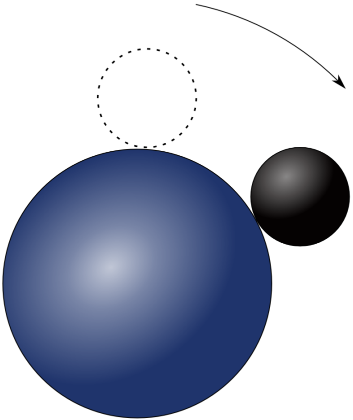
G_2 and the Rolling Ball



G_2 and the Rolling Ball

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The Cartan–Killing classification

Up to choice of cover and real form, the simple Lie groups are:

- Three infinite families, $\mathrm{SO}(n)$, $\mathrm{SU}(n)$, and $\mathrm{Sp}(n)$.
- Five exceptions:

$$\mathrm{G}_2, \quad \mathrm{F}_4, \quad \mathrm{E}_6, \quad \mathrm{E}_7, \quad \mathrm{E}_8.$$

- The infinite families are the respective symmetry groups of $\mathbb{R}^n$, $\mathbb{C}^n$, $\mathbb{H}^n$ with inner product.
- *Where do the exceptions come from?* They're all related to $\mathbb{O}$.

The split real form of G_2

We will relate two models for the split real form of G_2, both essentially due to Cartan:

- $G_2 = \mathrm{Aut}(\mathbb{O}')$, where $\mathbb{O}'$ is the 'split octonions'.
- $\mathfrak{g}_2 = \mathrm{Lie}(G_2)$ acts locally as symmetries of one ball rolling on another without slipping or twisting, *provided the ratio of radii is 3:1 or 1:3*.

Relating the two will explain

Why 1:3?

Split octonions

The split octonions are pairs of quaternions:

$$\mathbb{O}' = \mathbb{H} \oplus \mathbb{H}$$

with product $(a, b)(c, d) = (ac + d\overline{b}, \overline{a}d + cb)$.

They form a **composition algebra**: there is a quadratic form Q on $\mathbb{O}'$ such that

$$Q(xy) = Q(x)Q(y), \quad x, y \in \mathbb{O}'.$$

On pairs of quaternions, this is given by:

$$Q(a, b) = |a|^2 - |b|^2, \quad (a, b) \in \mathbb{H} \oplus \mathbb{H}.$$

G_2 acts on . . .

- $\mathbb{O}'$, fixing $1 \in \mathbb{O}'$ and preserving Q;
- $\mathrm{Im}(\mathbb{O}') = \mathrm{Im}(\mathbb{H}) \oplus \mathbb{H}$, the subspace orthogonal to 1;
- $C = \{x \in \mathrm{Im}(\mathbb{O}') : Q(x) = 0\}$, the space of null vectors in $\mathrm{Im}(\mathbb{O}')$;
- $\mathrm{P}C =$ 1d null subspaces of $\mathrm{Im}(\mathbb{O}')$, the projectivization of C.

We will see that this last space is closely related to the rolling ball, provided the ratio of radii is 1:3.

Rolling balls

The configuration space of the rolling ball is $S^2 \times \mathrm{SO}(3)$.

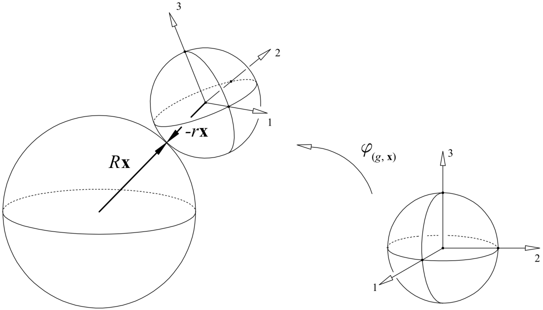


Figure : Bor and Montgomery, 2009.

We will consider a ball of unit radius rolling on a fixed ball of radius R, and see why $R = 3$ is special.

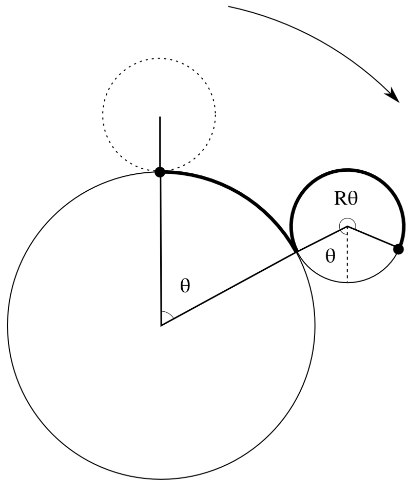
Without slipping or twisting

We encode the constraint in an **incidence geometry**, a barebones geometry with **points**, **lines**, and an **incidence relation**, telling us which points lie on which lines.

There is an incidence geometry with:

- Points are elements of $S^2 \times \mathrm{SO}(3)$;
- Lines are given by rolling without slipping or twisting along great circles.

Without slipping or twisting



Without slipping or twisting

If the central angle changes by θ, the rolling ball rotates by $(R+1)\theta$.

- Points are elements of $S^2 \times \mathrm{SO}(3)$;
- Lines are given by subsets of the form:

$$L = \{(\cos(\theta)u + \sin(\theta)v,\ \mathbf{R}(u \times v, (R+1)\theta)g) : \theta \in \mathbb{R}\}$$

where u, v are orthonormal, $g \in \mathrm{SO}(3)$ and $\mathbf{R}(w, \alpha)$ denotes the right-handed rotation about the w-axis by angle α.

Hiding inside $\mathrm{Im}(\mathbb{O}')$ ...

Remember:

$$\begin{aligned} \mathrm{P}C &= \text{1d null subspaces of } \mathrm{Im}(\mathbb{O}') \\ &= \{\langle x \rangle : \text{nonzero } x \in \mathrm{Im}(\mathbb{O}'),\ Q(x) = 0\} \\ &= \{\langle (a,b) \rangle : \text{nonzero } (a,b) \in \mathrm{Im}(\mathbb{H}) \oplus \mathbb{H},\ |a|^2 = |b|^2\} \\ &= \frac{S^2 \times S^3}{(a,b) \sim (-a,-b)}. \end{aligned}$$

This last space

$$\frac{S^2 \times S^3}{\mathbb{Z}_2}$$

tells us $\mathrm{P}C$ is awfully similar to the rolling ball configuration space:

$$S^2 \times \mathrm{SO}(3).$$

Hiding inside $\mathrm{Im}(\mathbb{O}')$ ...

Recall:

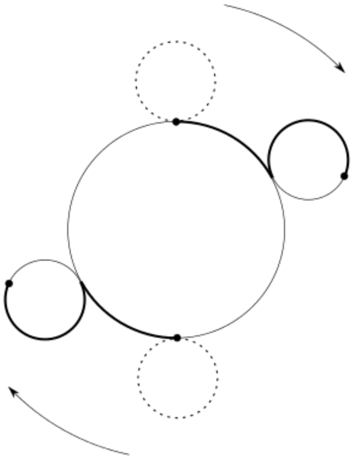
- $S^3 \subset \mathbb{H}$ is the group of unit quaternions.
- $\dfrac{S^3}{\mathbb{Z}_2} \cong \mathrm{SO}(3)$.
- Alas:
$$\frac{S^2 \times S^3}{\mathbb{Z}_2} \not\cong S^2 \times \mathrm{SO}(3).$$
- Instead:
$$\frac{S^2 \times S^3}{\mathbb{Z}_2} \cong \mathbb{RP}^2 \times S^3.$$

We will think of $\mathbb{RP}^2 \times S^3$ as the configuration space of a *spinor rolling on a projective plane.*

Spinor rolling on a projective plane

- $\mathbb{RP}^2$ is S^2 with antipodal points identified; so instead of one ball, we consider a pair, rolling in sync.
- The ball is a spinor: it is rotated by elements of S^3 instead of $\mathrm{SO}(3)$. Since
$$S^3 \to \mathrm{SO}(3)$$
is a double-cover, it takes a 720° rotation to get back where you started.

Spinor rolling on a projective plane



Without slipping or twisting

There is an incidence geometry where:

- Points are elements of $\mathbb{RP}^2 \times S^3$.
- Lines are given by a spinor rolling without slipping or twisting along lines of $\mathbb{RP}^2$. Explicitly, lines are given by subsets of the form:

$$L = \left\{ (\pm e^{\theta w} u,\ e^{\frac{R+1}{2}\theta w} q)) : \theta \in \mathbb{R} \right\}$$

where u, w are orthonormal, $q \in S^3$ and the exponentiation takes place in $\mathbb{H}$.

When $R = 3$

Remember, $\mathbb{RP}^2 \times S^3 \cong \mathrm{P}C$, the space of null 1d subspaces in $\mathrm{Im}(\mathbb{O}')$.

Theorem

If and only if $R = 3$, the incidence geometry of a spinor rolling on a projective plane coincides with the incidence geometry where

- Points are 1d null subspaces of $\mathrm{Im}(\mathbb{O}')$, i.e. elements of $\mathrm{P}C$.
- Lines are 2d null subspaces of $\mathrm{Im}(\mathbb{O}')$ on which the product vanishes.

G_2 acts as symmetries of this incidence geometry, hence of the spinor rolling on the projective plane when $R = 3$.

Coda

- A spinor needs to turn twice to get back where it started.
- On a projective plane, we get back where we started by going half way around.
- For what ratio of radii do we turn twice as we roll half way around?

Only 1:3

Not in this talk

- Null triples: generators $x, y, z \in \mathbb{O}'$ such that

$$g\colon (x, y, z) \mapsto (x', y', z')$$

  for a unique $g \in G_2$.
- Reconstructing $\mathbb{O}'$ from the rolling ball system using geometric quantization.

For more, see our paper.

Thank you!