$$E_{-i} e_t = 0 \quad \text{if } i \not\equiv t \mod n \tag{0.6}$$
$$K_i^{\pm 1} e_t = q^{\pm 1} e_t \quad \text{if } i \equiv t \mod n \tag{0.7}$$
$$K_i^{\pm 1} e_t = e_t \quad \text{if } i \not\equiv t \mod n \tag{0.8}$$
$$R^{\pm 1} e_t = e_{t\pm 1} \quad \text{for all } t \in \mathbb{Z}. \tag{0.9}$$

- For any $r \in \mathbb{N}$, $V^{\otimes r}$ is a $\widehat{\dot{\mathbf{U}}}(\widehat{\mathfrak{gl}}_n)$-weight representation with weights in

$$\Lambda(n,r) = \{\lambda \in \mathbb{N}^n \colon \sum_{i=1}^{n} \lambda_i = r\}.$$

The affine q-Schur algebra

- For any $r \in \mathbb{N}$, $V^{\otimes r}$ is a $\widehat{\dot{\mathbf{U}}}(\widehat{\mathfrak{gl}}_n)$-weight representation with weights in

$$\Lambda(n, r) = \{\lambda \in \mathbb{N}^n \colon \sum_{i=1}^{n} \lambda_i = r\}.$$

- Green defined a right action of $\widehat{\mathcal{H}}_{\widehat{A}_{r-1}}$ on $V^{\otimes r}$ which commutes with the left action of $\widehat{\dot{\mathbf{U}}}(\widehat{\mathfrak{gl}}_n)$.

Definition (Green)

Let $n, r \geq 3$. The *affine q-Schur algebra* $\widehat{\mathbf{S}}(n, r)$ is the centralizing algebra

$$\mathrm{End}_{\widehat{\mathcal{H}}_{\widehat{A}_{r-1}}}(V^{\otimes r}).$$

Affine Schur-Weyl duality (part of)

Theorem (Green)

For $n, r \geq 3$, the image of $\psi_{n,r} : \widehat{\dot{\mathbf{U}}}(\widehat{\mathfrak{gl}}_n) \to End(V^{\otimes r})$ is always isomorphic to $\widehat{\mathbf{S}}(n, r)$. If $n > r$, we even have

$$\psi_{n,r}(\dot{\mathbf{U}}(\widehat{\mathfrak{sl}}_n)) \cong \psi_{n,r}(\dot{\mathbf{U}}(\widehat{\mathfrak{gl}}_n)) \cong \widehat{\mathbf{S}}(n, r).$$

For $n = r$, this is no longer true.

The case $n > r \geq 3$

Theorem (Doty-Green)

For $n > r \geq 3$, $\widehat{\mathbf{S}}(n, r)$ is isomorphic to the quotient of $\dot{\mathbf{U}}(\widehat{\mathfrak{gl}}_n)$ by the ideal generated by all idempotents whose weight does not belong to $\Lambda(n, n)$.

The case $n = r \geq 3$

As vector spaces, we have

$$\widehat{\mathbf{S}}(n,n) \cong \psi_{n,n}(\dot{\mathbf{U}}(\widehat{\mathfrak{gl}}_n)) \oplus \bigoplus_{t \neq 0} \mathbb{Q}[R^t, R^{-t}].$$

However, this is not an algebra isomorphism.

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However, this is not an algebra isomorphism.

Definition (Deng-Du-Fu)

Define

$$R^{-1} := E_{+\delta} 1_n + \sum_{i=1}^{n} \sum_{a_i=0} E_{i-1}^{(a_{i-1})} \cdots E_1^{(a_1)} E_n^{(a_n)} \cdots E_{i+1}^{(a_{i+1})} 1_{(a_n, a_1, \ldots, a_{n-1})}$$

and

$$R := E_{-\delta} 1_n + \sum_{i=1}^{n} \sum_{a_i=0} E_{-(i-1)}^{(a_{i-1})} \cdots E_{-1}^{(a_1)} E_{-n}^{(a_n)} \cdots E_{-(i+1)}^{(a_{i+1})} 1_{(a_1, \ldots, a_n)}.$$

The second sum is over $(a_1, \ldots, a_n) \in \Lambda(n,n)$.

Theorem (Deng-Du-Fu)

$\widehat{\mathbf{S}}(n,n)$ is generated by $E_{\pm\delta}$, $E_{\pm i}$ and 1_λ, for $i = 1, \ldots, n$ and $\lambda \in \Lambda(n,n)$, with the usual relations and $(i = 1, \ldots, n)$:

i) $E_{\pm\delta}1_\lambda = 1_\lambda E_{\pm\delta} = 0$ *for all* $\lambda \neq (1^n)$;

ii) $E_{\pm\delta}1_n = 1_n E_{\pm\delta}$;

iii) $E_{+\delta}E_{-\delta}1_n = E_{-\delta}E_{+\delta}1_n = 1_n$;

iv) $E_i E_{+\delta}1_n = E_i^{(2)} E_{i-1} \ldots E_1 E_n \cdots E_{i+1}1_n$;

v) $1_n E_{+\delta}E_i = 1_n E_{i-1} \ldots E_1 E_n \cdots E_{i+1} E_i^{(2)}$;

vi) $E_{-i}E_{+\delta}1_n = E_{i-1} \cdots E_1 E_n \cdots E_{i+1}1_n$;

vii) $1_n E_{+\delta}E_{-i} = 1_n E_{i-1} \cdots E_1 E_n \cdots E_{i+1}$;

viii) $E_{-i}E_{-\delta}1_n = E_{-i}^{(2)} E_{-(i+1)} \cdots E_{-n}E_{-1} \cdots E_{-(i-1)}1_n$;

ix) $1_n E_{-\delta}E_{-i} = 1_n E_{-(i+1)} \cdots E_{-n}E_{-1} \cdots E_{-(i-1)}E_{-i}^{(2)}$;

x) $E_i E_{-\delta}1_n = E_{-(i+1)} \cdots E_{-n}E_{-1} \cdots E_{-(i-1)}1_n$;

xi) $1_n E_{-\delta}E_i = 1_n E_{-(i+1)} \cdots E_{-n}E_{-1} \cdots E_{-(i-1)}$.

Some interesting homomorphisms

Lemma (Doty-Green, Deng-Du-Fu)

There exists an injection $\sigma_{n,r} \colon \widehat{\mathcal{H}}_{\widehat{A}_{r-1}} \to \widehat{\mathbf{S}}(n,r)$ *defined by*

$$\begin{aligned} \sigma_{n,r}(b_i) &:= 1_r E_i E_{-i} 1_r = 1_r E_{-i} E_i 1_r \\ \sigma_{n,r}(b_r) &:= 1_r E_{-n} \cdots E_{-r} E_r \cdots E_n 1_r \\ \sigma_{n,r}(T_\rho) &:= 1_r E_{-n} \cdots E_{-r-1} E_{-1} \cdots E_{-r} 1_r \\ \sigma_{n,r}(T_\rho^{-1}) &:= 1_r E_r \cdots E_1 E_{r+1} \cdots E_n 1_r, \end{aligned}$$

for $i = 1, \ldots, r-1$ *and* $n > r \geq 3$, *and*

$$\begin{aligned} \sigma_{n,r}(b_i) &:= 1_r E_i E_{-i} 1_r = 1_r E_{-i} E_i 1_r \\ \sigma_{n,r}(T_\rho^{\pm 1}) &:= 1_r E_{\pm\delta} 1_r, \end{aligned}$$

for $i = 1, \ldots, r$ *and* $n = r \geq 3$.

Some interesting homomorphisms

Lemma (Deng-Du)

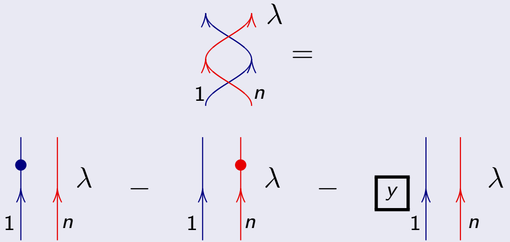
For $n \geq 3$, there exists an injection $\iota_n \colon \widehat{\mathbf{S}}(n,n) \to \widehat{\mathbf{S}}(n+1,n)$ defined by

$$\begin{array}{rcl}
1_\lambda & \mapsto & 1_{(\lambda,0)} \\
E_{\pm i}1_\lambda & \mapsto & E_{\pm i}1_{(\lambda,0)} \\
E_n 1_\lambda & \mapsto & E_n E_{n+1} 1_{(\lambda,0)} \\
E_{-n}1_\lambda & \mapsto & E_{-(n+1)}E_{-n}1_{(\lambda,0)} \\
E_{+\delta}1_n & \mapsto & E_n E_{n-1}\cdots E_1 E_{n+1} 1_{(1^n,0)} \\
E_{-\delta}1_n & \mapsto & E_{-(n+1)}E_{-1}\cdots E_{-n}1_{(1^n,0)}
\end{array}$$

Categorification for $n > r \geq 3$

Definition (M.M.-Thiel)

Define $\mathcal{U}(\widehat{\mathfrak{gl}}_n)_{[y]}$ by tensoring Khovanov and Lauda's $\mathcal{U}(\widehat{\mathfrak{gl}}_n)$ with $\mathbb{Q}[y]$ and deforming the relation



and the analogous relation with 1 and n switched and an additional minus sign.

As a consequence some bubble slide relations get deformed.

Categorification for $n > r \geq 3$

Definition (M.M-Thiel)

$\widehat{\mathcal{S}}(n,r)_{[y]}$ is the quotient of $\mathcal{U}(\widehat{\mathfrak{gl}}_n)_{[y]}$ by the ideal generated by all diagrams with regions whose label is not contained in $\Lambda(n,n)$.

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Theorem (M.M-Thiel)

The $\mathbb{Q}(q)$-linear algebra homomorphism

$$\gamma_{n,r} \colon \widehat{\mathbf{S}}(n,r) \to K_0^{\mathbb{Q}(q)}(\mathrm{Kar}\widehat{\mathcal{S}}(n,r)_{[y]})$$

defined by

$$\gamma_{n,r}(E_{\pm i}1_\lambda) := [\mathcal{E}_{\pm i}\mathbf{1}_\lambda] \otimes 1 \quad \textit{and} \quad \gamma_{n,r}(E_{\pm\delta}1_n) := [\mathcal{E}_{\pm\delta}\mathbf{1}_n] \otimes 1$$

for any $i = 1, \ldots, n$, is a well-defined isomorphism.

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for any $i = 1, \ldots, n$, is a well-defined isomorphism.

Note that the ideal generated by y is virtually nilpotent, so the categorification theorem also holds for $y = 0$.

Corollary

The restriction of

$$\gamma_{n,r} : \widehat{\mathbf{S}}(n,r) \to K_0^{\mathbb{Q}(q)}(\mathrm{Kar}\widehat{\mathcal{S}}(n,r)_{[y]})$$

gives rise to the composite isomorphism

$$\gamma_{n,r} \circ \sigma_{n,r} : \widehat{\mathcal{H}}_{\widehat{A}_{r-1}} \to 1_n \widehat{\mathbf{S}}(n,r) 1_n \to K_0^{\mathbb{Q}(q)}(\mathrm{Kar}\widehat{\mathcal{S}}(n,r)_{[y]}((1^r),(1^r)))$$

for any $n > r \geq 3$.

Corollary

The restriction of

$$\gamma_{n,r} \colon \widehat{\mathbf{S}}(n,r) \to K_0^{\mathbb{Q}(q)}(\mathrm{Kar}\widehat{\mathcal{S}}(n,r)_{[y]})$$

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for any $n > r \geq 3$.

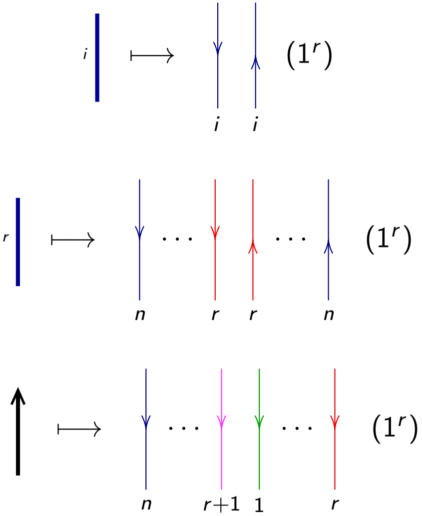
Theorem (M.M.-Thiel)

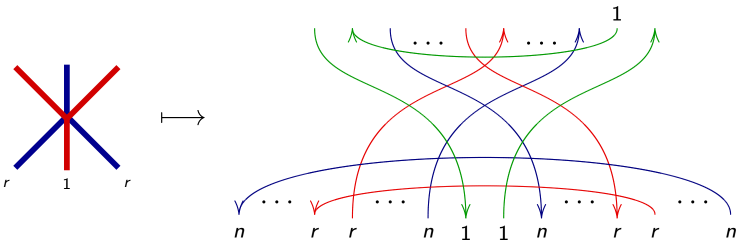
There exists a well-defined 2-functor

$$\Sigma_{n,r} \colon \mathcal{DEB}im^*_{\widehat{A}_{r-1}} \to \widehat{\mathcal{S}}(n,r)^*_{[y]}$$

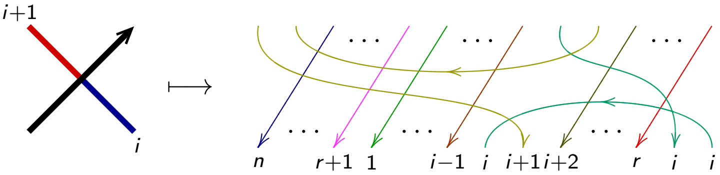
which categorifies $\gamma_{n,r} \circ \sigma_{n,r}$, *for any* $n > r \geq 3$.

The 2-functor $\Sigma_{n,r}$, e.g.:





(1^r)



(1^r)

Theorem (M.M.-Thiel)

For any $n > r \geq 3$, there exist a 2-category of extended singular affine Soergel bimodules $\mathcal{ESB}im_{\widehat{A}_{r-1}}$ and a 2-representation

$$\mathcal{F}' : \widehat{\mathcal{S}}(n,r)^*_{[y]} \to \mathcal{ESB}im^*_{\widehat{A}_{r-1}}.$$

such that the following diagram commutes

$$\begin{tikzcd} \mathcal{DEB}im^*_{\widehat{A}_{r-1}} \arrow[rr, "\mathcal{F}"] \arrow[dr, "\Sigma_{n,r}"'] & & \mathcal{ESB}im^*_{\widehat{A}_{r-1}} \\ & \widehat{\mathcal{S}}(n,r)^*_{[y]}((1^r),(1^r)) \arrow[ur, "\mathcal{F}'"'] & \end{tikzcd}$$

Categorification for $n = r \geq 3$ and $y = 0$

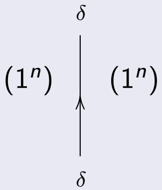
Definition (M.M.-Thiel)

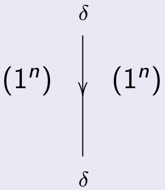
$\widehat{\mathcal{S}}(n,n)$ is the quotient of $\mathcal{U}(\widehat{\mathfrak{gl}}_n)$ by the ideal generated by all diagrams with regions whose label is not contained in $\Lambda(n,n)$ (taking $y = 0$ for simplicity), together with the generating 1-morphisms

$$\mathbf{1}_n\mathcal{E}_{+\delta}\mathbf{1}_n\{t\} \quad \text{and} \quad \mathbf{1}_n\mathcal{E}_{-\delta}\mathbf{1}_n\{t\},$$

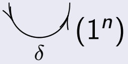
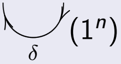
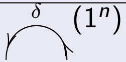
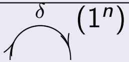
for $t \in \mathbb{Z}$, and the following generating 2-morphisms

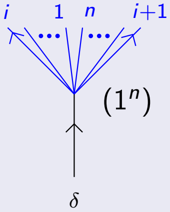
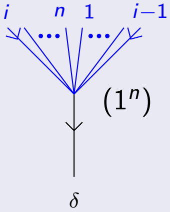
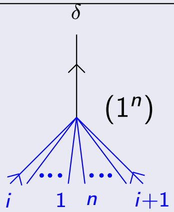
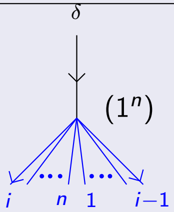
$1_{\mathcal{E}_{+\delta}\mathbf{1}_n\{t\}}$ $\qquad$ $1_{\mathcal{E}_{-\delta}\mathbf{1}_n\{t\}}$



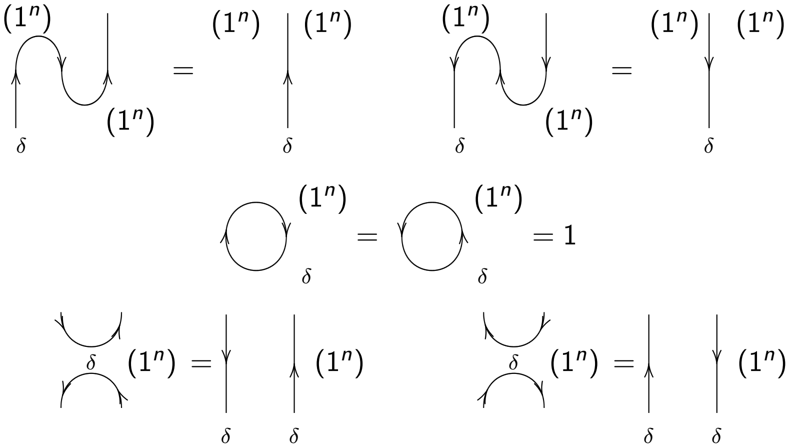


Generators

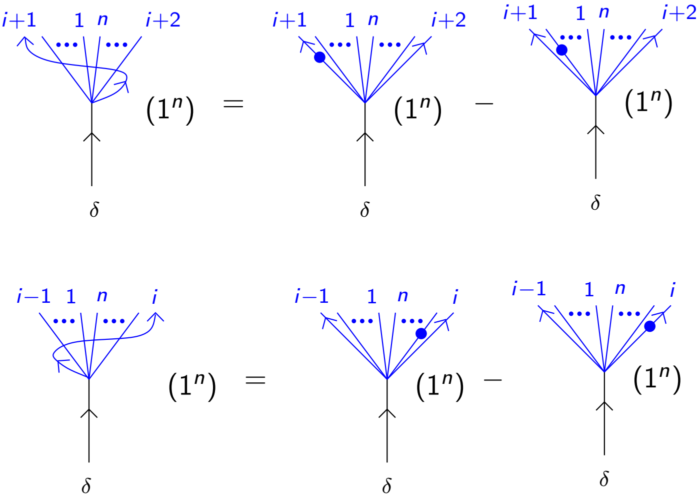
			
0	0	0	0

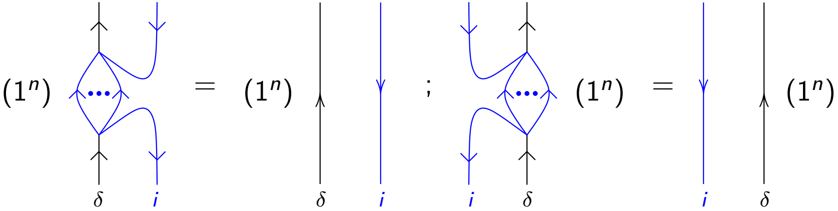
			
1	1	1	1

Extra relations

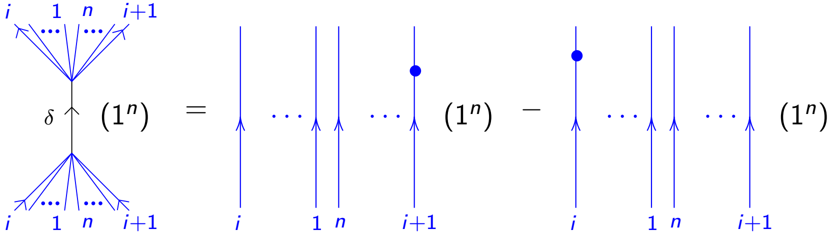


Extra relations



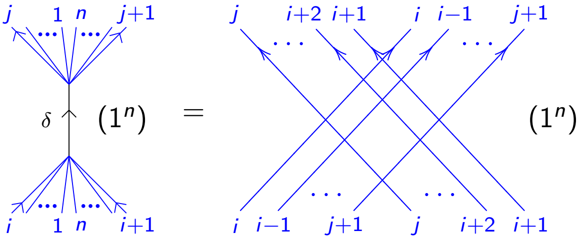


(1^n) = (1^n) ; (1^n) = (1^n)
δ i δ i i δ i δ



i 1 n $i{+}1$
δ (1^n) = (1^n) − (1^n)
i 1 n $i{+}1$ i 1 n $i{+}1$ i 1 n $i{+}1$

Extra relations



Impose cyclicity on all diagrams.

Categorification Theorem

Theorem (M.M.-Thiel)

For any $n \geq 3$, the $\mathbb{Q}(q)$-linear algebra homomorphism

$$\gamma_{n,n} \colon \widehat{\mathbf{S}}(n,n) \to K_0^{\mathbb{Q}(q)}(\mathrm{Kar}\widehat{\mathcal{S}}(n,n))$$

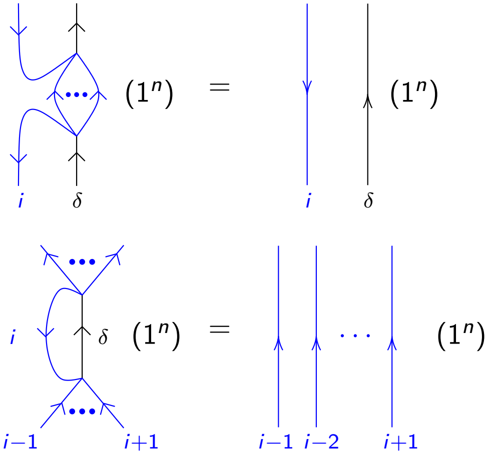
defined by

$$\gamma_{n,n}(E_{\pm i}1_\lambda) := [\mathcal{E}_{\pm i}\mathbf{1}_\lambda] \otimes 1 \quad \textit{and} \quad \gamma_{n,n}(E_{\pm\delta}1_n) := [\mathcal{E}_{\pm\delta}\mathbf{1}_n] \otimes 1$$

for any $i = 1, \ldots, n$, is a well-defined isomorphism.

Well-definedness, e.g.:

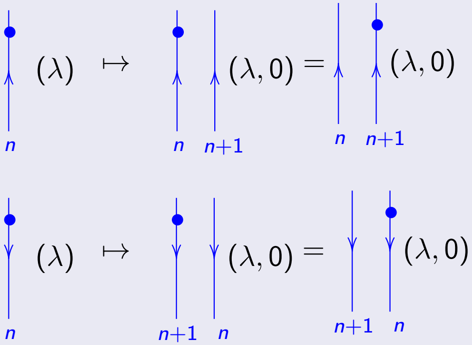
$$\mathcal{E}_{-i}\mathcal{E}_{+\delta}\mathbf{1}_n \cong \mathcal{E}_{i-1}\cdots\mathcal{E}_1\mathcal{E}_n\cdots\mathcal{E}_{i+1}\mathbf{1}_n$$



Where did we get the relations from?

Proposition (M.M.-Thiel)

The 2-functor $\mathcal{I}_n \colon \widehat{\mathcal{S}}(n,n) \to \widehat{\mathcal{S}}(n+1,n)$ *below is well-defined.*



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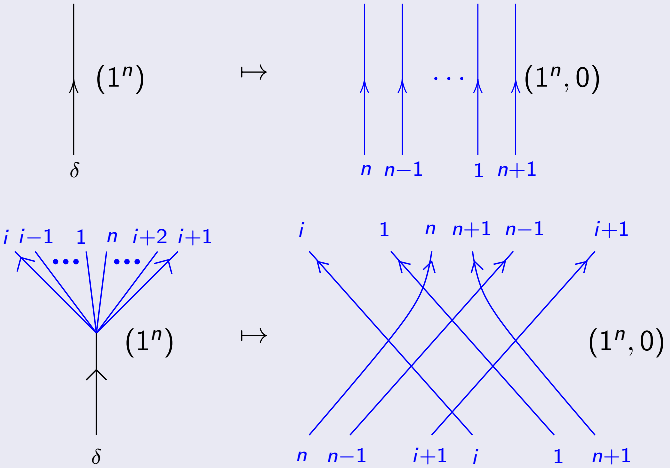

The End

THANKS!!!