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Symplectic Embeddings

Manuel Luís Henriques de Araújo

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Presidente: Professora Doutora Ana Leonor Mestre Vicente Silvestre
Orientador: Professor Doutor Gustavo Rui Gonçalves Fernandes de Oliveira Granja
Vogais: Professora Doutora Sílvia Nogueira da Rocha Ravasco dos Anjos
Professor Doutor João Paulo Neves Monteiro dos Santos

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Resumo

Seja (M, ω) uma variedade simpléctica compacta com $[\omega]$ integral. Provamos um Teorema de Tischler ([16]) e Gromov ([4]) que garante a existência de um mergulho simpléctico de (M, ω) em $\mathbb{C}P^n$, com a forma de Fubini-Study, para n suficientemente grande (corrigindo e simplificando a demonstração dada em [16]). Seja $\beta_0(M)$ o número de componentes conexas de M e $\beta_1(M)$ o primeiro número de Betti de M. Refinamos o Teorema de Tischler e Gromov, provando que o tipo de homotopia fraco do espaço dos mergulhos simplécticos de (M, ω) em $\mathbb{C}P^\infty$ é $(S^1)^{\beta_1(M)} \times (\mathbb{C}P^\infty)^{\beta_0(M)}$.

Palavras Chave: *Mergulho*, *Simpléctico*, *Espaço*, *Projectivo*, *Tipo*, *Homotopia*.

Abstract

Let (M, ω) be a closed symplectic manifold with $[\omega]$ integral. We prove a Theorem of Tischler ([16])) and Gromov ([4]) that (M, ω) symplectically embeds into $\mathbb{C}P^n$ with the Fubini-Study symplectic form, for n large enough (correcting and simplifying the argument given in [16]). Let $\beta_0(M)$ be the number of connected components of M and $\beta_1(M)$ the first Betti number of M. We refine this result of Tischler and Gromov by showing that the weak homotopy type of the space of symplectic embeddings of such a symplectic manifold into $\mathbb{C}P^\infty$ is $(S^1)^{\beta_1(M)} \times (\mathbb{C}P^\infty)^{\beta_0(M)}$.

Keywords: *Symplectic*, *Embedding*, *Projective*, *Space*, *Homotopy*, *Type*.

Contents

Introduction

Symplectic geometry first appeared as a theoretical tool for studying Hamiltonian mechanics and has since become a very active area of Mathematics, interesting in its own right. In this thesis we study an embedding problem in symplectic geometry. It is often useful, in Mathematics, to view an object as a subobject of one of particularly simple type. For instance, it is a basic fact in Differential Topology that any manifold can be embedded into $\mathbb{R}^n$, for sufficiently large n (Whitney's embedding Theorem), and embeddings in projective space play a fundamental role in Algebraic Geometry.

If (M, α) and (N, β) are symplectic manifolds, then a smooth map $f : M \to N$ is a *symplectic embedding* if it is an embedding and $f^*\beta = \alpha$. In analogy with Whitney's embedding theorem, it is natural to ask whether there exists a "universal" family of symplectic manifolds, in which any compact symplectic manifold can be embedded. A first guess would be $\mathbb{R}^{2n}$ with the standard symplectic form $\lambda_n = \sum_i dx_i \wedge dy_i$, but this does not work, for topological reasons. Indeed, if M is a compact manifold and $f : M \to \mathbb{R}^{2n}$ is an embedding, then the cohomology class of the pullback is $[f^*\lambda_n] = f^*([\lambda_n]) = f^*(0) = 0$. However, a symplectic form on a compact manifold is never exact, because its top exterior power is a volume form, therefore $f^*\lambda_n$ can never be a symplectic form. The next natural guess would be $(\mathbb{C}P^n, \Omega_n)$, where Ω_n is the Fubini-Study form. It is in fact a Theorem, due to Tischler ([16]) and Gromov ([4]), that every compact symplectic manifold (M, Ω), with Ω integral, admits a symplectic embedding into $(\mathbb{C}P^n, \Omega_n)$, for sufficiently large n. The condition that Ω be integral is necessary, because Ω_n is integral and therefore, for any smooth map $f : M \to \mathbb{C}P^n$, the form $f^*\Omega_n$ is also integral.

In this thesis, we give a proof of this Theorem of Tischler and Gromov, simplifying and correcting a mistake in Tischler's proof ([16]). The main idea in the proof is a method for "symplectic approximation" of smooth maps satisfying the necessary cohomological condition $[f^*\Omega_n] = [\Omega]$. We will also prove a parametric version of this symplectic approximation result and use it to prove our main Theorem, which is the following refinement of the symplectic embedding Theorem of Gromov and Tischler:

Theorem 0.0.1. *Let (M, Ω) be a compact symplectic manifold with Ω integral, $\beta_0(M)$ be the number of connected components of M and $\beta_1(M)$ the first Betti number of M. The space $\mathrm{SympEmb}(M, \mathbb{C}P^\infty)$ of symplectic embeddings of M into $\mathbb{C}P^\infty$ is weakly homotopy equivalent to $(S^1)^{\beta_1(M)} \times (\mathbb{C}P^\infty)^{\beta_0(M)}$.*

This is proved below as Theorem 3.0.2. The space $\mathrm{SympEmb}(M, \mathbb{C}P^\infty)$ of symplectic embeddings of M into $\mathbb{C}P^\infty$ is the union of the spaces $\mathrm{SympEmb}(M, \mathbb{C}P^n)$ of symplectic embeddings of M into $\mathbb{C}P^n$ with the colimit topology induced by the C^∞ topology on $\mathrm{SympEmb}(M, \mathbb{C}P^n)$. Theorem 0.0.1 should be compared with the following well known refinement of Whitney's embedding Theorem: the space of embeddings of a compact manifold into $\mathbb{R}^\infty$ is contractible.

It should be possible to compute the weak homotopy type of $\mathrm{SympEmb}(M, \mathbb{C}P^\infty)$ (as well as some homotopy groups of other spaces of symplectic embeddings) using Gromov's h-principle for symplectic embeddings (see [5] section 3.4.2 and [3], section 12.1), but we have decided to use more elementary methods, inspired by Tischler's proof.

In the same vein, we include as an appendix an elementary proof of the fact that it is possible to find continuous families of primitives for continuous families of exact forms on a compact manifold, a result which is needed for the proof of the parametric version of symplectic approximation in Chapter 3. The standard proof of this fact uses Hodge theory, but we have chosen to spell out an argument using partitions of unity which is alluded to (somewhat misleadingly) in [12, p. 96].

Organization of the Thesis: In the first chapter, we explain the construction of the standard symplectic structure on $\mathbb{C}P^n$, the Fubini-Study form, and we examine the properties of this symplectic form that will be relevant for the proof of the main theorems. In the second Chapter, we give a detailed proof of Tischler and Gromov's Theorem, following [16]. In the third chapter, we prove Theorem 0.0.1. Finally, there is an appendix where we show that it is possible to find a continuous family of primitives for any continuous family of exact forms on a compact manifold. See the introductions to the individual chapters for a more detailed description of their contents.

Prerequisites: We assume that the reader is familiar with Differential Topology at the level of [7], Algebraic Topology at the level of [6] and elementary Symplectic Geometry ([2]). Specifically, the reader will need to be familiar with singular and de Rham cohomology and the (weak) topology on $C^k(M, N)$.

Chapter 1

The symplectic structure on $\mathbb{C}P^n$

In this chapter, we describe the canonical symplectic structure Ω_n on $\mathbb{C}P^n$, called the *Fubini-Study* form and prove some properties which will be useful in the following chapters. In the first section, we define the *Fubini-Study* form and describe explicit Darboux coordinates on $\mathbb{C}P^n$ centered at each point following [16] (these are coordinates in which Ω_n becomes the standard symplectic form $\frac{1}{\pi}\sum_i dx_i \wedge dy_i$). We also prove that the forms Ω_n and the Darboux coordinates are natural with respect to the standard inclusions $\mathbb{C}P^n \hookrightarrow \mathbb{C}P^{n+1}$ and $\mathbb{C}^n \hookrightarrow \mathbb{C}^{n+1}$. In the second section, we show that Ω_n is integral and that $[\Omega_n]$ is a generator of $H^2(\mathbb{C}P^n;\mathbb{Z})$.

1.1 The Fubini-Study form

Let $\omega_n \in \Omega^2(\mathbb{C}^n)$ be defined by

$$\omega_n = \frac{i}{2\pi}\partial\bar{\partial}\log(|z|^2+1), \tag{1.1}$$

where $d = \partial + \bar{\partial}$ is the usual decomposition of the exterior derivative on a complex manifold (see [2, Lecture 15]). By [2, Section 16.3], this is a symplectic form on $\mathbb{C}^n$. If we consider the standard charts $\phi_i^n : U_i^n \to \mathbb{C}^n$ given by

$$[z_0 : \cdots : z_n] \mapsto \frac{1}{z_i}(z_0, \cdots, z_{i-1}, z_{i+1}, \cdots, z_n),$$

where

$$U_i^n = \{[z_0, \cdots, z_n] \in \mathbb{C}P^n : z_i \neq 0\},$$

we get a form $(\phi_i^n)^*\omega_n$ on U_i^n, for each i. It is easy to see that these forms agree on the intersections $U_i^n \cap U_j^n$ (see [2, Homework 12]) so the following definition makes sense:

Definition 1.1.1. The Fubini-Study form Ω_n is the unique 2-form on $\mathbb{C}P^n$ such that $\Omega_n|_{U_i} = (\phi_i^n)^*(\omega_n)$.

Now let $B^n(1)$ be the open unit ball in $\mathbb{C}^n$, let $H^n : \mathbb{C}^n \to B^n(1)$ be given by

$$z \mapsto \frac{z}{(1+|z|^2)^{1/2}}$$

and define coordinates $\varphi_i^n : U_i^n \to B^n(1)$, by

$$\varphi_i^n = H^n \circ \phi_i^n.$$

Proposition 1.1.2. *If $x_j + iy_j$ are the usual coordinates on $B^n(1) \subset \mathbb{C}^n$, then*

$$\Omega_n|_{U_i^n} = (\varphi_i^n)^*(\frac{1}{\pi}\sum_{j=1}^n dx_j \wedge dy_j).$$

Proof. Since $\Omega_n|_{U_i^n} = (\phi_i^n)^*\omega_n$, we only need to show that

$$\omega_n = (H^n)^*(\frac{1}{\pi}\sum_{j=1}^n dx_j \wedge dy_j).$$

We begin by computing ω_n:

$$\begin{aligned}
\omega_n &= \frac{i}{2\pi}\partial\bar{\partial}\log(1+|z|^2) \\
&= \frac{i}{2\pi}\partial\bar{\partial}\log\left(1+\sum_{\ell=1}^n z_\ell\bar{z}_\ell\right) \\
&= \frac{i}{2\pi}\partial\left(\sum_j \frac{z_j}{1+\sum_\ell z_\ell\bar{z}_\ell}d\bar{z}_j\right) \\
&= \frac{i}{2\pi}\sum_j \partial\left(\frac{z_j}{1+\sum_\ell z_\ell\bar{z}_\ell}\right)\wedge d\bar{z}_j.
\end{aligned}$$

Now

$$\partial\left(\frac{z_j}{1+\sum_\ell z_\ell\bar{z}_\ell}\right) = \sum_k \frac{\delta_{jk}(|z|^2+1) - z_j\bar{z}_k}{(|z|^2+1)^2}dz_k,$$

so we have

$$\begin{aligned}
\omega_n &= \frac{i}{2\pi}\frac{\sum_j\sum_k(\delta_{jk}(|z|^2+1) - z_j\bar{z}_k)dz_k\wedge d\bar{z}_j}{(|z|^2+1)^2} \\
&= \frac{i}{2\pi(|z|^2+1)^2}\left((|z|^2+1)\sum_\ell dz_\ell\wedge d\bar{z}_\ell - \sum_{j,k} z_j\bar{z}_k dz_k\wedge d\bar{z}_j\right) \\
&= \frac{i}{2\pi}\left(\frac{\sum_\ell dz_\ell\wedge d\bar{z}_\ell}{|z|^2+1} - \frac{\sum_{j,k} z_j\bar{z}_k dz_k\wedge d\bar{z}_j}{(|z|^2+1)^2}\right).
\end{aligned}$$

Now we compute $(H^n)^*\left(\frac{1}{\pi}\sum_{j=1}^n dx_j\wedge dy_j\right)$.

We have

$$\frac{1}{\pi}\sum_{j=1}^n dx_j\wedge dy_j = \frac{i}{2\pi}\sum_{j=1}^n dw_j\wedge d\bar{w}_j,$$

where $w_j = x_j + iy_j$ are the usual coordinates.

To compute $(H^n)^*(\frac{1}{\pi}\sum_{j=1}^n dx_j\wedge dy_j)$, we set $w_j = \frac{z_j}{(|z|^2+1)^{1/2}}$ and we get

$$
\begin{aligned}
dw_j &= d\left(\frac{z_j}{(1+|z|^2)^{1/2}}\right)\\
&= \frac{dz_j(|z|^2+1)^{1/2} - z_j\frac{1}{2}(|z|^2+1)^{-1/2}d\left(|z|^2\right)}{|z|^2+1}
\end{aligned}
$$

and

$$
\begin{aligned}
d\bar{w}_j &= d\left(\frac{\bar{z}_j}{\left(1+|z|^2\right)^{1/2}}\right)\\
&= \frac{d\bar{z}_j(|z|^2+1)^{1/2} - \bar{z}_j\frac{1}{2}(|z|^2+1)^{-1/2}d\left(|z|^2\right)}{|z|^2+1}.
\end{aligned}
$$

Therefore,

$$
dw_j \wedge d\bar{w}_j = \frac{dz_j \wedge d\bar{z}_j}{|z|^2+1} - \frac{\bar{z}_j\frac{1}{2}dz_j \wedge d\left(|z|^2\right)}{(|z|^2+1)^2} - \frac{z_j\frac{1}{2}d\left(|z|^2\right)\wedge d\bar{z}_j}{(|z|^2+1)^2}.
$$

Putting everything together, we get

$$
(H^n)^*\left(\frac{1}{\pi}\sum_{j=1}^{n} dx_j \wedge dy_j\right) = \frac{i}{2\pi}\left(\frac{\sum_j dz_j \wedge d\bar{z}_j}{|z|^2+1} - \frac{\frac{1}{2}\sum_j\left(\bar{z}_j dz_j \wedge d\left(|z|^2\right) + z_j d\left(|z|^2\right)\wedge d\bar{z}_j\right)}{(|z|^2+1)^2}\right).
$$

So, in order to finish the proof, we need to show

$$
\frac{1}{2}\sum_j\left(\bar{z}_j dz_j \wedge d\left(|z|^2\right) + z_j d\left(|z|^2\right)\wedge d\bar{z}_j\right) = \sum_{j,k} z_j\bar{z}_k dz_k \wedge d\bar{z}_j.
$$

We have

$$
\begin{aligned}
d(|z|^2) &= d\left(\sum_k z_k\bar{z_k}\right)\\
&= \sum_k(\bar{z_k}dz_k + z_k d\bar{z_k})
\end{aligned}
$$

and so

$$
\bar{z_j}dz_j \wedge d\left(|z|^2\right) = \bar{z_j}\sum_k(\bar{z_k}dz_j \wedge dz_k + z_k dz_j \wedge d\bar{z_k})
$$

and

$$
z_j d\left(|z|^2\right)\wedge d\bar{z_j} = z_j\sum_k(\bar{z_k}dz_k \wedge d\bar{z_j} + z_k d\bar{z_k}\wedge d\bar{z_j}).
$$

Putting both computations together, we get

$$
\begin{aligned}
&\frac{1}{2}\sum_j \left(\bar{z}_j dz_j \wedge d\left(|z|^2\right) + z_j d\left(|z|^2\right) \wedge d\bar{z}_j\right) \\
=&\frac{1}{2}\sum_{j,k}(\bar{z}_j\bar{z}_k dz_j \wedge dz_k + \bar{z}_j z_k dz_j \wedge d\bar{z}_k + z_j\bar{z}_k dz_k \wedge d\bar{z}_j + z_j z_k d\bar{z}_k \wedge d\bar{z}_j) \\
=&\sum_{j,k} z_j\bar{z}_k dz_k \wedge d\bar{z}_j,
\end{aligned}
$$

because

$$
\sum_{j,k}\bar{z}_j\bar{z}_k dz_j \wedge dz_k = \sum_{j,k} z_j z_k d\bar{z}_k \wedge d\bar{z}_j = 0.
$$

□

More generally, we have *standard coordinates* in a neighbourhood of each point in $\mathbb{C}P^n$, in which Ω_n takes the above form. These are constructed as follows (as done in [16]).

For each $p \in \mathbb{C}P^n$, we choose an $x = x(p) \in S^{2n+1} \subset \mathbb{C}^{n+1}$ which represents p. We fix this choice for the rest of the text and we make the choice in such a way that it doesn't depend on n, meaning that if $p \in \mathbb{C}P^n$, then $j_{n+1}(x(p)) = x(i_n(p))$, where $i_n : \mathbb{C}P^n \hookrightarrow \mathbb{C}P^{n+1}$ is the inclusion given by

$$
[z_0 : \cdots : z_n] \to [z_0 : \cdots : z_n : 0]
$$

and $j_{n+1} : \mathbb{C}^{n+1} \to \mathbb{C}^{n+2}$ is the inclusion given by

$$
(z_0, \cdots, z_n) \to (z_0, \cdots, z_n, 0).
$$

We let T_x^n be the complex hyperplane (in $\mathbb{C}^{n+1}$) that passes through x and is orthogonal to x. Let D_p^n be the subspace of $\mathbb{C}P^n$ consisting of those lines that intersect T_x^n (note that this does not depend on the choice of x) and denote by $S_x^n : D_p^n \to T_x^n$ the map sending each line to its intersection point with T_x^n.

We can identify T_x^n with $\mathbb{C}^n$ by using a unitary transformation of $\mathbb{C}^{n+1}$ that sends x to $(1, 0, \cdots, 0)$ and then forgetting the first coordinate (which is always 1). We choose these unitary transformations $A_x^m : \mathbb{C}^{m+1} \to \mathbb{C}^{m+1}$ sending x to $(1, 0, \cdots, 0)$ in a compatible way, meaning that $A_x^{m+1}|_{\mathbb{C}^{m+1}} = A_x^m$. Composing $S_x^n : D_p^n \to T_x^n$ with this identification of T_x^n with $\mathbb{C}^n$, we get a map $\phi_x^n : D_p^n \to \mathbb{C}^n$. Another way to describe this map is that given $q = [z_0 : \cdots : z_n] \in D_p^n$, we write

$$
\lambda(z_0, \cdots, z_n) = x + a_1 w_1 + \cdots + a_n w_n,
$$

for some $\lambda \in \mathbb{C}$, where $\{x, w_1, \cdots, w_n\}$ is a unitary basis of $\mathbb{C}^{n+1}$ and we map q to $(a_1, \cdots, a_n) \in \mathbb{C}^n$ (note that $\lambda(z_0, \cdots, z_n)$ is the intersection point of the line $q \in \mathbb{C}P^n$ with T_x^n). Note that the choice of unitary basis corresponds to the choice of unitary transformation A_x^n of $\mathbb{C}^{n+1}$ sending x to $(1, 0, \cdots, 0)$, the correspondence being $A_x^n(w_i) = e_i$, where $\{e_0, \cdots, e_n\}$ is the standard basis of $\mathbb{C}^{n+1}$.

Lemma 1.1.3. *Let $p \in \mathbb{C}P^n$ and $\tilde{x} = \tilde{x}(p) \in S^{2n+1} \subset \mathbb{C}^{n+1}$ be a representative of of p. Let*

$$
\tilde{A}_{\tilde{x}}^n : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}
$$

be a unitary transformation of $\mathbb{C}^{n+1}$ sending $\tilde{x}$ to $(1, 0, \cdots, 0)$ and define $\tilde{\phi}^n_{\tilde{x}} : D^n_p \to \mathbb{C}^n$ as above, using $\tilde{A}^n_{\tilde{x}}$ to identify $T^n_{\tilde{x}}$ with $\mathbb{C}^n$, i.e, $\tilde{\phi}^n_{\tilde{x}} = \tilde{A}^n_{\tilde{x}} \circ S^n_{\tilde{x}}$. Then $\tilde{\phi}^n_{\tilde{x}} = B \circ \phi^n_x$, where $B : \mathbb{C}^n \to \mathbb{C}^n$ is unitary.

Proof. We have $\alpha\tilde{x} = x$, for some $\alpha \in \mathbb{C}$ with absolute value 1. Consider the unitary basis $\{\tilde{x}, v_1, \cdots, v_n\}$ of $\mathbb{C}^{n+1}$ used to define $\tilde{\phi}^n_{\tilde{x}}$. Then $\{x, \alpha v_1, \cdots, \alpha v_n\}$ is a unitary basis of $\mathbb{C}^{n+1}$ and if $\bar{\phi}^n_x$ is defined using this basis, then $\bar{\phi}^n_x = \tilde{\phi}^n_{\tilde{x}}$. Let $\{x, w_1, \cdots, w_n\}$ be the unitary basis of $\mathbb{C}^{n+1}$ used to define ϕ^n_x and let B be the unitary matrix whose j^{th} column is the vector w_j written in the basis $\{\alpha v_1, \cdots, \alpha v_n\}$ of $\{x\}^\perp \subset \mathbb{C}^{n+1}$. Then, clearly, $\bar{\phi}^n_x = B \circ \phi^n_x$. □

Lemma 1.1.4. *The form ω_n defined in* (1.1) *is invariant under unitary transformations of $\mathbb{C}^n$.*

Proof. Unitary transformations are holomorphic, so they commute with ∂ and $\bar{\partial}$. Furthermore, they preserve norms, so they preserve $\omega_n = \frac{i}{2\pi}\partial\bar{\partial}\log(1 + |z|^2)$.

□

In view of the two previous Lemmas, $(\phi^n_x)^*\omega_n$ does not depend on the particular choices of x and A^{n+1}_x. In fact, we will now prove that $(\phi^n_x)^*\omega_n = \Omega_n|_{D_p}$.

Lemma 1.1.5. *Take $p \in \mathbb{C}P^n$ and suppose $D^n_p \cap U^n_i \neq \emptyset$, for some i. Then $\phi^n_x \circ (\phi^n_i)^{-1}$ is the composite*

$$\psi_i \circ A^n_x \circ \xi_i : \phi^n_i(U^n_i \cap D^n_p) \to \phi^n_x(U^n_i \cap D^n_p),$$

where $\xi_i : \mathbb{C}^n \to \mathbb{C}^{n+1}$ is defined by

$$(z_1, \cdots, z_n) \mapsto (z_1, \cdots, z_{i-1}, 1, z_i, \cdots, z_n)$$

and $\psi_i : \{(z_0, \cdots, z_n) \in \mathbb{C}^{n+1} : z_i \neq 0\} \to \mathbb{C}^n$ is defined by

$$(z_0, \cdots, z_n) \mapsto \frac{1}{z_i}(z_0, \cdots, \hat{z}_i, \cdots, z_n).$$

Proof. We do only the case $i = 0$, because it simplifies the notation. Take

$$x = (x_1, \cdots, x_n) \in \phi^n_0(U^n_0 \cap D^n_p) \subset \mathbb{C}^n.$$

Then

$$(\phi^n_0)^{-1}(x) = [1 : x_1 : \cdots : x_n] \in D^n_p,$$

so we have $\lambda(1, x_1, \cdots, x_n) \in T^n_x$, for some $\lambda \in \mathbb{C}$, therefore we can write

$$\lambda(1, x_1, \cdots, x_n) = x + a_1 w_1 + \cdots + a_n w_n,$$

where $\{x, w_1, \cdots, w_n\}$ is the unitary basis of $\mathbb{C}^{n+1}$ defined by $A^n_x(w_i) = e_i$. By definition,

$$\phi^n_x \circ (\phi^n_0)^{-1}(x_1, \cdots, x_n) = (a_1, \cdots, a_n).$$

We have

$$A_x^n(\lambda(1, x_1, \cdots, x_n)) = A_x^n(x + a_1 w_1 + \cdots + a_n w_n) = (1, a_1, \cdots, a_n),$$

so

$$A_x^n(1, x_1, \cdots, x_n) = (1/\lambda, a_1/\lambda, \cdots, a_n/\lambda),$$

and then

$$\psi_0(A_x^n(\xi_0(x))) = \psi_0(A_x^n(1, x_1, \cdots, x_n)) = \psi_0(1/\lambda, a_1/\lambda, \cdots, a_n/\lambda) = (a_1, \cdots, a_n)$$

and this completes the proof.

□

Proposition 1.1.6. *Let $p \in \mathbb{C}P^n$ and $x = x(p) \in S^{2n+1} \subset \mathbb{C}^{n+1}$ be the chosen representative of p. Then*

$$\Omega_n|_{D_p} = (\phi_x^n)^* \omega_n.$$

Proof. By the definition of Ω_n and the previous Lemma, we only need to check that ω_n is invariant under $\psi_i \circ A_x^n \circ \xi_i$, for any i.

We do it for $i = 0$, for simplicity of notation. We have

$$\psi_0^* \partial\bar{\partial} \log(1 + |z|^2) = \partial\bar{\partial} \log(1 + |z_1/z_0|^2 + \cdots + |z_n/z_0|^2) = \partial\bar{\partial}(\log |z|^2 - \log |z_0|^2)$$

and

$$\partial\bar{\partial} \log |z_0|^2 = \partial\bar{\partial} \log(z_0 \bar{z}_0) = \partial \frac{z_0}{z_0 \bar{z}_0} = \partial \frac{1}{\bar{z}_0} = 0,$$

so

$$\psi_0^* \partial\bar{\partial} \log(1 + |z|^2) = \partial\bar{\partial} \log |z|^2.$$

Since A_x^n is unitary, we have

$$(A_x^n)^* \partial\bar{\partial} \log |z|^2 = \partial\bar{\partial} \log |z|^2.$$

Finally, it is clear that

$$\xi_0^* \partial\bar{\partial} \log |z|^2 = \partial\bar{\partial} \log(1 + |z|^2)$$

and this completes the proof.

□

Composing $\phi_x^n : D_p^n \to \mathbb{C}^n$ with $H^n : \mathbb{C}^n \to B^n(1)$ yields a diffeomorphism $\varphi_x^n : D_p^n \to B^n(1)$. In these coordinates the symplectic form Ω_n on $\mathbb{C}P^n$ is

$$\frac{1}{\pi}\sum_{i=1}^{n} dx_i \wedge dy_i.$$

Definition 1.1.7. The *standard coordinates* on D_p^n are the coordinates defined by

$$\varphi_x^n = H^n \circ \phi_x^n : D_p^n \to B^n(1),$$

where $x = x(p) \in S^{2n+1}$ is the chosen representative of p.

Now consider $i_n : \mathbb{C}P^n \hookrightarrow \mathbb{C}P^{n+1}$ the inclusion given by

$$[z_0 : \cdots : z_n] \to [z_0 : \cdots : z_n : 0].$$

Proposition 1.1.8. $i_n^*\Omega_{n+1} = \Omega_n$.

Proof. First notice that if we use coordinates

$$\phi_i^n : U_i \to \mathbb{C}^n \text{ and } \phi_i^{n+1} : V_i \to \mathbb{C}^{n+1},$$

where

$$U_i = \{[z_0 : \cdots : z_n] : z_i \neq 0\}, V_i = \{[z_0 : \cdots : z_{n+1}] : z_i \neq 0\}$$

and ϕ_i^n, ϕ_i^{n+1} are the usual coordinates on these open sets, then i_n becomes the map $j_n : \mathbb{C}^n \to \mathbb{C}^{n+1}$ given by $(z_1, \cdots, z_n) \to (z_1, \cdots, z_n, 0)$. So it is enough to check that

$$j_n^*(\partial\bar{\partial}\log(1 + \sum_{k=1}^{n+1} |z_k|^2)) = \partial\bar{\partial}\log(1 + \sum_{k=1}^{n} |z_k|^2).$$

This follows easily from the fact that ∂ and $\bar{\partial}$ commute with f^*, for any holomorphic map f.

□

Note that in the previous proof, it was useful to know that the following square commutes:

$$\begin{array}{ccc} \mathbb{C}P^n \supset U_i^n & \xrightarrow{\phi_i^n} & \mathbb{C}^n \\ \downarrow i_n & & \downarrow j_n \\ \mathbb{C}P^{n+1} \supset U_i^{n+1} & \xrightarrow{\phi_i^{n+1}} & \mathbb{C}^{n+1}. \end{array}$$

In the following chapters, it will be useful to know that similar properties hold for the charts $\phi_x^n : D_p^n \to \mathbb{C}^n$ we defined above.

Proposition 1.1.9. *For any $p \in \mathbb{C}P^n$, consider $x = x(p) \in S^{2n+1} \subset \mathbb{C}^{n+1}$ the chosen representative. Then, the following diagram commutes:*

$$\begin{array}{ccc} \mathbb{C}P^n \supset D_p^n & \xrightarrow{\phi_x^n} & \mathbb{C}^n \\ \Big\downarrow{\scriptstyle i_n} & & \Big\downarrow{\scriptstyle j_n} \\ \mathbb{C}P^{n+1} \supset D_p^{n+1} & \xrightarrow{\phi_x^{n+1}} & \mathbb{C}^{n+1}. \end{array}$$

Proof. This is clear, because we picked the unitary transformations $A_x^m : \mathbb{C}^{m+1} \to \mathbb{C}^{m+1}$ sending x to $(1, 0, \cdots, 0)$ so that $A_x^{m+1}|_{\mathbb{C}^{m+1}} = A_x^m$ for all m.

□

It is now clear that the following diagram also commutes:

$$\begin{array}{ccc} \mathbb{C}P^n \supset D_p^n & \xrightarrow{\varphi_x^n} & B^n(1) \\ \Big\downarrow{\scriptstyle i_n} & & \Big\downarrow{\scriptstyle j_n} \\ \mathbb{C}P^{n+1} \supset D_p^{n+1} & \xrightarrow{\varphi_x^{n+1}} & B^{n+1}(1). \end{array}$$

Note that $D_p^{n+1} \cap \mathbb{C}P^n = D_p^n$, for $p \in \mathbb{C}P^n$, because $T_x^{n+1} \cap \mathbb{C}^{n+1} = T_x^n$.

1.2 The cohomology class of the Fubini-Study form

Now we show that the Fubini-Study form Ω_n is integral and $[\Omega_n]$ generates $H^2(\mathbb{C}P^n; \mathbb{Z})$. This allows us to fix a generator $\chi \in H^2(\mathbb{C}P^\infty)$ restricting to $[\Omega_n]$ for all n.

We start by explaining what it means for a form to be integral and to generate $H^2(\mathbb{C}P^n; \mathbb{Z})$. Consider a compact orientable manifold M. The inclusion $\mathbb{Z} \hookrightarrow \mathbb{R}$ induces a map

$$H^k(M; \mathbb{Z}) \to H^k(M; \mathbb{R}).$$

We say that a cohomology class $\alpha \in H^k(M; \mathbb{R})$ is *integral* if it is in the image of this map. By the *de Rham Theorem* there is a canonical isomorphism

$$H^k_{dR}(M) \xrightarrow{\sim} H^k(M; \mathbb{R}),$$

defined as usual by integration. Composing these two maps, we obtain a map

$$H^k(M; \mathbb{Z}) \to H^k_{dR}(M).$$

We say that a cohomology class $[\omega] \in H^k_{dR}(M)$ is *integral* if it is in the image of this map. We say that a closed form $\omega \in \Omega^k(M)$ is integral if $[\omega]$ is integral. Considering the diagram

$$\begin{array}{ccc} H^k(M; \mathbb{Z}) & \longrightarrow & H^k(M; \mathbb{R}) \\ \Big\downarrow{\scriptstyle e} & & \Big\downarrow{\scriptstyle \wr} \\ \mathrm{Hom}(H_k(M), \mathbb{Z}) & \hookrightarrow & \mathrm{Hom}(H_k(M), \mathbb{R}) \end{array}$$

we see that a class $\alpha \in H^k(M;\mathbb{R})$ is integral if and only if it takes integer values on all integral cycles. Therefore, a class $[\omega] \in H^k_{dR}(M)$ is integral if and only if its integral on every closed integral k-chain is an integer.

In the case when $M = \mathbb{C}P^n$, the map e in the above diagram is an isomorphism (by the Universal Coefficients Theorem), therefore the map

$$H^k(\mathbb{C}P^n;\mathbb{Z}) \hookrightarrow H^k(\mathbb{C}P^n;\mathbb{R}) = H^k_{dR}(\mathbb{C}P^n)$$

is a monomorphism. So we can think of $H^k(\mathbb{C}P^n;\mathbb{Z})$ as a subgroup of $H^k_{dR}(\mathbb{C}P^n)$ and then it makes sense to say that $[\Omega_n]$ is a generator of $H^2(\mathbb{C}P^n;\mathbb{Z})$.

Proposition 1.2.1. *$[\Omega_n]$ is integral and it is a generator of $H^2(\mathbb{C}P^n;\mathbb{Z})$.*

Proof. Since

$$H^2(\mathbb{C}P^n;\mathbb{Z}) \cong \mathbb{Z} \text{ and } H^2(\mathbb{C}P^n;\mathbb{R}) \cong \mathbb{R},$$

we can pick a generator $[\omega]$ of $H^2(\mathbb{C}P^n;\mathbb{Z}) \subset H^2_{dR}(\mathbb{C}P^n)$ and then choose $\lambda \in \mathbb{R}$ such that $[\Omega_n] = \lambda[\omega]$. Then

$$\int_{\mathbb{C}P^n} \Omega_n^n = \lambda^n \int_{\mathbb{C}P^n} \omega^n = \pm\lambda^n,$$

because

$$\int_{\mathbb{C}P^n} \omega^n = \pm 1,$$

since $[\omega^n]$ is a generator of $H^{2n}(\mathbb{C}P^n;\mathbb{Z})$. So, if we can show that

$$\int_{\mathbb{C}P^n} \Omega_n^n = \pm 1,$$

then we will have $\lambda^n = \pm 1$ and therefore $\lambda = \pm 1$ and so $[\Omega_n]$ is integral and furthermore it is a generator of $H^2(\mathbb{C}P^n;\mathbb{Z})$.

It is very easy to check that

$$\int_{\mathbb{C}P^n} \Omega_n^n = 1,$$

doing the computation in one of the standard coordinate charts (note that D^n_p has full measure) and using the fact the the volume of the unit ball in $\mathbb{R}^{2n} = \mathbb{C}^n$ is $\pi^n/n!$. □

Consider

$$\mathbb{C}P^\infty = \operatorname{colim}(\cdots \to \mathbb{C}P^n \xrightarrow{i_n} \mathbb{C}P^{n+1} \to \cdots)$$

and let $\iota_n : \mathbb{C}P^n \to \mathbb{C}P^\infty$ be the canonical maps. Since $\iota_n^* : H^2(\mathbb{C}P^\infty;\mathbb{Z}) \to H^2(\mathbb{C}P^n;\mathbb{Z})$ are isomorphisms of infinite cyclic groups, $\iota_n = \iota_{n+1} \circ i_n$ and $i_n^*\Omega_{n+1} = \Omega_n$, we can choose a generator $\chi \in H^2(\mathbb{C}P^\infty;\mathbb{Z})$ such that

$$\iota_n^*\chi = [\Omega_n] \text{ for all } n. \tag{1.2}$$

We leave this χ fixed for the rest of the text.

Chapter 2

Existence of Symplectic Embeddings into $\mathbb{C}P^n$

Recall that Ω_n is the Fubini-Study form on $\mathbb{C}P^n$ (see Definition 1.1.1). We will abuse notation and not distinguish between a map $f : M \to \mathbb{C}P^n$ and its composite with a canonical inclusion $\mathbb{C}P^n \hookrightarrow \mathbb{C}P^{n+k}$. This will cause no trouble, in view of the results of Chapter 1. The aim of this chapter is to prove the following Theorem of Tischler ([16]) and Gromov ([4]):

Theorem 2.0.1. *Let (M, Ω) be a closed symplectic manifold, with $[\Omega]$ integral. Then, there exists $N \in \mathbb{N}$ and a symplectic embedding $(M, \Omega) \hookrightarrow (\mathbb{C}P^N, \Omega_N)$.*

Proof. We follow the proof in [16] (while also simplifying and correcting a mistake in this proof). The proof is done in two steps. The first step is to show that there is an embedding $f_0 : M \to \mathbb{C}P^n$, for some n, with $[f_0^*\Omega_n] = [\Omega]$. This is the main result of Section 2.1 and appears below as Proposition 2.1.1. It is a purely topological result, involving only basic Differential and Algebraic Topology. The second and main step is to show that, for N large enough, the embedding f_0 can be approximated by a symplectic embedding $f : (M, \Omega) \to (\mathbb{C}P^N, \Omega_N)$. This is the main result of Section 2.2 and appears below as Theorem 2.2.1. □

2.1 Realization of a cohomology class by an embedding

The aim of this section is to prove the following result:

Proposition 2.1.1. *Let (M, Ω) be a closed symplectic manifold, with $[\Omega]$ integral. Then, there is $n \in \mathbb{N}$ and an embedding $f_0 : M \hookrightarrow \mathbb{C}P^n$ such that $f_0^*[\Omega_n] = [\Omega]$.*

Proof. Recall that $\chi \in H^2(\mathbb{C}P^\infty; \mathbb{Z})$ is the generator described in (1.2). Since $\mathbb{C}P^\infty$ is a $K(\mathbb{Z}, 2)$ (see [6], Example 4.50, page 380) and M has the homotopy type of a cell complex (see [6, p. 529]) we can find a continuous map $f : M \to \mathbb{C}P^\infty$ such that $f^*\chi = [\Omega]$, and this map is unique up to homotopy (see [6], Section 4.3, Theorem 4.57). Since M is compact, we have $f(M) \subset \mathbb{C}P^n$, for large enough n (this follows, e.g., from Lemma 3.2.6 below). For any such n, we get $f : M \to \mathbb{C}P^n$ such that $f^*[\Omega_n] = [\Omega]$, because $\iota_n^* \chi = [\Omega_n]$. Taking $n > \dim M$, we can approximate f by an embedding $f_0 : M \to \mathbb{C}P^n$ (see [7, Theorems 2.2.6 and 2.2.13]). If f_0 is close enough to f in the C^0 topology, then, by Lemma 2.1.2 below, it will be homotopic to f and therefore $f^*[\Omega_n] = f_0^*[\Omega_n] = [\Omega]$. □

The following Lemma implies that the space $C^k(M, N)$ is locally path connected. This is more than what we used in the proof of Proposition 2.1.1, but we will need this more detailed result later.

Lemma 2.1.2. *Let M, N be closed manifolds, $f : M \to N$ a C^k map ($0 \leq k \leq \infty$) and $\mathcal{N} \subset C^k(M, N)$ a neighbourhood of f in the C^k topology. Then, there exists $\tilde{\mathcal{N}} \subset \mathcal{N}$ open, such that $f \in \tilde{\mathcal{N}}$ and for all $g \in \tilde{\mathcal{N}}$ there exists a path $\gamma : [0, 1] \to \mathcal{N}$ with $\gamma(0) = f$ and $\gamma(1) = g$.*

Proof. Let $i : N \hookrightarrow \mathbb{R}^\ell$ be an embedding (ℓ large enough) and let $\pi : U \to N$ be the projection of a tubular neighbourhood. We have continuous maps

$$i_* : C^k(M, N) \to C^k(M, U)$$

and

$$\pi_* : C^k(M, U) \to C^k(M, N).$$

Therefore $(\pi_*)^{-1}(\mathcal{N}) \subset C^k(M, U)$ is open, and $i_*(f) \in (\pi_*)^{-1}(\mathcal{N})$. If g is close enough to $i_* f$, then

$$(1 - t)i_* f + tg \in (\pi_*)^{-1}(\mathcal{N})$$

for all $t \in [0, 1]$. So there is a neighbourhood $\mathcal{N}'$ of $i_* f$ such that if $g \in \mathcal{N}'$, then $(1-t)i_* f + tg \in (\pi_*)^{-1}(\mathcal{N})$ for all $t \in [0, 1]$. Finally, $\tilde{\mathcal{N}} = (i_*)^{-1}(\mathcal{N}')$ is the desired neighbourhood, since for any $g \in \tilde{\mathcal{N}}$ one can take $\gamma(t) = \pi_*((1 - t)i_* f + t i_* g)$.

□

2.2 Symplectic approximation

The goal of this section is to prove the following Theorem:

Theorem 2.2.1. *Let $f_0 : M \to \mathbb{C}P^n$ be an embedding, such that $[f_0^* \Omega_n] = [\Omega]$. Then there is a $p \in \mathbb{N}$ and a homotopy through embeddings $f_t : M \to \mathbb{C}P^{n+p}$ ($t \in [0, p]$) such that each f_t is C^0-close to f_0 and f_p is a symplectic embedding.*

Note that, since $[f_0^* \Omega_n] = [\Omega]$, there is a 1-form ω such that $f_0^* \Omega_n + d\omega = \Omega$. The idea of the proof is to "correct" f_0 so as to add $d\omega$ to $f_0^* \Omega_n$. To do this, we write $d\omega$ as a sum of simpler forms supported on small enough open sets and then we proceed inductively, correcting our embedding to add one term of this sum to the pullback at each step.

This first Lemma shows that we can write $d\omega$ as a sum of forms with controlled supports.

Lemma 2.2.2. *Given a* 1*-form* ω *on a closed manifold* M *and a finite open cover* $\{W_\ell\}$ *of* M*, there is* $p \in \mathbb{N}$ *and functions* $h_k, t_k : M \to \mathbb{R}$ *with* $k = 1, \cdots, p$ *such that*

1. $\sum_{k=1}^{p} dh_k \wedge dt_k = d\omega$*;*
2. *For each* k *there is an* ℓ *such that* $\operatorname{supp} h_k$ *and* $\operatorname{supp} t_k$ *are contained in* W_ℓ*.*

Proof. Let $\{\rho_i\}_{i=1}^{q}$ be a finite partition of unity subordinate to a covering $\{U_i\}_{i=1}^{q}$ by coordinate charts. Then

$$d\omega = \sum_i d(\rho_i \omega).$$

In coordinates on U_i, we get

$$\rho_i \omega = \sum_{j=1}^{m} \bar{h}_j^i dx_j,$$

for some functions $\bar{h}_j^i$ with support contained in $\operatorname{supp} \rho_i$. Now we pick $\lambda_i : M \to \mathbb{R}$ that is 1 on a neighbourhood $\operatorname{supp} \rho_i$ and 0 outside a slightly larger neighbourhood of $\operatorname{supp} \rho_i$ contained in U_i and we define $\bar{t}_j^i = \lambda_i x_j$ on U_i, extending it to M by zero. Then

$$\rho_i \omega = \sum_{j=1}^{m} \bar{h}_j^i d(\bar{t}_j^i),$$

so

$$d(\rho_i \omega) = \sum_{j=1}^{m} d\bar{h}_j^i \wedge d\bar{t}_j^i$$

for each i and hence

$$d\omega = \sum_{i,j} d\bar{h}_j^i \wedge d\bar{t}_j^i.$$

We rewrite this as

$$d\omega = \sum_i d\bar{h}_i \wedge d\bar{t}_i.$$

Now take a partition of unity $\{\phi_k\}$ subordinate to $\{W_k\}$ and write

$$\bar{h}_i = \sum_j \phi_j \bar{h}_i$$

and

$$\bar{t}_i = \sum_k \phi_k \bar{t}_i.$$

Then

$$d\omega = \sum_{i,j,k} d(\phi_j \bar{h}_i) \wedge d(\phi_k \bar{t}_i).$$

Reindexing, we get

$$d\omega = \sum_{k=1}^{p} dh_k \wedge dt_k,$$

where the functions h_k, t_k satisfy condition 2. □

At each step of the induction, it will be necessary to control the size of the corrections we make to the previous embeddings. This is where there is a mistake in [16], specifically in the proof of Lemma 2 (on page 233, there would have to be N^2 copies instead of N and this invalidates the rest of the argument). The following Lemmas will be used to effectively control the size of the modifications we make at each step.

Lemma 2.2.3. *For any $r > 0$, denote by D^2_r the disk of radius r in $\mathbb{R}^2$. Given $R \geq \epsilon > 0$, there exists a smooth map $\phi : D^2_R \to D^2_\epsilon$ that sends 0 to 0 and preserves the standard area form, i.e. satisfies*

$$\phi^*(dx \wedge dy) = dx \wedge dy.$$

Proof. Given $n \in \mathbb{N}$, consider the map

$$\phi_R : D^2_R \setminus \{0\} \to D^2_{R/\sqrt{n}}$$

defined in polar coordinates by

$$(\theta, r) \mapsto (\alpha(\theta), \rho(r)) = (n\theta, r/\sqrt{n}).$$

It is easy to compute $\rho d\rho \wedge d\alpha = r dr \wedge d\theta$, so this map preserves the area form.

Now given any disk D^2_R we can translate it away from the origin inside $\mathbb{R}^2$, so that its translation $T(D^2_R)$ sits inside $D^2_{3R} \setminus \{0\}$. Then the restriction of ϕ_{3R} to $T(D^2_R)$ gives a map

$$D^2_R \to D^2_{3R/\sqrt{n}}$$

preserving the area form.

However, this map does not send 0 to 0. To correct this, pick a much smaller disk D^2_δ inside D^2_ϵ and construct an area preserving map $\tilde{f} : D^2_R \to D^2_\delta$ as above. If δ is small enough, we can find a translation of D^2_δ inside D^2_ϵ sending $\tilde{f}(0)$ to $0 \in D^2_\epsilon$.

□

Lemma 2.2.4. *Let M be a compact manifold, $\epsilon > 0$ and $F : M \to \mathbb{R}^2$ a smooth map. Then, there exists a smooth map $f : M \to \mathbb{R}^2$ such that:*

1. $||f(x)|| < \epsilon$ *for all* $x \in M$*;*
2. $f^*(dx \wedge dy) = F^*(dx \wedge dy)$*;*
3. $\operatorname{supp} f \subset \operatorname{supp} F$*.*

Proof. Since M is compact, the image of F is inside some disk of radius R centered at the origin and so this is a direct consequence of Lemma 2.2.3.

□

Recall the notation defined on page 6, before Lemma 1.1.3.

For $p \in \mathbb{C}P^n$, $x = x(p) \in S^{2n+1} \subset \mathbb{C}^{n+1}$ and $\epsilon > 0$, define

$$T_x^n(\epsilon) = \{y \in T_x^n : |y - x| < \epsilon\} \tag{2.1}$$

and

$$V(p, \epsilon) = (S_x^n)^{-1}(T_x^n(\epsilon)). \tag{2.2}$$

For $[x], [y] \in \mathbb{C}P^n$, we define $\alpha([x], [y]) \in [0, \pi/2]$ to be the angle between $[x]$ and $[y]$, so

$$\cos\alpha = |\langle x, y\rangle|/(|x||y|)$$

and we define

$$\mathcal{V}(p, \epsilon, \delta) = \{q \in \mathbb{C}P^n : \alpha(q, p') < \delta \text{ for some } p' \in V(p, \epsilon)\},$$

for $p \in \mathbb{C}P^n$. Note that α is the usual metric on $\mathbb{C}P^n$. We do not make the dependence of $V(p, \epsilon)$ and $\mathcal{V}(p, \epsilon, \delta)$ on n explicit in the notation, since this will cause no confusion.

Lemma 2.2.5. *Given $R > \epsilon > 0$, there exists $\delta > 0$ such that $\mathcal{V}(p, \epsilon, \delta) \subset V(p, R)$ for all $p \in \mathbb{C}P^n$. Furthermore, δ can be chosen independently from n.*

Proof. This is an easy exercise, cf. [16]. □

Now we can prove the main Theorem of this section:

Proof. (of Theorem 2.2.1)

We have

$$\Omega = f_0^*\Omega_n + d\omega,$$

for some 1-form ω. Let $\epsilon > 0$ be an arbitrary positive number, pick $x_1, \cdots, x_q \in \mathbb{C}P^n$ such that $\{V(x_1, \epsilon), \cdots, V(x_q, \epsilon)\}$ is an open cover of $\mathbb{C}P^n$ and let $W_i = f_0^{-1}(V(x_i, \epsilon))$. By Lemma 2.2.2, we can write

$$d\omega = \frac{1}{\pi}\sum_{k=1}^{p} dh_k \wedge dt_k$$

where $\operatorname{supp} h_k$ and $\operatorname{supp} t_k$ are compact and contained in W_k (we may need to reindex the open sets to include repetitions).

Pick arbitrary positive numbers $0 < \epsilon_1 < \epsilon_2 < \cdots < \epsilon_p$, with $\epsilon_1 = \epsilon$. By Lemma 2.2.5 there are $\delta_1, \cdots, \delta_{p-1} > 0$ such that

$$\mathcal{V}(x, \epsilon_k, \delta_k) \subset V(x, \epsilon_{k+1}),$$

for all x and for $k = 1, \cdots, p-1$. Taking $\delta = \min(\delta_k)$, we get

$$\mathcal{V}(x, \epsilon_k, \delta) \subset V(x, \epsilon_{k+1}),$$

for all x and for $k = 1, \cdots, p-1$. These choices of ϵ_i and δ will be used to guarantee that whenever we make a “correction” to our map on an open set W_i, the resulting map will still carry each W_k to a standard coordinate neighbourhood centered on x_k, allowing the construction to proceed.

Define $f_1 : M \to \mathbb{C}P^{n+1}$ as follows. On $M \setminus W_1$ we set $f_1 = i_{n+1} \circ f_0$. On W_1, we use the standard coordinates on D_{x_1} to view f_0 as a map to $B^n(1)$ and then define f_1 by adding an extra complex coordinate. This extra coordinate will be $a+ib$, where $g = (a, b) : W_1 \to \mathbb{R}^2$ is a smooth function with compact support contained in W_1, such that $g^*(dx \wedge dy) = dh_1 \wedge dt_1$ and g is small enough that $\alpha(f_1(x), f_0(x)) < \delta$ for all $x \in W_1$, so, in particular,

$$f_1(x) = (f_0(x), a(x) + ib(x)) \in B^{n+1}(1).$$

To construct this map g, we start with $G = (h_1, t_1) : M \to \mathbb{R}^2$ and we apply Lemma 2.2.4.

The map f_1 is well defined and smooth, since $\operatorname{supp} g \subset W_1$.

We also define f_t, for $t \in [0, 1]$, by using $t(a + ib)$ as the extra coordinate.

Claim 2.2.6. *Each f_t is an embedding, C^0-close to f_0.*

Proof. Since M is compact, it is enough to prove that each f_t is an injective immersion, C^0-close to f_0.

f_t is injective:

Suppose $f_t(x) = f_t(y)$.

If $x, y \in W_1$, then we can use coordinates on D_{x_1} and we have

$$f_t(x) = f_t(y) \Leftrightarrow (f_0(x), t(a(x) + ib(x))) = (f_0(y), t(a(y) + ib(y))),$$

so in particular $f_0(x) = f_0(y)$, so $x = y$.

If $x, y \in M \setminus W_1$, then $f_t(x) = f_0(x)$ and $f_t(y) = f_0(y)$, so $x = y$.

If $x \in W_1$ and $y \notin W_1$, then we must have $g(x) = 0$, because $f_t(x) = f_t(y) = f_0(y) \in \mathbb{C}P^n$ (if $g(x) \neq 0$, then $f_t(x) \in \mathbb{C}P^{n+1} \setminus \mathbb{C}P^n$, for $t > 0$). Then $f_t(x) = f_0(x)$ and $f_t(y) = f_0(y)$, so $x = y$.

f_t is an immersion:

For $x \notin \operatorname{supp} g$, we have $f_0 = f_t$ in a neighbourhood of x, so f_t has injective derivative at x because f_0 has injective derivative at x.

For $x \in \operatorname{supp} g \subset W_1$, we can use the coordinates on D_{x_1} and write $f_t(y) = (f_0(y), t(a(y)+ib(y)))$, for y in a neighbourhood of x. Then the fact that f_0 is immersive at x obviously implies that f_t is immersive at x.

f_t is C^0-close to f_0:

In fact, we can make g arbitrarily small, and then obviously f_t will be arbitrarily close to f_0.

□

Since the symplectic form Ω_n is equal to

$$\frac{1}{\pi} \sum_{j=1}^{n} dx_j \wedge dy_j$$

in standard coordinates, we have

$$f_1^*\Omega_{n+1} = f_0^*\Omega_n + \frac{1}{\pi} dh_1 \wedge dt_1$$

(note that $f_0^*\Omega_n = (i_{n+1} \circ f_0)^*\Omega_{n+1}$). Furthermore, we have

$$f_1(W_k) \subset \mathcal{V}(x_k, \epsilon_1, \delta) \subset V(x_k, \epsilon_2)$$

for all k (because $\alpha(f_1(x), f_0(x)) < \delta$ for all $x \in M$), so we can repeat the process, adding an extra coordinate to f_1 on W_2 to get f_2.

Continuing in the same way, we obtain a homotopy $f_t : M \to \mathbb{C}P^{n+p}$ through embeddings, with

$$f_p^*\Omega_{n+p} = f_0^*\Omega_n + \frac{1}{\pi} \sum_{k=1}^{p} dh_k \wedge dt_k = f_0^*\Omega_n + d\omega = \Omega,$$

and such that each f_t is C^0-close to f_0.

□

Chapter 3

The space of symplectic embeddings into $\mathbb{C}P^\infty$

In this chapter we describe the weak homotopy type of the space of symplectic embeddings of a closed symplectic manifold into $\mathbb{C}P^\infty$. We begin by defining this space.

Definition 3.0.1. Let (M,Ω) be a closed symplectic manifold. We define the space $\mathrm{SympEmb}(M,\mathbb{C}P^\infty)$ by

$$\mathrm{SympEmb}(M,\mathbb{C}P^\infty)=\mathrm{colim}(\mathrm{SympEmb}(M,\mathbb{C}P^1)\hookrightarrow\mathrm{SympEmb}(M,\mathbb{C}P^2)\hookrightarrow\cdots),$$

with the colimit topology, induced by the C^∞ topology on $\mathrm{SympEmb}(M,\mathbb{C}P^n)\subset C^\infty(M,\mathbb{C}P^n)$ (see [7, page 34] for a definition of the C^∞ topology).

Note that an element in $\mathrm{SympEmb}(M,\mathbb{C}P^\infty)$ is a symplectic embedding $M\hookrightarrow\mathbb{C}P^n$, for some n. We define the spaces $\mathrm{Emb}(M,\mathbb{C}P^\infty)$ and $C^\infty(M,\mathbb{C}P^\infty)$ in an analogous way.

The purpose of this chapter is to prove the following Theorem, which is the main Theorem of the thesis. Here, we reduce its proof to four results, which will be proved in the four sections that follow.

Theorem 3.0.2. *Let M be a closed symplectic manifold, $\beta_0(M)$ the number of connected components of M and $\beta_1(M)$ the first Betti number of M. The space $\mathrm{SympEmb}(M,\mathbb{C}P^\infty)$ is weakly homotopy equivalent to $(S^1)^{\beta_1(M)}\times(\mathbb{C}P^\infty)^{\beta_0(M)}$.*

Proof. The computation of the weak homotopy type of $\mathrm{SympEmb}(M,\mathbb{C}P^\infty)$ is done in four steps. They are carried out in turn in the four sections of this chapter.

1. Let $\overline{\mathrm{Emb}}(M,\mathbb{C}P^\infty)$ be the subspace of $\mathrm{Emb}(M,\mathbb{C}P^\infty)$ consisting of embeddings $f:M\to\mathbb{C}P^\infty$ such that $f^*\chi=[\Omega]$ (recall that χ is the generator of $H^2(\mathbb{C}P^\infty;\mathbb{Z})$ chosen at the end of Chapter 1). We show that the inclusion

$$\mathrm{SympEmb}(M,\mathbb{C}P^\infty)\hookrightarrow\overline{\mathrm{Emb}}(M,\mathbb{C}P^\infty)$$

is a weak homotopy equivalence. This is Theorem 3.1.1.

2. We show that the inclusion
$$\mathrm{Emb}(M, \mathbb{C}P^\infty) \hookrightarrow C^\infty(M, \mathbb{C}P^\infty)$$
is a weak homotopy equivalence. This is Theorem 3.2.1.

3. We show that the inclusion
$$C^\infty(M, \mathbb{C}P^\infty) \hookrightarrow C(M, \mathbb{C}P^\infty)$$
is a weak homotopy equivalence. This is Theorem 3.3.1.

4. Finally, we show that $C(M, \mathbb{C}P^\infty)$ is homotopy equivalent to $H^2(M;\mathbb{Z}) \times (S^1)^{\beta_1(M)} \times (\mathbb{C}P^\infty)^{\beta_0(M)}$. This is Theorem 3.4.1.

Step 1 is the most important one, and the only one that involves any Symplectic Geometry. Steps 2 and 3 are well known results in Differential Topology, while Step 4 is a well known result in Algebraic Topology (however, we do not know of any references for these results, so we include proofs).

Since $\mathbb{C}P^\infty$ is a $K(\mathbb{Z}, 2)$ and χ is a generator of $H^2(\mathbb{C}P^\infty;\mathbb{Z})$, it follows from [6, Theorem 4.57] that the map
$$[M, \mathbb{C}P^\infty] \to H^2(M;\mathbb{Z})$$
$$[f] \mapsto f^*\chi$$
is a bijection. Hence, the space $\overline{C}(M, \mathbb{C}P^\infty)$ of continuous maps $f : M \to \mathbb{C}P^\infty$ such that $f^*\chi = [\Omega]$ is a path component of $C(M, \mathbb{C}P^\infty)$, therefore it is homotopy equivalent to $(S^1)^{\beta_1(M)} \times \mathbb{C}P^\infty$.

Now, since the inclusions
$$\mathrm{Emb}(M, \mathbb{C}P^\infty) \hookrightarrow C^\infty(M, \mathbb{C}P^\infty) \text{ and } C^\infty(M, \mathbb{C}P^\infty) \hookrightarrow C(M, \mathbb{C}P^\infty)$$
are weak homotopy equivalences, the subspaces $\overline{C}^\infty(M, \mathbb{C}P^\infty)$ and $\overline{\mathrm{Emb}}(M, \mathbb{C}P^\infty)$ defined by the same condition $f^*\chi = [\Omega]$ are path components of the respective spaces and therefore we have weak homotopy equivalences
$$\mathrm{SympEmb}(M, \mathbb{C}P^\infty) \hookrightarrow \overline{\mathrm{Emb}}(M, \mathbb{C}P^\infty) \hookrightarrow \overline{C}^\infty(M, \mathbb{C}P^\infty) \hookrightarrow \overline{C}(M, \mathbb{C}P^\infty).$$
□

In this chapter, we will always use the weak topology on spaces of smooth functions, even when the domain manifold is not compact.

3.1 Parametric symplectic approximation

The aim of this section is to prove the following Theorem:

Theorem 3.1.1. *Let (M, Ω) be a closed symplectic manifold, with Ω integral, χ the generator of $H^2(\mathbb{C}P^\infty;\mathbb{Z})$ defined by* (1.2) *and $\overline{\mathrm{Emb}}(M, \mathbb{C}P^\infty)$ the space of smooth embeddings f such that $f^*\chi = [\Omega]$. Then, the* *inclusion*

$$\mathrm{SympEmb}(M, \mathbb{C}P^\infty) \hookrightarrow \overline{\mathrm{Emb}}(M, \mathbb{C}P^\infty) \tag{3.1}$$

is a weak homotopy equivalence.

This Theorem will follow from a parametric version of Theorem 2.2.1 (Theorem 3.1.2) and a well known Lemma in algebraic topology (Lemma 3.1.3):

Theorem 3.1.2. *Let (M, Ω) be a closed symplectic manifold and $f_0 : D^k \to \overline{\mathrm{Emb}}(M, \mathbb{C}P^n)$ continuous in the C^∞ topology, such that $f_0(z)$ is a symplectic embedding for z in a neighbourhood of ∂D^k. Then there is $p \in \mathbb{N}$ and a C^0-small homotopy $f_t : D^k \to \overline{\mathrm{Emb}}(M, \mathbb{C}P^{n+p})$, with $t \in [0, p]$, fixed on ∂D^k, such that $f_p(z) \in \mathrm{SympEmb}(M, \mathbb{C}P^{n+p})$ for all $z \in D^k$.*

Lemma 3.1.3. *Consider a continuous map $\phi : X \to Y$ between topological spaces and let $k \geq 0$. Then the following are equivalent:*

1. *The map induced by ϕ on π_{k-1} is injective and the map induced by ϕ on π_k is surjective.*

2. *For every commutative diagram of the type*

$$\begin{array}{ccc} S^{k-1} & \xrightarrow{g} & X \\ \big\downarrow & & \big\downarrow \\ D^k & \xrightarrow{f} & Y \end{array}$$

   there is a map $h : D^k \to X$, such that in the diagram

$$\begin{array}{ccc} S^{k-1} & \xrightarrow{g} & X \\ \big\downarrow & \overset{h}{\nearrow} & \big\downarrow \\ D^k & \xrightarrow{f} & Y \end{array}$$

   the top left triangle commutes and the bottom right triangle commutes up to homotopy $rel\ S^{k-1}$.

Proof. See [11], page 70.

□

We postpone the proof of Theorem 3.1.2 and give the proof of the main Theorem of this section:

Proof. (of Theorem 3.1.1)

By Lemma 3.1.3, it is enough to show that given a diagram

$$\begin{array}{ccc} S^{k-1} & \xrightarrow{g} & \mathrm{SympEmb}(M, \mathbb{C}P^\infty) \\ \big\downarrow & & \big\downarrow \\ D^k & \xrightarrow{f} & \overline{\mathrm{Emb}}(M, \mathbb{C}P^\infty) \end{array}$$

there is a map $D^k \to \mathrm{SympEmb}(M, \mathbb{C}P^\infty)$, such that in the diagram

$$\begin{array}{ccc} S^{k-1} & \xrightarrow{\;g\;} & \mathrm{SympEmb}(M, \mathbb{C}P^\infty) \\ \big\downarrow & \nearrow & \big\downarrow \\ D^k & \xrightarrow{\;f\;} & \overline{\mathrm{Emb}}(M, \mathbb{C}P^\infty) \end{array}$$

the top left triangle commutes and the bottom right triangle commutes up to homotopy $rel\ S^{k-1}$.

The only detail that we need to deal with in order to be able to apply Theorem 3.1.2, is that we have a family of embeddings that are already symplectic for $z \in \partial D^k$ and we need them to be symplectic for z in a neighbourhood of ∂D^k. But f is clearly homotopic $rel\ S^{k-1}$ to a map which is constant in the radial direction near ∂D^k.

□

It remains to prove Theorem 3.1.2, which is the main contribution of this thesis. We fix a map $f_0 : D^k \to \overline{\mathrm{Emb}}(M, \mathbb{C}P^n)$, with $f_0(z)$ symplectic for z in a neighbourhood of ∂D^k, and denote by $\tilde{f}_0 : D^k \times M \to \mathbb{C}P^n$ the adjoint map. Since, for each $z \in D^k$, we have $[f_0(z)^*\Omega_n] = [\Omega]$, we have a continuous map

$$D^k \to d(\Omega^1(M))$$
$$z \mapsto \Omega - f_0(z)^*\Omega_n$$

that is zero for z in a neighbourhood of ∂D^k, since $f_0(z)$ is symplectic for z in a neighbourhood of ∂D^k. Then we can write $\Omega = f_0(z)^*\Omega_n + d\omega(z)$, for some continuous map $\omega : D^k \to \Omega^1(M)$ with $\omega(z) = 0$ for z in a neighbourhood of ∂D^k (see Theorem A.0.1).

Lemma 3.1.4. *Let $\omega : D^k \to \Omega^1(M)$ be a continuous map, with $\omega(z) = 0$ for z in a neighbourhood of ∂D^k. Given a finite open cover $\{W_i\}$ of $D^k \times M$, there is $p \in \mathbb{N}$ and continuous functions $h_k, t_k : D^k \to C^\infty(M, \mathbb{R})$, $k = 1, \cdots, p$, such that*

1. $\sum_{k=1}^{p} dh_k(z) \wedge dt_k(z) = d\omega(z)$*, for each* $z \in D^k$*;*

2. *For each k there is an ℓ such that $\operatorname{supp} \tilde{h_k}$ and $\operatorname{supp} \tilde{t_k}$ are contained in W_ℓ (where $\tilde{h_k}, \tilde{t_k} : D^k \times M \to \mathbb{R}$ are the adjoint maps to h_k and t_k);*

3. $h_k(z) = t_k(z) = 0$ *for* $z \in \partial D^k$*.*

Proof. Step 1.

We begin by finding functions $h_k, t_k : D^k \to C^\infty(M, \mathbb{R})$ such that $\sum_k dh_k(z) \wedge dt_k(z) = d\omega(z)$, for each $z \in D^k$ and $h_k(z) = t_k(z) = 0$ for $z \in \partial D^k$ (so for now ignore the condition on the supports). By picking a partition of unity for M, subordinate to a coordinate cover, we see that it is enough to write $d(\rho\omega(z)) = \sum_k dh_k(z) \wedge dt_k(z)$, for some $\rho : M \to \mathbb{R}_0^+$ with compact support contained in a coordinate chart $(U, (x_1, \cdots, x_m))$, with $h_k(z) = t_k(z) = 0$ for $z \in \partial D^k$.

We can write $\rho\omega(z) = h_1(z)dx_1 + \cdots + h_m(z)dx_m$ on U, where each $h_i : D^k \to C^\infty(U, \mathbb{R})$ is continuous. Each $h_i(z)$ has compact support contained in U, so we can extend it by 0 to all of M, getting $h_i : D^k \to C^\infty(M, \mathbb{R})$. Clearly $h_i(z) = 0$ for z in a neighbourhood of ∂D^k, because $\omega(z) = 0$ for z in a neighbourhood of ∂D^k. Since $\operatorname{supp} \tilde{h}_i$ is compact and $\operatorname{supp} \tilde{h}_i \subset (D^k \setminus \partial D^k) \times U$ (because $h_i(z) = 0$, for z in a neighbourhood of ∂D^k) we can find open sets V_1, V_2 such that

$$\operatorname{supp} \tilde{h}_i \subset V_1 \subset \overline{V}_1 \subset V_2 \subset (D^k \setminus \partial D^k) \times U.$$

Then, we let $\lambda_i : D^k \times M \to \mathbb{R}_0^+$ be a smooth function such that $\lambda_i = 1$ on $\overline{V_1}$ and $\lambda_i = 0$ outside V_2. We define $\tilde{t}_i(z, x) = \lambda_i(z, x) x_i(x)$, obtaining a smooth map $\tilde{t}_i : D^k \times M \to \mathbb{R}$. The adjoint map to $\tilde{t}_i$ is a continuous map $t_i : D^k \to C^\infty(M, \mathbb{R})$ with the desired properties.

Step 2.

Now consider a partition of unity $\{\rho_k\}$ for $D^k \times M$, subordinate to the cover $\{W_k\}$. This makes it possible to write each $\tilde{h}_i$ and each $\tilde{t}_i$ as a sum of functions supported in elements of the open cover and this finishes the proof.

□

Lemma 3.1.5. *Let M be a closed manifold, $\epsilon > 0$ and $F : D^k \to C^\infty(M, \mathbb{R}^2)$ a continuous map. Then, there exists a continuous map $f : D^k \to C^\infty(M, \mathbb{R}^2)$ such that:*

1. $||f(z)(x)|| < \epsilon$ *for all* $z \in D^k$ *and* $x \in M$*;*
2. $f(z)^*(dx \wedge dy) = F(z)^*(dx \wedge dy)$*;*
3. *Writing* $\tilde{f}, \tilde{F} \colon D^k \times M \to \mathbb{R}^2$ *for the adjoint maps to* f *and* F*, we have* $\operatorname{supp} \tilde{f} \subset \operatorname{supp} \tilde{F}$*.*

Proof. Since $M \times D^k$ is compact, the image of $\tilde{F}$ is contained in some disk of radius R centered at the origin and so this is a direct consequence of Lemma 2.2.3.

□

Proof. (of Theorem 3.1.2)

We take a finite open cover $\{V(x_j, \epsilon)\}$ of $\mathbb{C}P^n$ (see (2.2)), for some $\epsilon > 0$ and consider the open cover $\{W_j = \tilde{f_0}^{-1}(V(x_j, \epsilon))\}$ of $D^k \times M$. We use Lemma 3.1.4 to write $d\omega(z) = \frac{1}{\pi} \sum_{i=1}^{p} dh_i(z) \wedge dt_i(z)$, where $\tilde{h}_i, \tilde{t}_i$ have compact support contained in W_i (we may need to change the indices of the open cover and include repeated open sets for this) and $h_i(z) = t_i(z) = 0$ for $z \in \partial D^k$. Pick $0 < \epsilon_1 < \epsilon_2 < \cdots < \epsilon_p$, with $\epsilon_1 = \epsilon$. By Lemma 2.2.5 there is $\delta > 0$ such that

$$\mathcal{V}(x, \epsilon_j, \delta) \subset V(x, \epsilon_{j+1}),$$

for all x and for $j = 1, \cdots, p-1$. Now we inductively add new coordinates to $\tilde{f}_0$, just as in the non parametric case. Concretely, we define $\tilde{f}_1 : D^k \times M \to \mathbb{C}P^{n+1}$ by taking $\tilde{f}_1 = \tilde{f}_0$ outside W_1. On W_1, we use the standard coordinates on $V(x_1, \epsilon) \subset D^n_{x_1}$ to define $\tilde{f}_1(x) = (\tilde{f}_0(x), a(x) + ib(x))$, where $\tilde{g} = (a, b) : W_1 \to \mathbb{R}^2$ is a function with compact support contained in W_1, such that $g(z)^*(dx \wedge dy) = dh_1(z) \wedge dt_1(z)$ and $\tilde{g}$ is small enough that $\alpha(\tilde{f}_1(x), \tilde{f}_0(x)) < \delta$ for all $x \in W_1$, so that in particular

$$\tilde{f}_1(x) = (\tilde{f}_0(x), a(x) + ib(x)) \in B^{n+1}(1).$$

To construct this map g, we start with $G = (h_1, t_1) : D^k \to C^\infty(M, \mathbb{R}^2)$ and we apply Lemma 3.1.5.

This $\tilde{f}_1$ gives a well defined and continuous $f_1 : D^k \to \mathrm{Emb}(M, \mathbb{C}P^{n+1})$ (the proof that $f_1(z)$ is an embedding, for all z, is the same as the proof in the non parametric case).

We define $\tilde{f}_t$, for $t \in [0,1]$, by using $t(a+ib)$ as the extra coordinate and so we get $f_t : D^k \to \mathrm{Emb}(M, \mathbb{C}P^{n+1})$. Since $h_i(z) = t_i(z) = 0$ for $z \in \partial D^k$, we get $f_t(z) = f_0(z)$ for all $t \in [0,1]$ and $z \in \partial D^k$.

Since the symplectic form Ω_n is equal to

$$\frac{1}{\pi}\sum_{j=1}^{n} dx_j \wedge dy_j$$

in standard coordinates, we have

$$f_1(z)^*\Omega_{n+1} = f_0(z)^*\Omega_n + \frac{1}{\pi} dh_1(z) \wedge dt_1(z),$$

for all $z \in D^k$. Furthermore, we have

$$\tilde{f}_1(W_k) \subset \mathcal{V}(x_k, \epsilon_1, \delta) \subset V(x_k, \epsilon_2)$$

for all k (because $\alpha(\tilde{f}_1(x), \tilde{f}_0(x)) < \delta$ for all $x \in M \times D^k$), so we can repeat the process, adding an extra coordinate to $\tilde{f}_1$ on W_2 to get $\tilde{f}_2$.

Continuing in the same way, we obtain a homotopy $f_t : D^k \to \overline{\mathrm{Emb}}(M, \mathbb{C}P^{n+p})$ fixed on ∂D^k, with

$$f_p(z)^*\Omega_{n+p} = f_0(z)^*\Omega_n + \frac{1}{\pi}\sum_{k=1}^{p} dh_k(z) \wedge dt_k(z) = f_0^*\Omega_n + d\omega(z) = \Omega,$$

for all $z \in D^k$.

□

3.2 Parametric approximation by embeddings

The goal of this section is to prove the following Theorem:

Theorem 3.2.1. *Let M be a closed manifold. Then, the inclusion*

$$\mathrm{Emb}(M, \mathbb{C}P^\infty) \hookrightarrow C^\infty(M, \mathbb{C}P^\infty)$$

of the space of smooth embeddings $M \hookrightarrow \mathbb{C}P^\infty$ into the space of smooth maps $M \to \mathbb{C}P^\infty$ is a weak homotopy equivalence.

The main ingredient in the proof of Theorem 3.2.1 is the following Proposition:

Proposition 3.2.2. *Let M and N be C^∞ manifolds, with M compact. Then the inclusion $\mathrm{Emb}(M,N) \hookrightarrow C^\infty(M,N)$ of the space of smooth embeddings $M \hookrightarrow N$ into the space of smooth maps $M \to N$ induces an isomorphism on π_k, for k such that $\dim N \geq 2(\dim M + k + 1) + 1$.*

One possible proof of this would be to adapt the usual proof of the Whitney embedding theorem (see, e.g., [7, Theorem 2.2.13]) to make it work parametrically. This would give a better result: the map in question is an isomorphism on π_k for for k such that $\dim N \geq 2\dim M + 1 + k + 1$. For simplicity, since we do not need a sharp estimate, we will instead apply the usual Whitney embedding theorem to prove Proposition 3.2.2.

First we need to establish some notation. Define

$$A^{k+1}_{a,b} = \{z \in D^{k+1} : a \leq |z| \leq b\}, \text{ for } 0 \leq a \leq b \leq 1$$

the annulus with inner radius a and outer radius b;

$$S^k_a = \{z \in D^{k+1} : |z| = a\}, \text{ for } 0 \leq a \leq 1$$

the sphere of radius a and

$$D^{k+1}_a = \{z \in D^{k+1} : |z| \leq a\}, \text{ for } 0 \leq a \leq 1,$$

the disk of radius a.

Proof. (of Proposition 3.2.2)

By Lemma 3.1.3, it is enough to show that given a diagram

$$\begin{array}{ccc} S^k & \xrightarrow{g} & \mathrm{Emb}(M,N) \\ \big\downarrow & & \big\downarrow \\ D^{k+1} & \xrightarrow{f} & C^\infty(M,N) \end{array}$$

there is a map $D^{k+1} \to \mathrm{Emb}(M,N)$ such that, in the diagram

$$\begin{array}{ccc} S^k & \xrightarrow{g} & \mathrm{Emb}(M,N) \\ \big\downarrow & \nearrow & \big\downarrow \\ D^{k+1} & \xrightarrow{f} & C^\infty(M,N), \end{array}$$

the top left triangle commutes and the bottom right triangle commutes up to homotopy $rel\ S^k$.

So consider a diagram as above and denote by $\tilde{f} : D^{k+1} \times M \to N$ and $\tilde{g} : S^k \times M \to N$ the corresponding adjoint maps. Let R_1, R_2, R_3, R_4 be constants such that $1 > R_1 > R_2 > R_3 > R_4 > 0$.

By Proposition 3.2.5 below, there is a continuous map $g_1 : S^k \to \mathrm{Emb}(M,N)$, homotopic to $g_0 := g$, such that $\tilde{g}_1 : S^k \times M \to N$ is smooth. This homotopy determines a continuous map $h_1 : A^{k+1}_{R_1,1} \to \mathrm{Emb}(M,N)$, with $h_1|_{S^k_1} = g = g_0$ and $h_1|_{S^k_{R_1}}$ a suitable rescaling of g_1.

Now we approximate $\tilde{g}_1$ by an embedding $\tilde{g}_2 : S^k \times M \to N$ that is homotopic to $\tilde{g}_1$ through maps $\tilde{g}_t$ $(t \in [1,2])$ such that $\tilde{g}_t|_{\{z\}\times M}$ is an embedding for all $z \in S^k$ and $t \in [1,2]$. This can be done, using Lemma 2.1.2, because the set of embeddings $S^k \times M \to N$ is dense in $C^\infty(S^k \times M, N)$ (since $2(k+\dim M)+1 \leq \dim N$, see [7, Theorem 2.2.13]), the set of embeddings $M \to N$ is open in $C^\infty(M,N)$

([7, Corollary 2.1.6]) and S^k is compact (see the proof of Proposition 3.2.5 for a more detailed argument of the same sort). This homotopy determines a map $h_2 : A^{k+1}_{R_2,R_1} \to \mathrm{Emb}(M,N)$ with $h_2|_{S^k_{R_1}}$ and $h_2|_{S^k_{R_2}}$ suitable rescalings of g_1 and g_2, respectively.

Now we can use the transversality theorem ([7, Theorem 3.2.1]) to find a section of the normal bundle to the embedding $\tilde{g}_2 : S^k \times M \to N$ that is transverse to the zero section. Since $\dim N \geq 2(k + \dim M) + 1$, transversality implies that this section is never zero. Using this section, we can extend $\tilde{g}_2$ to an embedding $S^k \times M \times [0,\epsilon] \to N$. This determines a map $h_3 : A^{k+1}_{R_3,R_2} \to \mathrm{Emb}(M,N)$ such that $\tilde{h}_3 : A^{k+1}_{R_3,R_2} \times M \to N$ is an embedding and $h_3|_{S^k_{R_2}}$ is a suitable rescaling of g_2.

Since $h_3|_{S^k_{R_3}}$ is homotopic to g (as a map $S^k \to \mathrm{Emb}(M,N)$) we get a map $h_4 : A^{k+1}_{R_4,R_3} \to \mathrm{Emb}(M,N)$ with $h_4|_{S^k_{R_3}} = h_3|_{S^k_{R_3}}$ and $h_4|_{S^k_{R_4}}$ a suitable rescaling of g.

Since $\tilde{h}_4|_{S^k_{R_4} \times M}$ is a rescaling of $\tilde{g}$, the maps $\tilde{h}_4, \tilde{h}_3$ and a rescaling of $\tilde{f}$ determine a map $\tilde{f} \cup \tilde{h}_4 \cup \tilde{h}_3 : D^{k+1}_{R_2} \times M \to N$ which is an embedding on $A^{k+1}_{R_3,R_2} \times M$. We can now approximate this map by an embedding $\tilde{h} : D^{k+1}_{R_2} \times M \to N$ such that $\tilde{h}|_{S^k_{R_2} \times M} = \tilde{g}_2$. This can be done by relative approximation by embeddings, because $2(k+1+\dim M)+1 \leq \dim N$ (see [9, II.3.2]). If we pick $\tilde{h}$ sufficiently close to $\tilde{f} \cup \tilde{h}_4 \cup \tilde{h}_3$ then $\tilde{h}$ will be homotopic $rel\ S^k_{R_2}$ to $\tilde{f} \cup \tilde{h}_4 \cup \tilde{h}_3$ through maps whose restrictions to $\{z\} \times M$ are embeddings, for all $z \in D^{k+1}_{R_2}$ (again by Lemma 2.1.2, compactness of $D^{k+1}_{R_2}$ and openness of $Emb(M,N)$).

We obtain a map $\tilde{h}_1 \cup \tilde{h}_2 \cup \tilde{h} : D^{k+1} \times M \to N$ whose adjoint is the desired map $D^{k+1} \to \mathrm{Emb}(M,N)$.

□

Our next aim is to prove Proposition 3.2.5 below, so as to conclude the proof of Proposition 3.2.2. Recall that a *convolution kernel* is nonnegative function $\theta : \mathbb{R}^m \to \mathbb{R}$ with compact support and with integral equal to 1. Given a subset $X \subset \mathbb{R}^m$ and $r > 0$, we define

$$X_r = \{x \in \mathbb{R}^m : \overline{B_r(x)} \subset X\}. \tag{3.2}$$

Given a map $f : \mathbb{R}^k \times \mathbb{R}^m \to \mathbb{R}^n$ and $\alpha = (\alpha_1, \cdots, \alpha_k)$, $\beta = (\beta_1, \cdots, \beta_m)$ with $\alpha_i, \beta_i \in \mathbb{Z}^+_0$, we define $|\alpha| = \sum \alpha_i$, $|\beta| = \sum \beta_i$ and

$$\partial^{(\alpha,\beta)} f = \frac{\partial^{|\alpha|+|\beta|} f}{(\partial z_1)^{\alpha_1} \cdots (\partial z_k)^{\alpha_k} (\partial x_1)^{\beta_1} \cdots (\partial x_m)^{\beta_m}},$$

where $(z_1, \cdots, z_k, x_1, \cdots, x_m)$ are the usual coordinates in $\mathbb{R}^k \times \mathbb{R}^m$.

Lemma 3.2.3. *Let $U \subset \mathbb{R}^k$, $V \subset \mathbb{R}^m$ be open, θ a convolution kernel on $\mathbb{R}^k \times \mathbb{R}^m$ with $\operatorname{supp}\theta \subset B_{r_1}(0) \times B_{r_2}(0)$ and $F : U \to C^\infty(V, \mathbb{R}^n)$ a continuous map. Then,*

(i) $\theta * \tilde{F}$ is C^∞ on $U_{r_1} \times V_{r_2}$ and $\partial^{(\alpha,\beta)}(\theta * \tilde{F}) = (\partial^{(\alpha,\beta)}\theta) * \tilde{F}$;

(ii) $\partial^{(0,\beta)}(\theta * \tilde{F}) = \theta * (\partial^{(0,\beta)}\tilde{F})$, on $U_{r_1} \times V_{r_2}$;

(iii) The map $U_{r_1} \to C^\infty(V_{r_2}, \mathbb{R}^n)$ sending z to $(\theta * \tilde{F})|_{\{z\} \times V_{r_2}}$ is continuous.

(iv) Given $\epsilon > 0$ and $K \subset U \times V$ compact, there is $R > 0$ such that, for all $r_1, r_2 \leq R$, we have $K \subset U_{r_1} \times V_{r_2}$ and

$$|\partial^{(0,\beta)}(\theta * \tilde{F})(z,x) - \partial^{(0,\beta)}\tilde{F}(z,x)| < \epsilon,$$

for all $(z,x) \in K$.

Proof. (iii) follows from (i), while (i), (ii) and (iv) follow from the proof given in [7, Theorem 2.2.3].

□

Lemma 3.2.4. *Let M, S be closed manifolds, $U \subset \mathbb{R}^n$ be open, $F : S \to C^\infty(M,U)$ continuous and d a distance on $C^\infty(M,U)$ inducing the C^∞ topology. Then, for any $\epsilon > 0$, there exists $G : S \to C^\infty(M,U)$ continuous, such that*

- *$\tilde{G} : S \times M \to U$ is C^∞;*
- *$d(F(z), G(z)) < \epsilon$ for all $z \in S$.*

Proof. Let $\{V_i\}$ and $\{W_j\}$ be finite covers of S and M, respectively, by coordinate charts, such that there exist families of closed coordinate disks $\{A_i \subset V_i\}$ and $\{B_j \subset W_j\}$ which are still covers. Let $\{\mu_i\}$ be a partition of unity on S, with $\operatorname{supp} \mu_i \subset A_i$ and let $\{\lambda_j\}$ be a partition of unity on M, with $\operatorname{supp} \lambda_j \subset B_j$. The map

$$\prod_{ij} C(A_i, C^\infty(B_j, \mathbb{R}^n)) \to C(S, C^\infty(M, \mathbb{R}^n))$$

$$F_{ij} \mapsto \sum_{ij} \mu_i \lambda_j F_{ij}$$

is continuous. Therefore, we only need to find $G_{ij} : A_i \to C^\infty(B_j, \mathbb{R}^n)$ with $\tilde{G}_{ij}$ smooth and $d(G_{ij}(z), F(z))$ small enough. By Lemma 3.2.3, this is achieved by taking $\tilde{G}_{ij} = \theta_{ij} * \tilde{F}|_{V_i \times W_j}$, for θ_{ij} with small enough support.

□

Proposition 3.2.5. *Let M, N, S be closed manifolds and $F : S \to \mathrm{Emb}(M,N)$ a continuous map. Then, there exists a continuous map $G : S \to \mathrm{Emb}(M,N)$, homotopic to F, such that the adjoint map $\tilde{G}$ is C^∞.*

Proof. Let $i : N \hookrightarrow \mathbb{R}^\ell$ be an embedding and let $\pi : U \to N$ be the projection of a tubular neighbourhood. Then we have continuous maps

$$i_* : C^\infty(M,N) \to C^\infty(M,U)$$

and

$$\pi_* : C^\infty(M,U) \to C^\infty(M,N).$$

Let $F^{(1)} = i_* \circ F : S \to \mathrm{Emb}(M,U)$ and let $\mathcal{N} = (\pi_*)^{-1}(\mathrm{Emb}(M,N))$. Then $\mathcal{N}$ is open (because $\mathrm{Emb}(M,N)$ is open in $C^\infty(M,N)$) and we have $F^{(1)}(z) \in \mathcal{N}$ for all $z \in S$. So, if d is a distance in $C^\infty(M,U)$ inducing the C^∞ topology, then, for all $z \in S$, there exists $\delta_z > 0$ such that

$$B_d(F^{(1)}(z), \delta_z) \subset \mathcal{N}.$$

Since $F^{(1)}$ is continuous and S is compact, we can find $\delta > 0$ such that

$$B_d(F^{(1)}(z), \delta) \subset \mathcal{N}$$

for all $z \in S$. Now we can find $\epsilon > 0$ such that, for all $z \in S$, if $g \in B_d(F^{(1)}(z), \epsilon)$ then

$$(1-t)F^{(1)}(z) + tg \in B_d(F^{(1)}(z), \delta), \text{ for all } t \in [0,1]$$

(again, we start by finding ϵ_z for each z and then use compactness and continuity). By Lemma 3.2.4, there exists $G^{(1)} : S \to C^\infty(M, U)$ such that $\tilde{G}^{(1)}$ is C^∞ and $d(F^{(1)}(z), G^{(1)}(z)) < \epsilon$ for all $z \in S$. Then $G = \pi_* \circ G^{(1)}$ has the desired properties (the homotopy between F and G is $H_t = \pi_* \circ H_t^{(1)}$, where $H_t^{(1)}$ is the linear homotopy between $F^{(1)}$ and $G^{(1)}$).

□

This finishes the proof of Proposition 3.2.2. To prove Theorem 3.2.1, we will need the following results:

Lemma 3.2.6. *Let $X_0 \hookrightarrow X_1 \hookrightarrow \cdots$ be a sequence of T_1 topological spaces and closed embeddings. Suppose K is compact and $f : K \to \operatorname{colim} X_n$ is continuous. Then there exists m such that f factors through the canonical embedding $\iota_m : X_m \to \operatorname{colim} X_n$.*

Proof. Since ι_m is an embedding, it suffices to show that $f(K) \subset X_m \subset \operatorname{colim}_n X_n$, for some m. Suppose not. Then there exists $x_n \in f(K) \setminus X_n$, for each $n \in \mathbb{N}$. Let $F = \{x_n : n \in \mathbb{N}\}$ and note that F must be an infinite set. Since $F \cap X_n \subset \{x_1, \cdots, x_{n-1}\}$ is finite, for all n, any subset of F is closed in the colimit, hence F is discrete. But, since $F \subset f(K)$, it is also compact, contradicting the fact that it is an infinite discrete set.

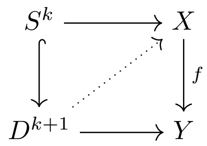
□

Lemma 3.2.7. *Let $X_0 \hookrightarrow X_1 \hookrightarrow \cdots$ and $Y_0 \hookrightarrow Y_1 \hookrightarrow \cdots$ be sequences of T_1 topological spaces and closed embeddings. Let $X = \operatorname{colim} X_n$ and $Y = \operatorname{colim} Y_n$. Suppose $f_n : X_n \to Y_n$ are continuous maps such that the obvious diagram commutes and f_n induces isomorphisms on π_k for $k \leq \phi(n)$, where $\phi : \mathbb{N}_0 \to \mathbb{N}_0$ is a function such that $\phi(n) \to \infty$ as $n \to \infty$. Then the map $f = \operatorname{colim} f_n : X \to Y$ is a weak homotopy equivalence.*

Proof. By Lemma 3.1.3, it is enough to show that given a diagram

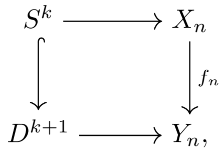
$$\begin{array}{ccc} S^k & \longrightarrow & X \\ \big\downarrow & & \big\downarrow f \\ D^{k+1} & \longrightarrow & Y \end{array}$$

there is a map $D^{k+1} \to X$, such that in the diagram



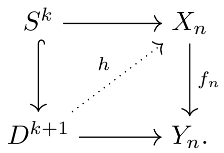
the top left triangle commutes and the bottom right triangle commutes up to homotopy $rel\ S^k$.

Now S^k and D^{k+1} are compact, so by Lemma 3.2.6, we have a diagram



for some n.

If we pick n large enough so that $\phi(n) \geq k+1$, then f_n induces an isomorphism on π_ℓ, for $\ell \leq k+1$. In particular, f_n is injective on π_k and surjective on π_{k+1}, which, according to Lemma 3.1.3, allows us to fill in the diagram:



Then clearly, writing $\iota_n : X_n \to \operatorname{colim} X_m$ for the canonical embedding, $\iota_n \circ h$ fills in the original diagram, as desired.

□

Now we can prove Theorem 3.2.1:

Proof. (of Theorem 3.2.1)

The map in question is the colimit of the maps $\mathrm{Emb}(M, \mathbb{C}P^n) \hookrightarrow C^\infty(M, \mathbb{C}P^n)$. According to Proposition 3.2.2 , these maps induce isomorphisms on π_k for k smaller than some $\phi(n)$, which grows to infinity as n grows. Lemma 3.2.7 then implies that the map in question is a weak homotopy equivalence.

□

3.3 Parametric smooth approximation

The aim of this section is to prove the following Theorem:

Theorem 3.3.1. *Let M be a closed manifold. Then, the inclusion $C^\infty(M, \mathbb{C}P^\infty) \hookrightarrow C(M, \mathbb{C}P^\infty)$ is a weak homotopy equivalence.*

This will be a consequence of the following Theorem, by passing to the colimit, as in the proof of Theorem 3.2.1:

Theorem 3.3.2. *Let M and N be closed manifolds. Then, the inclusion $C^\infty(M, N) \hookrightarrow C(M, N)$ is a weak homotopy equivalence.*

Proof. By Lemma 3.1.3, we need to show that, given a diagram

$$\begin{CD} S^k @>F>> C^\infty(M,N) \\ @VVV @VVV \\ D^{k+1} @>G>> C(M,N) \end{CD}$$

there is a map $D^{k+1} \to \mathrm{Emb}(M,N)$ such that in the following diagram

$$\begin{array}{ccc} S^k & \xrightarrow{F} & C^\infty(M,N) \\ \downarrow & \nearrow & \downarrow \\ D^{k+1} & \xrightarrow{G} & C(M,N) \end{array}$$

the top left triangle commutes and the bottom right triangle commutes up to homotopy $rel\ S^k$. We proceed in steps:

1. We start by embedding N into $\mathbb{R}^D$, for some large D and we let $U \subset \mathbb{R}^D$ be a tubular neighbourhood, with $\pi : U \to N$ the projection.

2. Let d be the euclidean distance on $\mathbb{R}^D$. Since N is compact, we can pick ϵ such that if $x \in N$, $y \in \mathbb{R}^D$ and $d(y,x) < \epsilon$, then $y \in U$.

3. Now let $G^{(1)} : D^{k+1} \to C(M, \mathbb{R}^D)$ be the obvious map obtained from G by composing with the embedding $N \hookrightarrow \mathbb{R}^D$. We will show (Proposition 3.3.3) that we can approximate $G^{(1)}$ by a continuous map $G^{(2)} : D^{k+1} \to C^\infty(M, \mathbb{R}^D)$ such that $d(G^{(2)}(z)(x), G^{(1)}(z)(x)) < \epsilon$, for all $z \in D^{k+1}$ and $x \in M$.

4. Now let $H_t : D^{k+1} \to C(M, \mathbb{R}^D)$ by the linear homotopy between $G^{(1)}$ and $G^{(2)}$. By the previous steps, we have
$$H_t(z)(x) = (1-t)G^{(1)}(z)(x) + tG^{(2)}(z)(x) \in U$$
for all $z \in D^{k+1}$ and $x \in M$. Then we can project this homotopy to N, using π, to get a homotopy $\pi_* \circ H_t$ of maps $D^{k+1} \to C(M,N)$ between G and a continuous map $G^{(3)} : D^{k+1} \to C^\infty(M,N)$.

5. The map $G^{(3)}$ almost solves the problem, but we don't necessarily have $G^{(3)}|_{S^k} = F$. Nevertheless, the homotopy $\pi_* \circ H_t$ restricts to a homotopy J_t between $F = G|_{S^k}$ and $G^{(3)}|_{S^k}$ as maps $S^k \to C^\infty(M,N)$. Let $z = (|z|, \theta) \in (0,1] \times S^k$ be the usual polar coordinates on $D^{k+1} \setminus \{0\}$. Define $G^{(4)} : D^{k+1} \to C^\infty(M,N)$ by
$$G^{(4)}(z) = \begin{cases} J_{2(1-|z|)}(\theta) & \text{if } |z| \geq 1/2 \\ G^{(3)}(2z) & \text{if } |z| \leq 1/2. \end{cases}$$
The map $G^{(4)}$ now has the desired properties.

□

The only thing left to explain is the approximation made in Step 3. What we need to prove is the following:

Proposition 3.3.3. *Let M be a closed manifold, $G : D^k \to C(M, \mathbb{R}^n)$ a continuous map and $\epsilon > 0$. Then, there is a continuous map $H : D^k \to C^\infty(M, \mathbb{R}^n)$ such that $|G(z)(x) - H(z)(x)| < \epsilon$ for all $z \in D^k$ and $x \in M$.*

To prove this, we will need a Lemma. Given a subset $X \subset \mathbb{R}^m$ and $r > 0$, we define

$$\overline{B_r}(X) = \bigcup_{x \in X} \overline{B_r(x)}.$$

Lemma 3.3.4. *Let $U \subset \mathbb{R}^m$ be open, $F : D^k \to C(U, \mathbb{R}^n)$ continuous and $\theta : \mathbb{R}^m \to \mathbb{R}$ a convolution kernel, with $\operatorname{supp} \theta \subset B_r(0)$. Then,*

(i) For each $z \in D^k$, the map $\theta * F(z)$ is smooth, and $\partial^\alpha(\theta * F(z)) = (\partial^\alpha \theta) * F(z)$.

(ii) Recall the notation defined in (3.2)*. The map*

$$\begin{aligned} D^k &\to C^\infty(U_r, \mathbb{R}^n) \\ z &\mapsto \theta * F(z) \end{aligned}$$

is continuous.

(iii) For any $\epsilon > 0$ and $K \subset U$ compact, there is $R > 0$ such that, if $r \leq R$, then $K \subset U_r$ and $|(\theta * F(z))(x) - F(z)(x)| < \epsilon$, for all $x \in K$.

Proof. (i) See [7, Theorem 2.2.3].

(ii) Let $z_n \to z$ be a converging sequence in D^k. We need to show that $\partial^\alpha(\theta * F(z_n)) \to \partial^\alpha(\theta * F(z))$ uniformly on each compact set $K \subset U_r$. So let $K \subset U_r$ be compact. We have

$$\begin{aligned} |\partial^\alpha(\theta * F(z_n))(x) - \partial^\alpha(\theta * F(z))(x)| &= |(\partial^\alpha \theta) * F(z_n)(x) - (\partial^\alpha \theta) * F(z)(x)| \\ &= |(\partial^\alpha \theta) * (F(z_n) - F(z))(x)| \\ &= \left| \int_{B_r(0)} \partial^\alpha \theta(y)(F(z_n)(x-y) - F(z)(x-y)) dy \right| \\ &= \left| \int_{B_r(x)} \partial^\alpha \theta(x-w)(F(z_n)(w) - F(z)(w)) dw \right| \\ &\leq \int_{B_r(x)} |\partial^\alpha \theta(x-w)(F(z_n)(w) - F(z)(w))| \, dw. \end{aligned}$$

Now, given $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that for $n \geq N$ we have $|F(z_n)(w) - F(z)(w)| < \epsilon$ for all $w \in \overline{B_r}(K)$, because $F(z_n) \to F(z)$ uniformly on compact sets (F is continuous) and $\overline{B_r}(K)$ is compact. Therefore, if we let $C = \int_{B_r(x)} |\partial^\alpha \theta(x-w)| dw$, then we have

$$|\partial^\alpha(\theta * F(z_n))(x) - \partial^\alpha(\theta * F(z))(x)| < C\epsilon, \text{ for } n \geq N, \, x \in K.$$

(iii) We have

$$\begin{aligned}|(\theta * F(z))(x) - F(z)(x)| &= \left|\int_{B_r(0)} \theta(y)F(z)(x-y)dy - \int_{B_r(0)} \theta(y)F(z)(x)dy\right| \\ &\leq \int_{B_r(0)} \theta(y)\,|F(z)(x-y) - F(z)(x)|\,dy \\ &= \int_{B_r(0)} \theta(y)\left|\tilde{F}(z,x-y) - \tilde{F}(z,x)\right| dy.\end{aligned}$$

Pick $\tilde{R} > 0$ such that $K \subset U_{\tilde{R}}$. By uniform continuity of $\tilde{F}$ on the compact set $D^k \times \overline{B_{\tilde{R}}}(K) \subset D^k \times U$, we know that, for r small enough, we will have

$$\left|\tilde{F}(z,x-y) - \tilde{F}(z,x)\right| < \epsilon,$$

for all $z \in D^k$, $x \in K$, $y \in \overline{B_r(0)}$. Then,

$$|(\theta * F(z))(x) - F(z)(x)| < \int_{B_r(0)} \theta(y)\epsilon dy = \epsilon.$$

□

Proof. (of Proposition 3.3.3)

Let $\{U_i\}$ be a finite cover of M by coordinate charts and let W_i be open sets, such that

$$W_i \subset \overline{W_i} \subset U_i$$

and the W_i still cover M. By Lemma 3.3.4, we can pick $R > 0$ such that $\overline{W_i} \subset (U_i)_R$. Furthermore, if we let θ be a convolution kernel with support contained in $B_R(0) \subset \mathbb{R}^m$, and we define

$$H_i(z) = \theta * G(z) : (U_i)_R \to \mathbb{R}^n,$$

then, again by Lemma 3.3.4,

$$H_i : D^k \to C^\infty((U_i)_R, \mathbb{R}^n)$$

is continuous and for R small enough, we will have

$$|H_i(z)(x) - G(z)(x)| < \epsilon,$$

for all $z \in D^k$ and $x \in \overline{W_i}$.

Now let $\{\lambda_i\}_i$ be a partition of unity subordinate to $\{W_i\}$ and define

$$H(z) = \sum_i \lambda_i H_i(z).$$

Now, each map $\lambda_i H_i : D^k \to C^\infty(M, \mathbb{R}^n)$ is continuous: the maps $\partial^\alpha \lambda_i H_i(w)$ are supported in W_i, for all $w \in D^k$. If we take $z_n \to z$ in D^k, then $\partial^\alpha \lambda_i H_i(z_n) \to \partial^\alpha \lambda_i H_i(z)$ uniformly on $\overline{W_i}$, because H_i is continuous. We conclude that H is continuous, because $C^\infty(M,\mathbb{R}^n)$ is a topological vector space.

Furthermore, it is clear that

$$|H(z)(x)-G(z)(x)|<\epsilon,$$

because $H(z)(x)$ is a convex combination of the $H_i(z)(x)$.

□

This finishes the proof of Theorem 3.3.2. We end with the proof of the main result of this section.

Proof. (of Theorem 3.3.1)

By Theorem 3.3.2, the inclusion maps maps

$$I_n : C^\infty(M,\mathbb{C}P^n)\hookrightarrow C(M,\mathbb{C}P^n)$$

are weak homotopy equivalences. The inclusion map

$$I : C^\infty(M,\mathbb{C}P^\infty)\hookrightarrow C(M,\mathbb{C}P^\infty)$$

is the colimit of the I_n, so, by Lemma 3.2.7, it is a weak homotopy equivalence.

□

3.4 The homotopy type of the space of maps to $\mathbb{C}P^\infty$

The purpose of this section is to prove the following Theorem:

Theorem 3.4.1. *Let M be a closed manifold, $\beta_0(M)$ the number of connected components of M and $\beta_1(M)$ the first Betti number of M. The space $C(M,\mathbb{C}P^\infty)$ of continuous maps $M\to\mathbb{C}P^\infty$ is homotopy equivalent to $H^2(M;\mathbb{Z})\times(S^1)^{\beta_1(M)}\times(\mathbb{C}P^\infty)^{\beta_0(M)}$, where and $H^2(M;\mathbb{Z})$ is regarded as a discrete space.*

The strategy of the proof is to show that $C(M,\mathbb{C}P^\infty)$ is homotopy equivalent to $C(X,A)$, where X is a finite cell complex and A is a topological abelian group and a cell complex. Then, we show that such a space is always homotopy equivalent to a product of Eilenberg-Maclane spaces and this implies that its homotopy groups determine its homotopy type. Finally, we compute the homotopy groups of $C(X,A)$, using the fact that we have a homeomorphism $C(X,A)\to C_*(X,A)\times A$, where $C_*(X,A)$ is the set of pointed continuous maps $X\to A$.

Lemma 3.4.2. *There is a topological abelian group A which is a cell complex and homotopy equivalent to $\mathbb{C}P^\infty$.*

Proof. It is enough to prove that there is a topological abelian group A that is an Eilenberg-Maclane space $K(\mathbb{Z},2)$, since $\mathbb{C}P^\infty$ is a $K(\mathbb{Z},2)$ and any two $K(\mathbb{Z},2)$'s are homotopy equivalent.

To construct A, consider the chain complex of abelian groups

$$\cdots\to 0\to\mathbb{Z}\to 0\to 0,$$

where the $\mathbb{Z}$ is in degree 2. By the Dold-Kan correspondence (see [17, Section 8.4]), there is a simplicial abelian group $\mathcal{A}$ whose homotopy groups are the homology groups of this complex, i.e,

$$\pi_k(\mathcal{A}) = \begin{cases} \mathbb{Z} & \text{if } k = 2 \\ 0 & \text{if } k \neq 2 \end{cases}.$$

Moreover, since we started with a countable chain complex (meaning that it is countable in each dimension), we get a countable simplicial abelian group $\mathcal{A}$ (meaning that $\mathcal{A}_n$ is countable, for all n).

Let A be the geometric realization of $\mathcal{A}$. Then A is a $K(\mathbb{Z}, 2)$ (see [10], 16.1 and 16.6). The sum in $\mathcal{A}$ is a map $\mu : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ and so it induces a map $|\mu| : |\mathcal{A} \times \mathcal{A}| \to |\mathcal{A}|$, between geometric realizations. Now, by a Theorem of Milnor (see [14]), the map $|\pi_1| \times |\pi_2| : |\mathcal{A} \times \mathcal{A}| \to |\mathcal{A}| \times |\mathcal{A}| = A \times A$, where $\pi_i : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ are the projections, is a homeomorphism (because $\mathcal{A}$ is countable) and so we have a map $A \times A \to A$. It can be checked easily that this map gives A the structure of a topological abelian group.

□

Lemma 3.4.3. *Let X, Y and Y' be topological spaces and $\phi : Y \to Y'$ a homotopy equivalence. Then the map $\phi_* : C(X, Y) \to C(X, Y')$ defined by $g \mapsto \phi \circ g$ is a homotopy equivalence.*

Proof. Let $\psi : Y' \to Y$ be a homotopy inverse to ϕ and let $H : [0, 1] \times Y' \to Y'$ be a homotopy between $\phi \circ \psi$ and $\mathrm{id}_{Y'}$. Clearly, ψ_* and ϕ_* are continuous.

Define

$$H_* : [0, 1] \times C(X, Y') \to C(X, Y')$$

by

$$(t, g) \mapsto H_t \circ g.$$

To prove that H_* is continuous, let $S(K, U)$ be a subbasic open set in $C(X, Y')$ and let $(t, g) \in H_*^{-1}(S(K, U))$. We have $\{t\} \times g(K) \subset H^{-1}(U)$ and $g(K)$ is compact, so, by the Tube Lemma, there is an $\epsilon > 0$ such that

$$[t - \epsilon, t + \epsilon] \times g(K) \subset H^{-1}(U).$$

Now, for each $x \in g(K)$, we have $[t - \epsilon, t + \epsilon] \times \{x\} \subset H^{-1}(U)$ and since $[t - \epsilon, t + \epsilon]$ is compact, we can use the Tube Lemma to find an open neighbourhood V_x of x in Y' such that $[t - \epsilon, t + \epsilon] \times V_x \subset H^{-1}(U)$. Let

$$V = \bigcup_{x \in g(K)} V_x$$

and note that

$$\{t\} \times g(K) \subset [t - \epsilon, t + \epsilon] \times V \subset H^{-1}(U).$$

Now suppose $(\tilde{t}, \tilde{g}) \in (t - \epsilon, t + \epsilon) \times S(K, V)$. Then

$$H(\{\tilde{t}\} \times \tilde{g}(K)) \subset H([t - \epsilon, t + \epsilon] \times V) \subset U,$$

so $H_*(\tilde{t}, \tilde{g}) \in S(K, U)$. Then $H_*((t-\epsilon, t+\epsilon) \times S(K, V)) \subset S(K, U)$. Moreover, $g(K) \subset V$, so we have $(t, g) \in (t-\epsilon, t+\epsilon) \times S(K, V)$ and so we conclude that H_* is continuous.

We conclude that H_* is a homotopy between $\phi_* \circ \psi_*$ and $\mathrm{id}_{C(X,Y')}$. Similarly, a homotopy between $\psi \circ \phi$ and id_Y determines a homotopy between $\psi_* \circ \phi_*$ and $\mathrm{id}_{C(X,Y)}$ and this proves that ϕ_* and ψ_* are homotopy equivalences.

□

A consequence of the two previous Lemmas is that $C(M, \mathbb{C}P^\infty)$ is homotopy equivalent to $C(M, A)$, where A is a topological abelian group and a $K(\mathbb{Z}, 2)$.

Now, since M is a compact manifold, it is homotopy equivalent to a finite cell complex X ([13, Theorem 3.5 and Corollary 2.3]).

Lemma 3.4.4. *Let X, X' and Y be topological spaces and $\phi : X \to X'$ a homotopy equivalence. Then the map $\phi^* : C(X', Y) \to C(X, Y)$ defined by $g \mapsto g \circ \phi$ is a homotopy equivalence.*

Proof. Let $\psi : X' \to X$ be a homotopy inverse to ϕ and note that ϕ^* and ψ^* are continuous. Let $H : [0,1] \times X' \to X'$ be a homotopy between $\phi \circ \psi$ and $\mathrm{id}_{X'}$ and define

$$H^* : [0,1] \times C(X', Y) \to C(X', Y)$$

by

$$(t, g) \mapsto g \circ H_t.$$

Let $S(K, U)$ be a subbasic open set in $C(X', Y)$ and let $(t, g) \in (H^*)^{-1}(S(K, U))$. Then $\{t\} \times K \subset H^{-1}(g^{-1}(U))$, so there is an $\epsilon > 0$ such that

$$[t-\epsilon, t+\epsilon] \times K \subset H^{-1}(g^{-1}(U)).$$

Then we have

$$H^*((t-\epsilon, t+\epsilon) \times S(H([t-\epsilon, t+\epsilon] \times K), U)) \subset S(K, U)$$

and

$$(t, g) \in (t-\epsilon, t+\epsilon) \times S(H([t-\epsilon, t+\epsilon] \times K), U),$$

so H^* is continuous.

□

This implies that $C(M, A)$ is homotopy equivalent to $C(X, A)$, where X is a finite cell complex, homotopy equivalent to M.

By a Theorem of Milnor (see [15]) the space $C(X, Y)$ is homotopy equivalent to a cell complex, whenever Y is a cell complex and X is a finite cell complex. In our case, this implies that $C(X, A)$ is homotopy equivalent to a cell complex.

Now we prove that $C(X, A)$ is a topological abelian group.

Lemma 3.4.5. *Let Y, Z be topological spaces, with Y Hausdorff and $\mathcal{B}$ a basis for the topology of Z. Then the colection $\mathcal{C}$ of all open sets of the form $S(K,U)$, for $K \subset Y$ compact and $U \in \mathcal{B}$ is a subbbasis for the compact-open topology on $C(Y,Z)$.*

Proof. By definition of the compact-open topology, the collection of sets of the form $S(K,U)$, for $K \subset Y$ compact and $U \subset Z$ open is a subbbasis for the compact-open topology on $C(Y,Z)$. So all we need to do is prove that any set in this subbasis can be written as a union of finite intersections of elements of $\mathcal{C}$.

So let $K \subset Y$ be compact, $U \subset Z$ open and $f \in S(K,U)$. For each $x \in K$, let $U_x \in \mathcal{B}$ be such that $f(x) \in U_x \subset U$. The sets $f^{-1}(U_x)$ form an open cover of K, so we can take a finite subcover $f^{-1}(U_1), \cdots, f^{-1}(U_n)$. Now let $K_i \subset K \cap f^{-1}(U_i)$ be compact sets that cover of K (these exist because K is a compact Hausdorff space, hence normal). Clearly $f \in S(K_1,U_1) \cap \cdots \cap S(K_n,U_n)$. Moreover, if $g \in C(Y,Z)$ and $g(K_i) \subset U_i$ for all i, then

$$g(K) = g(K_1) \cup \cdots \cup g(K_n) \subset U_1 \cup \cdots \cup U_n \subset U,$$

so we have $S(K_1,U_1) \cap \cdots \cap S(K_n,U_n) \subset S(K,U)$.

□

Lemma 3.4.6. *Let X be a Hausdorff space and A a topological abelian group. Then $C(X,A)$ is a topological abelian group with sum $(f+g)(x) = f(x)+g(x)$.*

Proof. Let $\mu : A \times A \to A$ be the sum in A. The map

$$\begin{aligned} C(X,A) \times C(X,A) &\to C(X,A) \\ (f,g) &\mapsto f+g \end{aligned}$$

is the composite

$$C(X,A) \times C(X,A) \overset{\Phi}{\to} C(X, A \times A) \overset{\mu_*}{\to} C(X,A),$$

where $\Phi(f,g) = f \times g$. So it suffices to prove that Φ is continuous.

The collection of all sets of the form $U \times V$, for U, V open in A is a basis for the topology of $A \times A$, so by Lemma 3.4.5 the collection of sets of the form $S(K, U \times V)$, for K compact and U, V open is a subbasis for the compact-open topology on $C(X, A \times A)$. So it is enough to show that $\Phi^{-1}(S(K, U \times V))$ is open. But

$$\Phi^{-1}(S(K, U \times V)) = S(K,U) \times S(K,V)$$

is open, so Φ is continuous.

□

Putting everything together, we get the following result.

Proposition 3.4.7. *Given a compact manifold M, there is a finite cell complex X and a cell complex A, which is a topological group, such that:*

- *X is homotopy equivalent to M;*

- *A is homotopy equivalent to $\mathbb{C}P^\infty$;*
- *$C(X,A)$ is homotopy equivalent to $C(M,\mathbb{C}P^\infty)$;*
- *$C(X,A)$ is a topological abelian group and homotopy equivalent to a cell complex;*

Now we compute the homotopy groups of $C(X,A)$. Let $C_*(X,A)$ be the subspace of $C(X,A)$ consisting of pointed maps, where we pick an arbitrary basepoint $x_0 \in X$ and we use 0 as the basepoint for A. Consider the map $s : A \to C(X,A)$ defined by $s(a)(y) = a$, for all $a \in A$ and $y \in X$.

Lemma 3.4.8. *Let (X, x_0) be pointed a topological space and A a topological abelian group. The map $C_*((X,x_0),(A,0)) \times A \to C(X,A)$ defined by $(f,a) \mapsto f + s(a)$ is a homeomorphism.*

Proof. It is clear that the map is continuous, because $C(X,A)$ is a topological group. Now consider the map $C(X,A) \to C_*((X,x_0),(A,0)) \times A$ defined by $g \mapsto (g - s(g(x_0)), g(x_0))$. It is clearly continuous and it is very easy to check that it is an inverse to the map in the statement.

□

We will now compute the homotopy groups of $C_*(X,A)$.

Proposition 3.4.9. *Let X be a finite cell complex, $\beta_0(X)$ the number of connected components of X and $\beta_1(X)$ its first Betti number. Then, the homotopy groups of $C_*(X, K(\mathbb{Z};2))$ are*

$$\pi_k(C_*(X, K(\mathbb{Z},2))) = \begin{cases} H^2(X;\mathbb{Z}) & \text{if } k = 0 \\ \mathbb{Z}^{\beta_1(X)} & \text{if } k = 1 \\ \mathbb{Z}^{\beta_0(X)-1} & \text{if } k = 2 \\ 0 & \text{if } k \geq 3. \end{cases}$$

Proof. Writing $[X,Y]_*$ for the set of pointed homotopy classes of maps $X \to Y$.

$$\begin{aligned} \pi_k(C_*(X, K(\mathbb{Z},2))) &= [S^k, C_*(X, K(\mathbb{Z},2))]_* \\ &= [S^k \wedge X, K(\mathbb{Z},2)]_* \\ &= H^2(S^k \wedge X; \mathbb{Z}) \\ &= \tilde{H}^{2-k}(X;\mathbb{Z}) \\ &= \begin{cases} H^2(X;\mathbb{Z}) & \text{if } k = 0 \\ H^1(X;\mathbb{Z}) & \text{if } k = 1 \\ \tilde{H}^0(X;\mathbb{Z}) & \text{if } k = 2 \\ 0 & \text{if } k \geq 3 \end{cases}. \end{aligned}$$

But

$$H^1(X;\mathbb{Z}) = \mathrm{Hom}(H_1(X),\mathbb{Z}) \oplus \mathrm{Ext}(H_0(X),\mathbb{Z}) = \mathbb{Z}^{\beta_1(X)} \oplus 0,$$

where $\beta_1(X)$ is the first Betti number of X, so we obtain the desired result.

□

Now we know the homotopy groups of $C(M, \mathbb{C}P^\infty)$:

Proposition 3.4.10. *Let M be a closed manifold, $\beta_0(M)$ the number of connected components of M and $\beta_1(M)$ the first Betti number of M. Then the homotopy groups of $C(M, \mathbb{C}P^\infty)$ are*

$$\pi_k(C(M, \mathbb{C}P^\infty)) = \begin{cases} H^2(M;\mathbb{Z}) & \text{if } k = 0 \\ \mathbb{Z}^{\beta_1(M)} & \text{if } k = 1 \\ \mathbb{Z}^{\beta_0(M)} & \text{if } k = 2 \\ 0 & \text{if } k \geq 3. \end{cases}$$

Proof. By Proposition 3.4.7, we have $C(M, \mathbb{C}P^\infty)$ homotopy equivalent to $C(X, A)$, where A is a topological abelian group homotopy equivalent to $\mathbb{C}P^\infty$ and X is a finite cell complex homotopy equivalent to M. By Lemma 3.4.8, we have $C(X, A) \cong C_*(X, A) \times A$. Now Lemma 3.4.9 applied to $C_*(X, A)$ and the fact that A is a $K(\mathbb{Z}, 2)$ finish the proof.

□

We will now prove that $C(X, A)$ is homotopy equivalent to a product of Eilenberg-Maclane spaces, so that its homotopy groups determine its homotopy type.

Lemma 3.4.11. *Let C be a chain complex of abelian groups. For each k, let*

$$g_k : \bigoplus_{\alpha \in I_k} \mathbb{Z} \to H_k(C)$$

be a surjective map and let

$$i_k : \bigoplus_{\beta \in J_k} \mathbb{Z} \hookrightarrow \bigoplus_{\alpha \in I_k} \mathbb{Z}$$

be the kernel of g_k. Let D_k be the chain complex

$$\cdots \to 0 \to \bigoplus_{\beta \in J_k} \mathbb{Z} \hookrightarrow \bigoplus_{\alpha \in I_k} \mathbb{Z} \to 0 \to \cdots$$

concentrated in degrees $k+1$ and k. Then, there is a quasi-isomorphism

$$\bigoplus_k D_k \to C.$$

Proof. In the diagram

$$\begin{array}{ccccc} & & \bigoplus_{\alpha \in I_k} \mathbb{Z} & & \\ & & \downarrow g_k & & \\ Z_k(C) & \xrightarrow{\pi_k} & H_k(C) & \longrightarrow & 0 \end{array}$$

we can find a lift f_k, because $\bigoplus_{\alpha \in I_k} \mathbb{Z}$ is free:

$$\begin{array}{ccccc} & & \bigoplus_{\alpha\in I_k}\mathbb{Z} & & \\ & \swarrow^{f_k} & \downarrow{\scriptstyle g_k} & & \\ Z_k(C) & \xrightarrow{\pi_k} & H_k(C) & \longrightarrow & 0. \end{array}$$

Now we have $\pi_k \circ f_k \circ i_k = g_k \circ i_k = 0$, so $\operatorname{im} f_k \circ i_k \subset \ker \pi_k = B_k(C)$, so we have a diagram

$$\begin{array}{ccccc} & & \bigoplus_{\beta\in J_k}\mathbb{Z} & & \\ & & \downarrow{\scriptstyle f_k\circ i_k} & & \\ C_{k+1} & \longrightarrow & B_k(C) & \longrightarrow & 0 \end{array}$$

and we can find a lift h_k, , because $\bigoplus_{\beta\in J_k}\mathbb{Z}$ is free:

$$\begin{array}{ccccc} & & \bigoplus_{\beta\in J_k}\mathbb{Z} & & \\ & \swarrow^{h_k} & \downarrow{\scriptstyle f_k\circ i_k} & & \\ C_{k+1} & \longrightarrow & B_k(C) & \longrightarrow & 0. \end{array}$$

Then the maps f_k and h_k determine a chain map $\phi_k : D_k \to C$ that is an isomorphism on H_k. Clearly, then, the map

$$\bigoplus_k \phi_k : \bigoplus_k D_k \to C$$

is a quasi-isomorphism.

□

Proposition 3.4.12. *Let Z be a topological abelian group that is homotopy equivalent to a cell complex. Suppose that Z only has finitely many nonzero homotopy groups and that these are finitely generated. Then Z is homotopy equivalent to a finite product of Eilenberg-Maclane spaces.*

Proof. Since Z is homotopy equivalent to a cell complex and a finite product of Eilenberg-Maclane spaces is a cell complex, it is enough, by [6, Theorem 4.5], to show that Z is weakly homotopy equivalent to a finite product of Eilenberg-Maclane spaces.

The singular complex $Sing(Z)$ of Z ([17, Example 8.2.4]) is a simplicial abelian group. We have a weak homotopy equivalence $|Sing(Z)| \to Z$ (see [10, Theorem 16.6 (ii)]) so it is enough to show that $|Sing(Z)|$ is weakly homotopy equivalent to a finite product of Eilenberg-Maclane spaces.

Suppose we have a weak homotopy equivalence of simplicial abelian groups

$$\phi : Sing(Z) \to \mathcal{K}(G_0, 0) \times \cdots \times \mathcal{K}(G_n, n),$$

where $\mathcal{K}(G_i, i)$ is a countable simplicial abelian group whose only nonzero homotopy group is $\pi_i(\mathcal{K}(G_i, i)) = G_i$. Then its geometric realization

$$|\phi| : |Sing(Z)| \to |\mathcal{K}(G_0, 0) \times \cdots \times \mathcal{K}(G_n, n)|$$

is a weak homotopy equivalence of topological spaces (see [10, Theorems 16.1 and 16.6]). But $|\mathcal{K}(G_0, 0) \times \cdots \times \mathcal{K}(G_n, n)|$ is homeomorphic to $|\mathcal{K}(G_0, 0)| \times \cdots \times |\mathcal{K}(G_n, n)|$ (see [14], noting that $\mathcal{K}(G_i, i)$ is countable) and $|\mathcal{K}(G_i, i)|$ is a $K(G_i, i)$ (see [10, Theorems 16.1 and 16.6]), so we get the desired weak homotopy equivalence

$$|Sing(Z)| \to K(G_0, 0) \times \cdots \times K(G_n, n).$$

So we see that it is enough to prove that such a ϕ exists. By the Dold-Kan correspondence (see [17, Section 8.4]), this is equivalent to showing that the normalized chain complex $N(Sing(Z))$ is quasi-isomorphic to a finite product of countable chain complexes with only one nonzero homology group. This is true by Lemma 3.4.11.

□

Now we can prove the main Theorem of this section:

Proof. (of Theorem 3.4.1)

By Lemma 3.4.7, the space $C(M, \mathbb{C}P^\infty)$ is homotopy equivalent to $C(X, A)$, where X is a finite cell complex homotopy equivalent to M and A is a topological abelian group and a cell complex, homotopy equivalent to $\mathbb{C}P^\infty$. By Proposition 3.4.12, the space $C(X, A)$ is homotopy equivalent to a product of Eilenberg-Maclane spaces, so its homotopy type is determined by its homotopy groups. Since S^1 is a $K(\mathbb{Z}, 1)$ and $\mathbb{C}P^\infty$ is a $K(\mathbb{Z}, 2)$, the desired result follows from Proposition 3.4.10.

□

Appendix A

Primitives of families of exact forms

Let M be a closed manifold and denote by $\Omega^*(M)$ the de Rham complex of M. The space $\Omega^k(M)$ of smooth k-forms on M is given the C^∞ topology, as a subspace of $C^\infty(M, \bigwedge^k T^*M))$. In this appendix, we prove the following Theorem:

Theorem A.0.1. *Let M be a closed manifold. Then there is a continuous linear map*

$$P_M : d(\Omega^*(M)) \to \Omega^*(M)$$

such that $dP_M = \mathrm{id}$.

One way to prove this Theorem is to use Hodge theory, as mentioned in [12, p. 96]. In [12] it is also said that one can prove this by using the Poincaré Lemma for compactly supported cohomology, the Mayer-Vietoris sequence and induction on the number of open sets in a good cover. An induction argument doesn't seem possible, but below we will use the Poincaré Lemma and basic Homological Algebra to find an explicit formula for the primitive of an exact form on M.

We note that the method described in this appendix also proves the following result, which is often useful in Differential Geometry. Recall that a family $(\omega_t)_{t\in\mathbb{R}^\ell}$ of k-forms on an m-manifold M is called *smooth*, if in every coordinate chart we have

$$\omega_t = \sum_{i_1<\cdots<i_k} f_{i_1\cdots i_k}(x_1, \cdots, x_m, t) dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

with $f_{i_1\cdots i_k}$ smooth, for all $i_1 < \cdots < i_k$.

Theorem A.0.2. *Let $(\omega_t)_{t\in\mathbb{R}^\ell}$ be a smooth family of exact $(k+1)$-forms on a closed manifold M. Then, there is a smooth family $(\alpha_t)_{t\in\mathbb{R}^\ell}$ of k-forms on M such that $d\alpha_t = \omega_t$ for all t.*

A.1 The local case

We start with the local case $M = \mathbb{R}^n$, and find explicit compactly supported primitives for exact compactly supported forms, following [1, pp. 37 - 39]. For a manifold V, let $\Omega^k_c(V) \subset \Omega^k(V)$ be the subspace of compactly supported k-forms. Let

$$\pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$$
$$(x_1, \cdots, x_{n-1}, t) \mapsto (x_1, \cdots, x_{n-1})$$

be the canonical projection and define

$$\pi_*^n : \Omega_c^k(\mathbb{R}^n) \to \Omega_c^{k-1}(\mathbb{R}^{n-1})$$

by

$$\begin{aligned}(f(x,t)dx_{i_1} \wedge \cdots \wedge dx_{i_k}) &\mapsto 0 \\ (f(x,t)dx_{i_1} \wedge \cdots \wedge dx_{i_{k-1}} \wedge dt) &\mapsto \left(\int_{\mathbb{R}} f(x,t)dt\right) dx_{i_1} \wedge \cdots \wedge dx_{i_{k-1}}.\end{aligned} \tag{A.1}$$

Let $e = e(t)dt$ be a compactly supported form on $\mathbb{R}$, with integral equal to 1. Define

$$e_*^n : \Omega_c^{k-1}(\mathbb{R}^{n-1}) \to \Omega_c^k(\mathbb{R}^n)$$

by

$$\omega \mapsto (\pi^*\omega) \wedge e. \tag{A.2}$$

It is easy to see that e_*^n and π_*^n commute with d and that $\pi_*^n e_*^n = \mathrm{id}$. Now define $A(t) = \int_{-\infty}^t e(s)ds$ and define

$$K_n : \Omega_c^k(\mathbb{R}^n) \to \Omega_c^{k-1}(\mathbb{R}^n)$$

by

$$\begin{aligned}f(x,t)dx_{i_1} \wedge \cdots \wedge dx_{i_k} &\mapsto 0 \\ f(x,t)dx_{i_1} \wedge \cdots \wedge dx_{i_{k-1}} \wedge dt &\mapsto (-1)^{k-1}\left(\int_{-\infty}^t f(x,s)ds - A(t)\int_{\mathbb{R}} f(x,s)ds\right) dx_{i_1} \wedge \cdots \wedge dx_{i_{k-1}}.\end{aligned} \tag{A.3}$$

One can see ([1, Proposition 4.6]) that $dK_n - K_n d = \mathrm{id} - e_*^n \pi_*^n$. From these identities, it is easy to see that if we define

$$P_n : d(\Omega_c^*(\mathbb{R}^n)) \to \Omega_c^*(\mathbb{R}^n)$$

inductively by $P_1 = K_1$ and $P_n = e_*^n P_{n-1} \pi_*^n + K_n$ then we will have $dP_n = \mathrm{id}$ for all n. It is also easy to see that this inductive formula can be written as

$$P_n = \sum_{\ell=1}^n e_*^n \circ \cdots \circ e_*^{\ell+1} \circ K_\ell \circ \pi_*^{\ell+1} \circ \cdots \circ \pi_*^n, \tag{A.4}$$

with the convention that $\Omega_c^k(\mathbb{R}^m) = 0$, for $k < 0$. This is an explicit formula that gives a compactly supported primitive for any exact compactly supported form and it is a continuous linear map

$$P_n : d(\Omega_c^*(\mathbb{R}^n)) \to \Omega_c^*(\mathbb{R}^n).$$

The following Theorem summarizes this section:

Theorem A.1.1. *For each $n \geq 2$, let π_*^n, e_*^n and K_n be the maps defined by* (A.1)*,* (A.2) *and* (A.3)*, respectively. Then, the map*

$$P_n : d(\Omega_c^*(\mathbb{R}^n)) \to \Omega_c^*(\mathbb{R}^n)$$

defined by

$$P_n = \sum_{\ell=1}^{n} e_*^n \circ \cdots \circ e_*^{\ell+1} \circ K_\ell \circ \pi_*^{\ell+1} \circ \cdots \circ \pi_*^n$$

is a continuous linear map such that $dP_n = \mathrm{id}$.

A.2 Globalization

Now we let M be a closed manifold. Our aim is to use a partition of unity subordinate to a good cover and Theorem A.1.1 to find a primitive for an exact form on M. Recall that a *good cover* is an open cover where all finite intersections of open sets are diffeomorphic to $\mathbb{R}^m$. To prove that a good cover of M exists, one can consider a Riemannian metric on M and take a cover by geodesically convex open sets.

For the rest of this section we fix a finite good cover $\{U_0, \cdots, U_n\}$ of M, diffeomorphisms $\phi_i : U_i \to \mathbb{R}^m$ and a partition of unity $\{\rho_0, \cdots, \rho_n\}$ subordinate to $\{U_0, \cdots, U_n\}$.

Let $\omega \in \Omega^k(M)$ be exact and let's try to find an explicit primitive for ω. The goal is to write

$$\omega = \sum_{i=0}^{n} \omega_i,$$

where each $\omega_i \in \Omega_c^k(U_i)$ is exact as then Theorem A.1.1 gives us an explicit formula $P_m(\omega_i)$ for a primitive of ω_i. The first attempt at doing this would be to take $\omega_i = \rho_i \omega$, but unfortunately there is no reason why these forms should be exact. We will have to correct our initial guess so as to obtain exact forms.

We start by giving a simple example that illustrates the method of proof.

Example A.2.1*.* Suppose $\{U_0, U_1\}$ is a good cover of M (so in this example, M is not compact) and let $\{\rho_0, \rho_1\}$ be a partition of unity subordinate to this good cover. Let $\omega \in \Omega_c^k(M)$ be an exact form, with $k < m = \dim(M)$ (the case $k = m$ is simpler). We have

$$\rho_0 \omega \in \Omega_c^k(U_0) \text{ and } \rho_1 \omega \in \Omega_c^k(U_1),$$

with $\rho_0 \omega + \rho_1 \omega = \omega$, but these forms are not necessarily closed (which, for $k < m$, is equivalent to exact). Let

$$\beta = -d(\rho_0 \omega) = d(\rho_1 \omega)$$

and note that β is supported in $U_0 \cap U_1$. Note that, if

$$\partial : H_c^k(M) \to H_c^{k+1}(U_0 \cap U_1)$$

is the boundary map in the Mayer-Vietoris sequence for compactly suppported cohomology (see [1, Proposition 2.7]), then

$$\partial[\omega] = [\beta]$$

(this can easily be seen by chasing the diagram in the snake Lemma). Since ω is exact, we find that β is exact, as a form in $\Omega_c^{k+1}(U_0 \cap U_1)$. Then, $\gamma = P_m(\beta)$ is an explicit primitive of β that is also supported in $U_0 \cap U_1$. Note that $\gamma \neq \rho_1\omega$, since the latter is not necessarily supported in $U_0 \cap U_1$. Now

$$d(\gamma) = -d(\rho_0\omega) = d(\rho_1\omega),$$

so $\omega_0 = \rho_0\omega + \gamma$ and $\omega_1 = \rho_1\omega - \gamma$ are closed (hence exact) and we have $\omega_i \in \Omega_c^k(U_i)$ and $\omega_0 + \omega_1 = \omega$, as desired.

We will describe how to generalize the argument in the previous example to an arbitrary finite good cover. In order to decompose an exact form $\omega \in \Omega^k(M)$ as a sum of exact forms $\omega_i \in \Omega_c^k(U_i)$, we will apply the Generalized Mayer-Vietoris Sequence (see [1, Proposition 8.5]), adapted to the case of compactly supported forms.

Consider the double cochain complex $\mathcal{C}^{*,*}$, where

$$\mathcal{C}^{-p,q} = \bigoplus_{\alpha_0 < \cdots < \alpha_p} \Omega_c^q(U_{\alpha_0 \cdots \alpha_p}), \tag{A.5}$$

for $p = 0, \cdots, n$ and $q = 0, \cdots, m = \dim M$, where

$$U_{\alpha_0 \cdots \alpha_p} = U_{\alpha_0} \cap \cdots \cap U_{\alpha_p}.$$

For

$$\lambda \in \bigoplus_{\alpha_0 < \cdots < \alpha_p} \Omega_c^q(U_{\alpha_0 \cdots \alpha_p}),$$

we write $\lambda_{\alpha_0 \cdots \alpha_p}$ to denote its component in $\Omega_c^q(U_{\alpha_0 \cdots \alpha_p})$, for $0 \leq \alpha_0 < \cdots < \alpha_p \leq n$. If we have $0 \leq \alpha_0 < \cdots < \alpha_p \leq n$ and σ is a permutation of the set $\{\alpha_0, \cdots, \alpha_p\}$, then we define

$$\lambda_{\sigma(\alpha_0) \cdots \sigma(\alpha_p)} = (-1)^{|\sigma|} \lambda_{\alpha_0 \cdots \alpha_p}. \tag{A.6}$$

If $\beta_i = \beta_j$ for some $i \neq j$, then we define $\lambda_{\beta_0 \cdots \beta_p} = 0$.

The horizontal differential of the double complex (A.5) is

$$D' : \mathcal{C}^{-p,q} \to \mathcal{C}^{-(p-1),q}$$

defined by

$$(D'\lambda)_{\beta_0 \cdots \beta_{p-1}} = \sum_\alpha \lambda_{\alpha\beta_0 \cdots \beta_{p-1}}. \tag{A.7}$$

Note that $(D')^2 = 0$. The vertical differential of the double complex (A.5) is

$$D'' : \mathcal{C}^{-p,q} \to \mathcal{C}^{-p,q+1},$$

given by

$$(D''\lambda)_{\beta_0\cdots\beta_p} = (-1)^p d(\lambda_{\beta_0\cdots\beta_p}). \tag{A.8}$$

Clearly, $(D'')^2 = 0$. It is easy to check that $D'D'' + D''D' = 0$. The total complex of $\mathcal{C}^{*,*}$ is the cochain complex defined by

$$\mathcal{C}^n = \bigoplus_{p+q=n} \mathcal{C}^{p,q},$$

with differential $D = D' + D'' : \mathcal{C}^n \to \mathcal{C}^{n+1}$. It is easy to check that $D^2 = 0$, by using the identities $(D')^2 = 0$, $(D'')^2 = 0$ and $D'D'' + D''D' = 0$. Given $\omega \in \mathcal{C}^\ell$, we denote by $\omega^{(k)} \in \mathcal{C}^{-k,\ell+k}$ its components.

Define $K : \mathcal{C}^{-p,q} \to \mathcal{C}^{-(p+1),q}$ by

$$(K\omega)_{\alpha_0\cdots\alpha_{p+1}} = \sum_{i=0}^{p+1} (-1)^i \rho_{\alpha_i} \omega_{\alpha_0\cdots\hat{\alpha_i}\cdots\alpha_{p+1}}. \tag{A.9}$$

We have

$$\begin{aligned}
(D'K\omega)_{\alpha_0\cdots\alpha_p} &= \sum_\alpha (K\omega)_{\alpha\alpha_0\cdots\alpha_p} \\
&= \sum_\alpha \left(\rho_\alpha \omega_{\alpha_0\cdots\alpha_p} + \sum_{i=0}^{p} (-1)^{i+1} \rho_{\alpha_i} \omega_{\alpha\alpha_0\cdots\hat{\alpha_i}\cdots\alpha_p} \right) \\
&= \omega_{\alpha_0\cdots\alpha_p} - \sum_{i=0}^{p} (-1)^i \rho_{\alpha_i} \sum_\alpha \omega_{\alpha\alpha_0\cdots\hat{\alpha_i}\cdots\alpha_p} \\
&= \omega_{\alpha_0\cdots\alpha_p} - \sum_{i=0}^{p} (-1)^i \rho_{\alpha_i} (D'\omega)_{\alpha_0\cdots\hat{\alpha_i}\cdots\alpha_p} \\
&= \omega_{\alpha_0\cdots\alpha_p} - (KD'\omega)_{\alpha_0\cdots\alpha_p},
\end{aligned}$$

so

$$D'K + KD' = \mathrm{id}\,. \tag{A.10}$$

If we define $\mathcal{C}^{-1,q} = \Omega^q(M)$, then the formulas (A.9) and (A.7) still make sense when $p = -1$ and equation (A.10) still holds.

Proposition A.2.2. *For each $q \geq 0$, the sequence*

$$0 \to \bigoplus_{\alpha_0<\cdots<\alpha_n} \Omega_c^q(U_{\alpha_0\cdots\alpha_n}) \xrightarrow{D'} \cdots \xrightarrow{D'} \bigoplus_{\alpha_0} \Omega_c^q(U_{\alpha_0}) \xrightarrow{D'} \Omega^q(M) \to 0$$

is exact.

Proof. Given a cocycle ω , we have $D'K\omega = \omega - KD'\omega = \omega$, so ω is a coboundary.

□

The map $D' : \bigoplus_{\alpha_0} \Omega_C^k(U_{\alpha_0}) \to \Omega^k(M)$ is just the sum map. It induces a chain map $S : \mathcal{C}^* \to \Omega^*(M)$, which in degree k is

$$S : \bigoplus_{p+q=k} \mathcal{C}^{p,q} \to \Omega^k(M)$$

given by $S|_{\mathcal{C}^{0,k}} = D'$ and $S|_{\mathcal{C}^{p,q}} = 0$, for $p \neq 0$.

Proposition A.2.3. *The map $S : \mathcal{C}^* \to \Omega^*(M)$ induces isomorphisms on cohomology.*

Proof. S is injective on cohomology:

Let $\omega \in \mathcal{C}^\ell$ be a cocycle, with $S(\omega) = d\phi$. We need to show that ω is a D-coboundary. We write $\omega = (\omega^{(k)})_{k=0,\cdots,n}$, with

$$\omega^{(k)} \in \mathcal{C}^{-k,\ell+k} = \bigoplus_{\alpha_0<\cdots<\alpha_k} \Omega^{\ell+k}(U_{\alpha_0\cdots\alpha_k}).$$

By definition of S, we have $D'(\omega^{(0)}) = d\phi$. Since

$$D' : \bigoplus_{\alpha_0} \Omega_c^*(U_{\alpha_0}) \to \Omega^*(M)$$

is surjective, there exists

$$\psi^{(0)} \in \mathcal{C}^{0,\ell-1} = \bigoplus_{\alpha_0} \Omega_c^{\ell-1}(U_{\alpha_0})$$

such that $D'\psi^{(0)} = \phi$. Now, as D' commutes with d,

$$\begin{aligned} D'(\omega^{(0)} - D''\psi^{(0)}) &= d\phi - dD'\psi^{(0)} \\ &= d\phi - d\phi \\ &= 0, \end{aligned}$$

so there exists

$$\psi^{(1)} \in \mathcal{C}^{-1,\ell}$$

such that $D'\psi^{(1)} = \omega^{(0)} - D''\psi^{(0)}$, so $D''\psi^{(0)} + D'\psi^{(1)} = \omega^{(0)}$. Now

$$\begin{aligned} D'(\omega^{(1)} - D''\psi^{(1)}) &= D'\omega^{(1)} + D''D'\psi^{(1)} \\ &= D'\omega^{(1)} + D''(\omega^{(0)} - D''\psi^{(0)}) \\ &= D'\omega^{(1)} + D''\omega^{(0)} \\ &= 0, \end{aligned}$$

because $D\omega = 0$. Hence there exists

$$\psi^{(2)} \in \mathcal{C}^{-2,\ell+1}$$

such that $D'\psi^{(2)} = \omega^{(1)} - D''\psi^{(1)}$, so $D''\psi^{(1)} + D'\psi^{(2)} = \omega^{(1)}$. Continuing in this way, we obtain $\psi = (\psi^{(k)})_{k=0,\cdots,n}$ with $D\psi = \omega$.

S is surjective on cohomology:

Let $\phi \in \Omega^\ell(M)$ be closed. We need to find a cocycle $\omega \in \mathcal{C}^\ell$ such that $S(\omega) = \phi$. Let $\omega^{(0)} \in \mathcal{C}^{0,\ell}$ be such that $D'(\omega^{(0)}) = \phi$. If we took $\omega^{(k)} = 0 \in (C)^{-k,\ell+k}$ for all $k > 0$, we would have $S(\omega) = \phi$, but unfortunately ω would not necessarily be a cocycle. We will need to choose the components $\omega^{(k)}$ more carefully, for $k > 0$.

As $D'D''\omega^{(0)} = dD'\omega^{(0)} = d\phi = 0$, there is $\omega^{(1)} \in (C)^{-1,\ell+1}$ such that $D'(\omega^{(1)}) = -D''(\omega^{(0)})$. With this choice of $\omega^{(1)}$, we have $(D\omega)^{(0)} = D''\omega^{(0)} + D'\omega^{(1)} = 0$. Now $D'D''\omega^{(1)} = -D''D'\omega^{(1)} = D''D''\omega^{(0)} = 0$, so we can repeat the process, picking $\omega^{(2)} \in (C)^{-2,\ell+2}$ such that $D'\omega^{(2)} = -D''\omega^{(1)}$. If we continue with this process, we get $\omega \in \mathcal{C}^\ell$ with $D\omega = 0$ and $S\omega = \phi$, as desired.

□

Now let's go back to the problem of finding an explicit primitive for $\omega \in \Omega^k(M)$ exact. By Proposition A.2.3, there is a cocycle $\beta \in \mathcal{C}^k$ such that $S(\beta) = \omega$. The following result gives an explicit formula for β.

Lemma A.2.4. *Let $\omega \in \Omega^k(M)$ be closed and K be the map defined by* (A.9). *Define $\beta_i^{(0)} = \rho_i\omega$ and $\beta^{(j)} = (-KD'')^j\beta^{(0)}$ for $j > 0$. Then $D\beta = 0$ and $S\beta = \omega$.*

Proof. We have $S(\beta) = D'(\beta^{(0)}) = \sum_i \rho_i\omega = \omega$. Now we need to show that $D\beta = 0$.

First, we show by induction that $D'D''\beta^{(j)} = 0$, for all j. For $j = 0$, we have $D'D''\beta^{(0)} = dD'(\beta^{(0)}) = d\omega = 0$. For $j > 0$, we have

$$\begin{aligned} D'D''\beta^{(j)} &= -D'D''KD''\beta^{(j-1)} \\ &= D''D'KD''\beta^{(j-1)} \\ &= D''(\mathrm{id} - KD')D''\beta^{(j-1)} \\ &= D''D''\beta^{(j-1)} - D''KD'D''\beta^{(j-1)} \\ &= 0, \end{aligned}$$

because $D'D''\beta^{(j-1)} = 0$, by induction hypothesis.

Then,

$$\begin{aligned} (D\beta)^{(j)} &= D'\beta^{(j+1)} + D''\beta^{(j)} \\ &= -D'KD''\beta^{(j)} + D''\beta^{(j)} \\ &= -(\mathrm{id} - KD')D''\beta^{(j)} + D''\beta^{(j)} \\ &= KD'D''\beta^{(j)} \\ &= 0. \end{aligned}$$

□

Note that $\beta^{(j)} = 0$ whenever $j > m - k$, simply because $\beta^{(j)} \in \mathcal{C}^{-j,k+j} = 0$, whenever $k + j > m$. In the following, we will write $\beta = (\beta^{(j)})_{j=0,\cdots,m-k}$.

Lemma A.2.4 gives an explicit cocycle $\beta \in \mathcal{C}^k$ representing ω. Since ω exact, Proposition A.2.3 implies that β is also exact. The plan is to find an explicit $\gamma \in \mathcal{C}^{k-1}$ such that $D\gamma = \beta$, as then $S(\gamma)$ will be an explicit primitive for ω.

We start by finding $\gamma^{(m-k+1)}$. The goal is to have $\beta^{(m-k)} - D'\gamma^{(m-k+1)}$ exact, so we can find an explicit primitive for it, using Theorem A.1.1, and then continue with the construction of γ.

For $V \subset M$ open, let

$$\mathcal{B}(V;\mathbb{R}) = \begin{cases} \mathbb{R} & \text{if } V \neq \emptyset \\ 0 & \text{if } V = \emptyset \end{cases}, \tag{A.11}$$

and consider the cochain complex $C^*(\mathcal{U},\mathbb{R})$, where $C^{-p}(\mathcal{U},\mathbb{R}) = \bigoplus_{\alpha_0<\cdots<\alpha_p} \mathcal{B}(U_{\alpha_0\cdots\alpha_p};\mathbb{R})$

$$0 \to \bigoplus_{\alpha_0<\cdots<\alpha_n} \mathcal{B}(U_{\alpha_0\cdots\alpha_n};\mathbb{R}) \xrightarrow{D'} \cdots \xrightarrow{D'} \bigoplus_{\alpha_0} \mathcal{B}(U_{\alpha_0};\mathbb{R}) \to 0 \tag{A.12}$$

with differential D' given by

$$(D'\lambda)_{\alpha_0\cdots\alpha_{p-1}} = \sum_{\alpha} \lambda_{\alpha\alpha_0\cdots\alpha_{p-1}},$$

using the sign conventions in (A.6).

Consider the map $I : \mathcal{C}^{*,m} \to C^*(\mathcal{U};\mathbb{R})$ given by integration. It is easy to see that this is a chain map and furthermore, we have

$$\ker I = \operatorname{im} D'',$$

since a compactly supported m-form on $\mathbb{R}^m$ has a compactly supported primitive if and only if its integral is zero ([1, Corollary 4.7.1]). We want to find $\gamma^{(m-k+1)}$ such that

$$I(\beta^{(m-k)} - D'\gamma^{(m-k+1)}) = 0.$$

This is equivalent to $D'I\gamma^{(m-k+1)} = I\beta^{(m-k)}$. The map $D' : C^{-(m-k+1)}(\mathcal{U},\mathbb{R}) \to C^{-(m-k)}(\mathcal{U},\mathbb{R})$ is a linear map between $\mathbb{R}$-vector spaces, so we can find a linear map $L : C^{-(m-k)}(\mathcal{U},\mathbb{R}) \to C^{-(m-k+1)}(\mathcal{U},\mathbb{R})$ such that $D'L|_{\operatorname{im} D'} = \operatorname{id}_{\operatorname{im} D'}$, or equivalently $D'LD' = D'$.

For any $\theta \in \mathcal{C}^{k-1}$ with $D\theta = \beta$, we have $I\beta^{(m-k)} = I(D''\theta^{(m-k)} + D'\theta^{(m-k+1)}) = ID'\theta^{(m-k+1)} = D'I\theta^{(m-k+1)}$, so $I\beta^{(m-k)} \in \operatorname{im} D'$, therefore $D'LI\beta^{(m-k)} = I\beta^{(m-k)}$. Now, for each sequence $\alpha_0 < \cdots < \alpha_{m-k+1}$, pick

$$\lambda_{\alpha_0\cdots\alpha_{m-k+1}} \in \Omega^m_c(U_{\alpha_0\cdots\alpha_{m-k+1}})$$

such that

$$\int_{U_{\alpha_0\cdots\alpha_{m-k+1}}} \lambda_{\alpha_0\cdots\alpha_{m-k+1}} = 1$$

and define

$$(\gamma^{(m-k+1)})_{\alpha_0\cdots\alpha_{m-k+1}} = (LI\beta^{(m-k)})_{\alpha_0\cdots\alpha_{m-k+1}} \lambda_{\alpha_0\cdots\alpha_{m-k+1}}. \tag{A.13}$$

Clearly, $I\gamma^{(m-k+1)} = LI\beta^{(m-k)}$. Since

$$\begin{aligned} I(\beta^{m-k} - D'\gamma^{(m-k+1)}) &= I\beta^{(m-k)} - D'I\gamma^{(m-k+1)} \\ &= I\beta^{(m-k)} - D'LI\beta^{(m-k)} \\ &= I\beta^{(m-k)} - I\beta^{(m-k)} \\ &= 0, \end{aligned}$$

we have found $\gamma^{(m-k+1)}$ with the desired properties.

Now we have $\beta^{(m-k)} - D'\gamma^{(m-k+1)}$ exact, so we can find an explicit primitive

$$\gamma^{(m-k)} = (-1)^{m-k} P_m(\beta^{(m-k)} - D'\gamma^{(m-k+1)}),$$

with P_m the map defined by (A.4). Then we have $D''\gamma^{(m-k)} + D'\gamma^{(m-k+1)} = \beta^{(m-k)}$ (recall that $dP_m = \mathrm{id}$ and $D''\gamma^{(m-k)} = (-1)^{m-k} d\gamma^{(m-k)}$). Using this, it is easy to see that $D''(\beta^{(m-k-1)} - D'\gamma^{(m-k)}) = 0$, so we can find an explicit primitive $\gamma^{(m-k-1)}$ as before, and we have $D''\gamma^{(m-k-1)} + D'\gamma^{(m-k)} = \beta^{(m-k-1)}$.

Repeating the process, we get $\gamma = (\gamma^{(j)})_{j=0,\cdots,m-k+1} \in \mathcal{C}^{k-1}$ such that $D\gamma = \beta$ and we have explicit formulas for all the $\gamma^{(j)}$. In particular, $D''\gamma^{(0)} + D'\gamma^{(1)} = \beta^{(0)}$, so, since $SD' = 0$, we have $SD''\gamma^{(0)} = S\beta^{(0)} = \sum_i \rho_i\omega = \omega$ and hence $dS(\gamma^{(0)}) = \omega$. Therefore

$$S(\gamma^{(0)}) = \sum_i (\gamma^{(0)})_i$$

is an explicit primitive for ω.

Note that we have indeed written ω as a sum of exact forms

$$\omega_i = \beta_i^{(0)} - (D'\gamma^{(1)})_i = \rho_i\omega - (D'\gamma^{(1)})_i \in \Omega_c^k(U_i),$$

so $(D'\gamma^{(1)})_i$ is the “correction” that we need to subtract from $\rho_i\omega$ to get an exact form.

In the end, we obtain a (very complicated) formula for a primitive of any exact form on M, that defines a continuous linear map

$$P_M : d(\Omega_c^*(M)) \to \Omega_c^*(M)$$

such that $dP_M = \mathrm{id}$.

The following Theorem summarizes the construction of an explicit primitive for an exact form:

Theorem A.2.5. *Let M be a closed manifold of dimension m. Let $\{\rho_0, \cdots, \rho_n\}$ be a partition of unity subordinate to a good cover $\mathcal{U} = \{U_0, \cdots, U_n\}$. Let $C^*(\mathcal{U}, \mathbb{R})$ be the cochain complex defined in* (A.12) *and pick linear maps $L : C^{-p}(\mathcal{U}, \mathbb{R}) \to C^{-(p+1)}(\mathcal{U}, \mathbb{R})$ such that $D'L|_{\mathrm{im}\, D'} = \mathrm{id}_{\mathrm{im}\, D'}$. For each $\alpha_0 < \cdots < \alpha_{m-k+1}$, pick $\lambda_{\alpha_0\cdots\alpha_{m-k+1}} \in \Omega_c^m(U_{\alpha_0\cdots\alpha_{m-k+1}})$ with $\int \lambda_{\alpha_0\cdots\alpha_{m-k+1}} = 1$.*

Consider the maps D', D'' and K defined by (A.7), (A.8) *and* (A.9)*, respectively and for each $\alpha_0 < \cdots < \alpha_j$, let $P_{\alpha_0\cdots\alpha_j} : d(\Omega_c^*(U_{\alpha_0\cdots\alpha_j})) \to \Omega_c^*(U_{\alpha_0\cdots\alpha_j})$ be a linear map such that $D''P_{\alpha_0\cdots\alpha_j} = \mathrm{id}$, given by the formula in Theorem A.1.1. Let $I : \mathcal{C}^{*,m} \to C^*(\mathcal{U}, \mathbb{R})$ be given by componentwise integration.*

For $\omega \in \Omega^k(M)$ an exact form, define $\beta^{(j)} \in \bigoplus_{\alpha_0<\cdots<\alpha_j} \Omega_c^{k+j}(U_{\alpha_0\cdots\alpha_j})$ by

$$\beta^{(j)} = \begin{cases} (\rho_\alpha\omega) & \text{if } j = 0 \\ (-KD'')^j \beta^{(0)} & \text{if } j > 0. \end{cases}$$

For $\alpha_0 < \cdots < \alpha_{m-k+1}$, let

$$\gamma^{(m-k+1)}_{\alpha_0\cdots\alpha_{m-k+1}} = (LI\beta^{(m-k)})_{\alpha_0\cdots\alpha_{m-k+1}}\lambda_{\alpha_0\cdots\alpha_{m-k+1}}$$

and for $j \leq m-k$ and $\alpha_0 < \cdots < \alpha_j$ let

$$\gamma^{(j)}_{\alpha_0\cdots\alpha_j} = P_{\alpha_0\cdots\alpha_j}(\beta^{(j)}_{\alpha_0\cdots\alpha_j} - (D'\gamma^{(j+1)})_{\alpha_0\cdots\alpha_j}).$$

Then, the map

$$\begin{aligned} P_M : d(\Omega^*(M)) &\to \Omega^*(M) \\ \omega &\mapsto \sum_\alpha \gamma^{(0)}_\alpha \end{aligned}$$

is linear, continuous and satisfies

$$dP_M = \mathrm{id}\,.$$

Bibliography

[1] Raoul Bott and Loring W. Tu. *Differential Forms in Algebraic Topology*. Springer-Verlag, 1982.

[2] A. Cannas da Silva. *Lectures on Symplectic Geometry.* Springer-Verlag, 2006.

[3] Y. Eliashberg and N. Mishachev. *Introduction to the h-Principle.* American Mathematical Society, 2002.

[4] M. Gromov. A topological technique for the construction of solutions of differential equations and inequalities. *Actes, Congrès intern. Math., Tome 2*, pages 221 – 225, 1970.

[5] M. Gromov. *Partial Differential Relations.* Springer-Verlag, 1986.

[6] A. Hatcher. *Algebraic Topology*. Cambridge University Press, 2002.

[7] Morris W. Hirsch. *Differential Topology*. Springer-Verlag, 1991.

[8] Sze-Tsen Hu. *Homotopy Theory.* Academic Press, 1959.

[9] Antoni A. Kozinski. *Differential Manifolds*. Academic Press, 1993.

[10] J. P. May. *Simplicial Objects in Algebraic Topology*. The University of Chicago Press, 1992.

[11] J. P. May. *A Concise Course in Algebraic Topology*. The University of Chicago Press, `http://www.math.uchicago.edu/~may/CONCISE/ConciseRevised.pdf` online edition, 1999.

[12] D. McDuff and D. Salamon. *Introduction to Symplectic Topology*. Clarendon Press, 2nd edition, 1998.

[13] J. Milnor. *Morse Theory*. Princeton University Press, 1969.

[14] John Milnor. The geometric realization of a semi-simplicial complex. *The Annals of Mathematics, 2nd Ser., Vol. 65*, (2):357 – 362, 1957.

[15] John Milnor. On spaces having the homotopy type of a cw-complex. *Transactions of the American Mathematical Society, Vol. 90*, (2):272 – 280, 1959.

[16] D. Tischler. Closed 2-forms and an embedding theorem for symplectic manifolds. *Journal of Differential Geometry*, (12):229–235, 1977.

[17] C. Weibel. *An introduction to Homological Algebra*. Cambridge University Press, 1994.