TWO DIMENSIONAL TQFTS

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1. Introduction

This is an expository article, with the goal of classify two dimensional Topological Quantum Field Theories (TQFTs) in terms of Frobenius Algebras. TQFTs arose from physical interests. In fact, they constitute a simplified model for a Quantum Field Theory, that does not depend on additional geometric structure on the space-time, such as Riemannian metrics, curvatures, etc., but only on its topological structure. TQFTs are also interest in Mathematics, indeed it is an important subject of top research nowadays.

Atiyah's definiton of a n-dimensional TQFT is the following: A TQFT is a monoidal functor from the symmetric monoidal category of n-dimensional Cobordisms to the monoidal category $\mathbf{Vect}_k$ with symmetric monoidal structure given by the tensor product of vector spaces. In the case $n = 2$ there is a bijection between TQFTs and commutative Frobenius algebras. In fact, such a bijection gives rise to a natural equivalence of categories between the category of two dimensional TQFTs and the category of commutative Frobenius algebras, cFa.

We begin by giving a brief review on some definitions and examples concerning monoidal categories, that will allow us to understand Atiyah's definition of a TQFT. We spend some time in constructing the monoidal category Cob_n whose objects are $n-1$ orientable closed smooth manifolds and the corresponding morphisms are equivalence classes of diffeomorphic cobordisms relatively to their boundaries, with the tensor product given by disjoint union of cobordisms. We also present in more detail the case $n = 2$.

We continue by studying Frobenius algebras and their properties, showing some analogies between the algebraic structure of a Frobenius algebra and the categorical relations on Cob_2.

We finish our discussion by introduce TQFTs and examples in the two dimensional case, and by proving the correspondence between two dimensional TQFTs and Frobenius algebras.

2. Preliminares

2.1. Monoidal Categories. Category Theory is essential to understand TQFTs, following Atiyah's definition. In this section we will introduce only those notions that will be necessary to our discussion throughout this exposition.

Definition 2.1. *A monoidal category is a sextuple $(\mathcal{C}, \otimes, \alpha, 1, \iota, \lambda)$, where:*

- *i.* *$\mathcal{C}$ is a category;*
- *ii.* *A bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, usually called tensor product;*
- *iii.* *A natural isomorphism,*

$$\alpha : \otimes \circ (\otimes \times id_{\mathcal{C}}) \to \otimes \circ (id_{\mathcal{C}} \times \otimes),$$

called the associator, such that the following diagram commutes for all $X,\ Y,\ Z,\ W \in \mathrm{Ob}(\mathcal{C})$,

$$\begin{array}{ccccc} & & (W \otimes (X \otimes Y)) \otimes Z & \xrightarrow{\alpha_{W,X\otimes Y,Z}} & W \otimes ((X \otimes Y) \otimes Z) \\ & \nearrow^{\alpha_{W,X,Y}\otimes id_Z} & & & \\ (W \otimes X) \otimes Y) \otimes Z & & & & \downarrow{\scriptstyle id_W \otimes \alpha_{X,Y,Z}} \\ & \searrow_{\alpha_{W\otimes X,Y,Z}} & & & \\ & & (W \otimes X) \otimes (Y \otimes Z) & \xrightarrow{\alpha_{W,X,Y\otimes Z}} & W \otimes (X \otimes (Y \otimes Z)) \end{array}$$

- *iv.* *A fixed object, 1, of $\mathcal{C}$;*
- *v.* *And natural isomorphisms, ι and λ, such that the following triangle diagram is commutative,*

$$\begin{array}{ccc} (X \otimes 1) \otimes Y & \xrightarrow{a_{X,1,Y}} & X \otimes (1 \otimes Y)\ , \\ {\scriptstyle \lambda_X \otimes id_Y}\downarrow & \swarrow_{id_X \otimes \iota_Y} & \\ X \otimes Y & & \end{array}$$

for all $X,\ Y \in \mathrm{Ob}(\mathcal{C})$.

When convenient, we drop the additional notation for a monoidal category, and denote it simply by the underlying category.

Example 2.2. The category **Set** admits a monoidal structure given by the Cartesian product $\times$, the unit object is the singleton set $\{*\}$ and the structural natural isomorphisms are the obvious ones.

Example 2.3. We can define other monoidal structures on the category **Set**. For example, if we let the tensor product be given by disjoint union with the obvious structure natural isomorphisms. In this case, the unit object is the empty set. More generally, any category with products or coproducts, admits a monoidal structure.

Example 2.4. Let k be a fixed field. The category $\mathbf{Vect}_k$ admits also a monoidal structure, given by the tensor product of vector spaces, with the unit object being the field k. In fact, given any commutative ring R, the category of R-modules admits a monoidal structure, given by the tensor product of R-modules.

Definition 2.5 (Strict Monoidal Category)**.** *We say that a monoidal category* $(\mathcal{C}, \otimes, \alpha, 1, \iota, \lambda)$ *is strict whenever the natural isomorphisms* α, ι, τ *are composed by identities morphisms. In this case, we represent the monoidal category by* $(\mathcal{C}, \otimes, 1)$.

In order to simplify our discussion we will be only concerned about strict monoidal categories in the next sections. Thus, whenever we say monoidal category, monoidal functors, etc. we refer to the strict versions, except when explicity stated otherwise.
The examples above are of non-strict monoidal categories, however, we can for all intents and purposes we can proceed as if the monoidal structure is strict as explained in Theorem 2.8 below. Note that in our everyday we, unconsciously, do not make the distinction between the sets $(X \times Y) \times Z$ and $X \times (Y \times Z)$.

Definition 2.6. *A (strict) monoidal functor* $F : (\mathcal{B}, \otimes, a, 1, \iota, \lambda) \to (\mathcal{C}, \otimes', a', 1', \iota', \lambda')$*, between two monoidal categories, is a functor on the underlying categories* $F : \mathcal{B} \to \mathcal{C}$ *such that:*

i. $F(X \otimes Y) = F(X) \otimes' F(Y)$*, for any two objects* $X, Y \in \mathcal{B}$*;*
ii. $F(f \otimes g) = F(f) \otimes' F(g)$*, for any two morphisms* f, g *of* $\mathcal{B}$*;*
iii. $F(1) = 1'$;
iv. $F(a_{X,Y,Z}) = a'_{F(X),F(Y),F(Z)}$*, for all* $X, Y, Z \in \mathrm{Ob}(\mathcal{B})$*;*
v. $F(\iota_X) = \iota'_{F(X)}$*, for all* $X \in \mathrm{Ob}(\mathcal{B})$*;*
vi. $F(\lambda_X) = \lambda'_{F(X)}$*, for all* $X \in \mathrm{Ob}(\mathcal{B})$.

There is non-strict monoidal functor between monoidal categories, however we do not need such a notion.

Definition 2.7. *Let* $(\mathcal{B}, \otimes, 1)$ *and* $(\mathcal{C}, \otimes', 1')$ *be two strict monoidal categories, and* $F, G : \mathcal{B} \to \mathcal{C}$ *be monoidal functors. A natural transformation* $\eta : F \to G$ *is monoidal if we have:*

i. $\eta_{X \otimes Y} = u_X \otimes' u_Y$*, for any two objects* $X, Y \in \mathrm{Ob}(\mathcal{B})$*;*
ii. $u_1 = id_{1'}$.

We can also define a more general notion of monoidal natural transformation between non-strict monoidal functors. We say that two monoidal categories $\mathcal{B}$ and $\mathcal{C}$ are monoidally equivalent if there exists two monoidal functors $F : \mathcal{B} \to \mathcal{C}$ and $G : \mathcal{C} \to \mathcal{B}$ such that the compositions $G \circ F$ and $F \circ G$ are monoidally isomorphic to $id_{mathcalB}$ and $id_{\mathcal{C}}$, respectively. In fact, by assuming that we are working only with strict monoidal categories we do not lose generality.

Theorem 2.8 (Strictification Theorem)**.** *Every monoidal category is monoidally equivalent to a strict monoidal category.*

2.2. **Monoids and Actions.** A monoid is a set endowed with a multiplication which is associative and admits an identity element. We can express such a definition in terms of arrows and the monoidal structure of **Set** given by the cartesian product. Thus the notion of monoid can be generalized to any monoidal category.

Definition 2.9. *Let* $(\mathcal{C}, \otimes, a, 1, \iota, \tau)$ *be a monoidal category. We say that* (M, μ, η) *is a monoid in* $\mathcal{C}$*, where* $M \in \mathcal{C}$*,* $\mu \in \mathrm{Hom}(M \otimes M, M)$*,* $\eta \in \mathrm{Hom}(1, M)$*, if the following diagrams commute:*

$$\begin{array}{ccccc} M\otimes(M\otimes M) & \xrightarrow{\alpha_{M,M,M}} & (M\otimes M)\otimes M & \xrightarrow{\mu\otimes id_M} & M\otimes M \\ \downarrow{\scriptstyle id_M\otimes\mu} & & & & \downarrow{\scriptstyle \mu} \\ M\otimes M & & \xrightarrow{\quad\mu\quad} & & M \end{array} \qquad \begin{array}{ccccc} 1\otimes M & \xrightarrow{\eta\otimes id_M} & M\otimes M & \xleftarrow{id_M\otimes\eta} & M\otimes 1 \\ & \searrow{\scriptstyle \iota_M} & \downarrow{\scriptstyle\mu} & \swarrow{\scriptstyle \tau_M} & \\ & & M & & \end{array}$$

We should think of μ as an associative multiplication on M, where the condition of associativity is expressed by the first diagram. We should also think of η as giving a unit for our multiplication, as stated in the second diagram.

Example 2.10. A monoid in **Set** is precisely a set endowed with an associative multiplication admitting an identity. Thus, this purely categorical definition of a monoid captures the notion stated in the first paragraph of this subsection.

Example 2.11. In the category of abelian groups, **Ab**, (with the monoidal structure given by the tensor product of abelian groups and the unit object being $\mathbb{Z}$), a monoid is a ring. In fact, the universal mapping property of $\otimes$ guarantees that the defined multiplication is distributive over addition.

Example 2.12. Consider the category $\mathbf{Vect}_k$, with the monoidal structure mentioned above. A monoid in this category is an unitary associative algebra over k, i.e., a vector space, which has a ring structure, compatible with the k-vector space structure.

Given a monoidal category and a module in this category, we can define a notion similar to the notion of a left-module over a ring.

Definition 2.13. *Let* (M, μ, η) *be a monoid over a monoidal category* $\mathcal{C}$*. We define a (left) action of* M *on an object* X *of* $\mathcal{C}$ *to be a morphism* $\nu : M \otimes X \to X$ *such that the following two diagrams commutes:*

$$\begin{array}{ccccc} (M\otimes M)\otimes X & \xrightarrow{\alpha_{M,M,X}} & M\otimes(M\otimes X) & \xrightarrow{id_M\otimes\nu} & M\otimes X \\ \downarrow{\scriptstyle \mu\otimes id_X} & & & & \downarrow{\scriptstyle \nu} \\ M\otimes X & & \xrightarrow{\quad\nu\quad} & & X \end{array} \qquad \begin{array}{ccc} 1\otimes X & \xrightarrow{\eta\otimes id_X} & M\otimes X \\ & \searrow{\scriptstyle \iota_X} & \downarrow{\scriptstyle\nu} \\ & & X \end{array}$$

Analogously, we can also define a (right) action of M on a given object X.

Example 2.14. Let M be a monoid in **Set**. An action of M on an object X is an action of a monoid in a set.

Example 2.15. A (left) action of a monoid R on an abelian group X, in **Ab**, gives X a left R-module structure. Similarly, a (right) action of R in an abelian group defines a right R-module.

Example 2.16. Given a monoid A in $\mathbf{Vect}_k$, an action of A on an object X defines a structure of a left module over the k-algebra A.

Given a monoidal category $\mathcal{C}$ we can also define the dual notion of a monoid, a comonoid, which is defined as a monoid but we reverse the arrows of the given commutative diagrams.

Definition 2.17. *Let* $(\mathcal{C}, \otimes, a, 1, \iota, \tau)$ *be a monoidal category. We say that* (C, δ, ϵ) *is a comonoid in* $\mathcal{C}$*, where* $C \in \mathcal{C}$*,* $\delta \in \mathrm{Hom}(M, M \otimes M)$*,* $\epsilon \in \mathrm{Hom}(C, 1)$*, the comultiplication and counit morphisms, resp., if the following diagrams commute:*

$$\begin{array}{ccccc} C\otimes(C\otimes C) & \xleftarrow{\alpha_{C,C,C}} & (C\otimes C)\otimes C & \xleftarrow{\delta\otimes id_C} & C\otimes C \\ \uparrow{\scriptstyle id_C\otimes\delta} & & & & \uparrow{\scriptstyle \delta} \\ C\otimes C & & \xleftarrow{\quad\delta\quad} & & C \end{array} \qquad \begin{array}{ccccc} 1\otimes C & \xleftarrow{\epsilon\otimes id_C} & C\otimes C & \xrightarrow{id_M\otimes\epsilon} & C\otimes 1 \\ & \nwarrow & \uparrow{\scriptstyle\delta} & \nearrow & \\ & & C & & \end{array}$$

A comonoid in the category $\mathbf{Vect}_k$ is called a coalgebra.

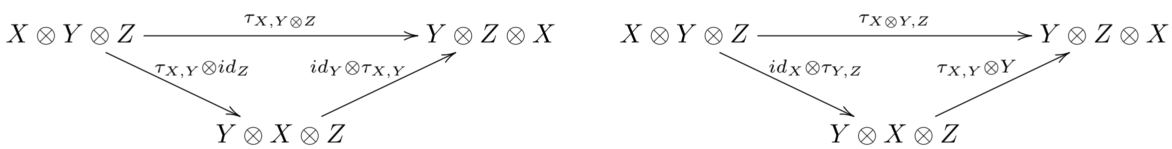
Example 2.18. Let G be a finite group. Consider the k-group algebra $k[G]$. Consider the linear map $k[G] \to k[G] \otimes k[G]$, given on basis elements by $g \mapsto \sum_{h,h',hh'=g} h \otimes h'$.
Such linear map defines a comultiplication on $k[G]$. The counit for this comultiplication is given by the linear map that assigns 1_k to the group identity, and 0 to all the others elements of $k[G]$.

2.3. **Symmetric Monoidal Categories.**

Definition 2.19. *A strict monoidal category* $(\mathcal{C}, \otimes, 1)$ *is said to be symmetric if there exists a natural isormophism* τ*, such that, for any pair of objects* X, Y*,*

$$\tau_{X,Y} : X \otimes Y \xrightarrow{\sim} Y \otimes X$$

and the following diagrams commute:



for any objects $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$.

Example 2.20. Note the that category **Set**, with the (strict) monoidal structure induced by the Cartesian Product is a symmetric monoidal category. In fact, the natural isomorphisms $\tau_{X,Y} : X \times Y \xrightarrow{\sim} Y \times X$, given by $(x, y) \mapsto (y, x)$, satisfy the commutative of the required diagrams.

Example 2.21. Note that, by the universal property of the tensor product, for any two k-vector spaces V and W, there is a natural isomorphism $\sigma_{V,W} : V \otimes W \xrightarrow{\sim} W \otimes V$. Therefore $\mathbf{Vect}_k$ is a symmetric monoidal category.

3. The Category of Cobordisms

In this section we define a family of categories, indexed by the positive integers, $\{\mathrm{Cob}_n\}$. For a given n, the objects of Cob_n are $n-1$ closed smooth orientable manifolds and the morphisms will be equivalence classes of *cobordisms* whose *in-boundary* and *out-boundary* are the source and target objects, respectively. We also define a monoidal structure on Cob_n and study in detail the case $n = 2$.

Definition 3.1. *Given an oriented smooth* n *manifold* M*, and an* $n-1$ *closed oriented submanifold* N *of* M*, we say that* $w \in T_xM$*, with* $x \in N$*, is a* ***positive normal*** *if given an oriented basis* $[v_1, \ldots, v_{n-1}]$ *of* T_xN*,* $[v_1, \ldots, v_{n-1}, w]$ *is an oriented basis for* T_xN*.*

If N is a connected component of the boundary of M, ∂M, a positive normal $w \in T_xN$, with $x \in N$, points inwards relatively to M if, in local coordinates, it is possible to find a differentiable curve $\gamma : [0, \epsilon) \to M$, such that $\gamma(0) = x$ and $\dot{\gamma}(0) = w$. Otherwise we say that w points outwards relatively to M. It is possible to verify that these definitions do not depend on the choice of local coordinates.

Definition 3.2. *Let* M *be a smooth manifold of dimension* n*. Suppose* N *is a connected component of* ∂M *with a chosen orientation, if a positive normal points inwards of* M *we say that* N *is an* ***in-boundary*** *of* M*, otherwise we say that* N *is an* ***out-boundary*** *of* M*:*

The definition of in-boundary, (resp., out-boundary) does not depend either on the choice of the point $x \in N$ or on the positive normal considered. Note also that given a manifold W, with M, N the in-boundary and out-boundary, respectively, we have $\partial W = M \coprod N$. Hereafter we will only consider *smooth oriented manifolds* when we refer simply the word manifold.

Definition 3.3 (Cobordism). *Let* M *and* N *be* $n-1$ *dimensional closed manifolds. We say that* (W, ι_0, ι_1)*, is a* ***cobordism*** *between* M *and* N*, if* W *is a compact* n *manifold such that the maps*

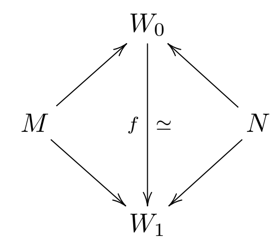
$$\iota_0 : M \to W, \ \iota_1 : N \to W$$

are orientation preserving diffeomorphisms onto the in-boundary and out-boundary of W*, respectively.*

We say that two $n-1$ dimensional manifolds are cobordant if there exists a cobordism between them. A cobordism allow us to "map" from its in-boundary to its out-boundary. Henceforth we will denote by $W : M \to N$, a cobordism W between M and N, however this is an abuse of notation since we did not refer the correspondent orientation preserving diffeomorphisms.

Since we can see a cobordism as map from its boundary components, we would like to be able to compose cobordisms. The gluing of cobordisms seems to be a natural choice for such a composition. However, this notion does not allow us to have a well defined composition, since there are few subtleties that we should take into account.

Definition 3.4. *Two oriented cobordisms $W_0,\ W_1 : M \to N$ are said to be* ***equivalent*** *if there exists an orientation preserving diffeomorphism $f : M \to N$ making the following diagram commute:*



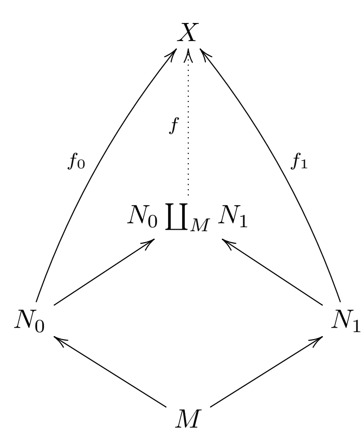
Remark 1. The equivalence of cobordisms defines an equivalence relation. The morphisms in our category Cob_n will consist precisely of equivalence classes of cobordisms. It is necessary to take the quotient by this equivalence relation in order for composition to be well defined.

We will denote by $W : M \to N$ not only a cobordism between M and N, but also its equivalence relation, when the context is clear.

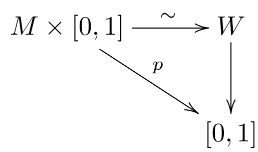
Definition 3.5 (Gluing along a common boundary)**.** *Let $f_0 : M \to W_0$ and $f_1 : M \to W_1$ be injective continuous maps between topological manifolds, such that M is mapped by f_0 homeomorphically onto the out-boundary of W_0 and mapped by f_1 homeomorphically onto the in-boundary of W_1. The gluing of W_0 and W_1 along M is the topological space $W_0 \coprod W_1/\sim$, where $\sim$ is the equivalence relation on $W_0 \coprod W_1$ defined by $w_0 \sim w_1$ if there exists $x \in M$ such that $w_0 = f_0(x)$ and $w_1 = f_1(x)$.*

We will represent $W_0 \coprod W_1/\sim$ by $W_0 \coprod_M W_1$, or whenever the context is clear simply by $W_0 W_1$. In fact, given any two topological manifolds, such that we can glue them along a common boundary, it is possible to prove that the resulting topological space is a manifold.

Note also, that the gluing of topological manifolds along a common boundary is a pushout in the category of topological manifold and continuous maps between them. In fact, if $f_0 : N_0 \to X$ and $f_1 : N_1 \to X$ are continuous maps to some topological space X, then there is a unique $f : N_0 \coprod_M N_1 \to X$ such that the following diagram commutes:



Theorem 3.6 (**Regular Interval Theorem**). *Let $W : M \to N$ be a cobordism, and $f : W \to [0,1]$ be a smooth map without critical points, such that $M = f^{-1}(0)$ and $N = f^{-1}(1)$. Then there exists a diffeomorphism between $M \times [0,1]$ and W such that the following diagram commutes:*



where $p: M \times [0,1] \xrightarrow{p} [0,1]$ *is the natural projection, onto the second factor.*

To a proof of this theorem, the reader should consult [3], Theorem 6.2.2.

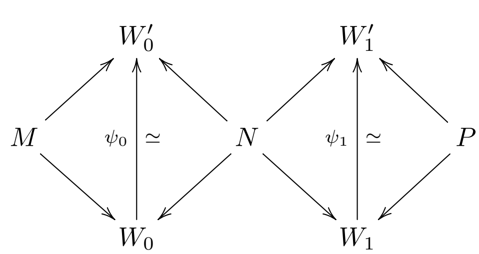
Corollary 3.7. *Let $W : M \to N$ be a cobordism, and $f : W \to [0,1]$ a Morse function such that $f^{-1}(0) = M$, $f^{-1}(1) = N$ and without critical points on the boundary of W. Then there exists $\epsilon > 0$, such that $f^{-1}([0,\epsilon])$ is diffeomorphic to the cylinder $M \times [0,1]$.*

Proof. Let $\epsilon > 0$ be such that $[0,\epsilon]$ does not contain any critical value of f, then the restriction of f to the interval $[0,\epsilon]$ is a Morse function without critical points. Hence, by the Regular Interval Theorem, $f^{-1}([0,\epsilon])$ is diffeomorphic to a cylinder. □

Consider cobordisms $W_0 : M \to N$ and $W_1 : N \to P$. Let $f_0 : W_0 \to [0,1]$ and $f_1 : W_1 \to [1,2]$ be two Morse functions, without critical points on the boundaries, and consider the topological manifold $W = W_0 \coprod_N W_1$, together with the Morse function $f : W \to [0,2]$, induced by f_0 and f_1. By the previous Corollary there is an $\epsilon > 0$ such that $[1-\epsilon, 1]$ and $[1, 1+\epsilon]$ are regular intervals for f_0, f_1, respectively, thus $f_0^{-1}([1-\epsilon,1])$ and $f_1^{-1}([1,1+\epsilon])$ are diffeomorphic to the cylinder $N \times [0,1]$. Since the gluing of two homeomorphic cylinders C_1 and C_2 is again a cylinder homeomorphic to C_1, we can define a homeomorphism $\phi : f_0^{-1}([1+\epsilon,1]) \coprod_{f_0^{-1}(\epsilon)} f_1^{-1}([1-\epsilon,1]) \to N \times [0,1]$, which is a diffeomorphism when restricted to each component. Therefore, we can define a smooth structure on the gluing to be the one induced by the smooth structure on $N \times [0,1]$. Also, since we have inclusion maps $\iota_0 : W_0 \to W_0 \coprod_N W_1$, $\iota_1 : W_1 \to W_0 \coprod_N W_1$, we can define a smooth structure on the gluing $W_0 \coprod_N W_1$ to be the one induced by the inclusion maps on each of the pieces $f^{-1}([0,1-\epsilon])$ and $f^{-1}([1+\epsilon,2])$, and to be the one induced by the smooth structure on the cylinder $N \times [0,1]$ on $f^{-1}([1-\epsilon,1+\epsilon])$. In fact, we have defined a smooth structure on all W, to see this we just need to check the compatibility of the different smooth structures on the intersections, but this follows by our choice of ϵ.

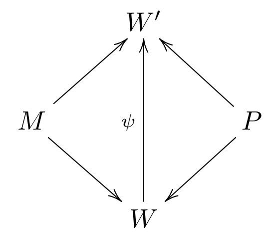
Proposition 3.8. *Let W_0 and W_1 be smooth manifolds, with W_0 having M as out-boundary, and W_1 having M as in-boundary. Consider the topogical manifold $W_0W_1 = W_0 \coprod_M W_1$. Then given two distinct smooth structures α and β on W_0W_1 which both induce the original smooth structures on M_0 and M_1, respectively, by pullback along the inclusion maps, there is a diffemorphism $\phi : (W_0W_1, \alpha) \xrightarrow{\sim} (W_0W_1, \beta)$ such that $\phi|_M = id_M$.*

We are now able to conclude that we can compose equivalence cobordism classes. Let $W_0,\ W_0' : M \to N$, $W_1, W_1' : N \to P$ be cobordisms, and $\psi_0 : W_0 \to W_0'$ and $\psi_1 : W_1 \to W_1'$ be diffeomorphisms rel boundary such that the following diagram commutes:



Then, by gluing, we obtain two homeomorphic topological manifolds $W = W_0W_1$ and $W' = W_0'W_1'$, with $\psi = \psi_0 \coprod \psi_1 : W \xrightarrow{\sim} W'$ a homeomorphism between them, which restricts to a diffeomorphism on W_0 and W_1.

Also, ψ is a homeomorphism such that the following diagram commutes:



We can regard the smooth structure defined above on W and W'. Also, since ψ is a homeomorphism between W and W', we can define a new smooth structure on W' to be the one induced by W, and in this case, ψ is, in fact, a diffeomorphism. Therefore, we are considering two different smooth structures on W', however they are in the conditions of the previous theorem, (since ψ is a diffeomorphism, when restricted to each piece), hence we have that such smooth structures are diffeomorphic rel boundary.
This means that W and W' are in fact equivalent cobordisms, which means, that we have defined a well-behaved composition on equivalence classes of cobordisms.

Definition 3.9. *Given two equivalence cobordism classes $W_0 : M \to N$, $W_1 : N \to P$, their composition is defined as $W_0W_1 : M \to P$, where $W_0W_1 : M \to P$ denotes the equivalence class defined by the gluing $W_0 \coprod_N W_1 : M \to P$.*

Such a composition is easily seen to be associative. This follows by noting that the pushout, as topological manifolds, is such that

$$(W_0 \textstyle\coprod_M W_1) \coprod_N W_2 \simeq W_0 \coprod_M (W_1 \coprod_P W_2)$$

By induction, we can extend Proposition 3.8., to any finite number of manifolds. Therefore, the associativity follows, when we consider the gluing of three manifolds. Let $W : M \to N$ be a cobordism. If we attach a cylinder $C = M \times [0, 1]$ to W, the resulted cobordism is equivalent to W. To show this, note that we have $CW = C(W_{[0,\epsilon]}W_{[\epsilon,1]}) = (CW_{[0,\epsilon]})W_{[\epsilon,1]} = W_{[0,\epsilon]}W_{[\epsilon,1]}$, where $W_{[0,\epsilon]}$, $W_{[\epsilon,1]}$ are as in the Corollary 3.7, and the second equality follows from the fact that $W_{[0,\epsilon]} \simeq M \times [0, 1]$ and $(M \times [0, 1])(M \times [0, 1]) \simeq M \times [0, 1]$.

Fact 3.10. *There is a family of categories, indexed by positive integers, $\{Cob_n\}$, such that, for each n, the objects are $n-1$ closed manifolds and the morphisms are equivalence classes of cobordisms between them.*

The following example illustrates an important construction, that it will be useful in through the exposition.

Example 3.11 (Cylinder Construction). Let $\mathbf{Diff}_{n-1}$ be the category whose objects are $n-1$ manifolds, and the morphisms are diffeomorphisms between them. Let $f : M \to N$ be a diffeomorphism. Then we can associate to f the cobordism class defined by the cobordism $M \times [0, 1] : M \to N$, where we map M to the in-boundary of $M \times [0, 1]$, by the identity map, and N to the out-boundary by the inverse of f, this association is called the *Cylinder Construction*. Moreover this association preserves composition and identity morphisms. To see this, let $f : M \to N$ and $g : N \to P$ be two diffeomorphisms, and consider their corresponding cobordisms $(M \times [0, 1], id, f^{-1})$ and $(N \times [0, 1], id, g^{-1})$. We can compose $(M \times [0, 1], id, f^{-1})$ with $(N \times [0, 1], f^{-1}, f^{-1} \circ g^{-1})$ to obtain a cobordism $(M \times [0, 1], id, f^{-1} \circ g^{-1})$, which is the cobordism given by the Cylinder Construction applied to $g \circ f : M \to P$. Also, $(N \times [0, 1], id, g^{-1}) \simeq (M \times [0, 1], f^{-1}, f^{-1} \circ g^{-1})$, since $f^{-1} \times id : N \times [0, 1] \to M \times [0, 1]$ gives us the equivalence. Therefore, we can define a functor $T : \mathbf{Diff}_{n-1} \to \mathrm{Cob}_n$, called the *Cylinder Construction Functor*. Note also, that since every morphism $f \in \mathbf{Diff}_{n-1}$ is an isomorphism, $T(f)$ is an invertible morphism in Cob_n

3.1. **Invertible Morphisms.** What are the isomorphisms in the category Cob_n? The reason to spend some time with this question is that the answer turns out to be very interesting. The isomorphisms are represented by equivalence classes of h-cobordisms.

Definition 3.12. *Given two differentiable manifolds X, Y, we say they are homotopy equivalent if there exist smooth maps $f : X \to Y$, $g : Y \to X$, such that $f \circ g \simeq id_Y$ and $g \circ f \simeq id_X$. We say that f and g are homotopy equivalences.*

Definition 3.13 (h-cobordism)**.** *Let $W : M_0 \to M_1$ be a cobordism between closed n dimensional manifolds. We say that W is an h cobordism if the inclusions maps $M_0 \hookrightarrow W \hookleftarrow M_1$ are homotopy equivalences.*

Proposition 3.14. *Let $W : M \to N$ be a cobordism defining an invertible morphism, then W is an h-cobordism.*

Proof. Let W' be an inverse of W, i.e. WW' and $W'W$ are the identity morphisms with respect to M and N, respectively. Consider the manifold X obtained by gluing in series a number of copies of WW' indexed by the integers, i.e., $X = \coprod_{n\in\mathbb{Z}}(WW')_n$. Since WW' is equivalent to $M \times [0,1]$ it follows that X is equivalent to $M \times \mathbb{R}$. Similarly, X is equivalent to $N \times \mathbb{R}$. Consider diffeormorphisms $\psi_0 : X \to M \times \mathbb{R}$ and $\psi_1 : X \to N \times \mathbb{R}$, such that $\psi_0(WW'_n)$ is $M \times [n, n+1]$, and $\psi_1(W'W_n)$ is $N \times [n, n+1]$.

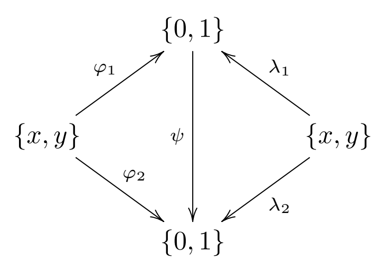
Note that $M \times \mathbb{R}$ deformation retracts onto $M \times [0,\infty)$, induced by the deformation retraction of $\mathbb{R}$ into $[0,\infty)$, therefore X deformation retracts onto $\psi_0^{-1}(M_0 \times [0,\infty))$. Reasoning similarly, X deformation retracts into $\psi_1^{-1}(N \times [0,\infty)$. Also, $M \times [0,\infty)$ contains $N \times [0,\infty)$, since by construction of the diffeomorphisms ψ_0 and ψ_1, $\psi_1(N \times [n, n+1]) \subset \psi_0(M \times [n, n+2])$. We have also $\psi_0(M \times [n, n+1]) \subset \psi_1(N \times [n-1, n+1])$.

This means that the deformation retract of $N \times [0,\infty)$ induces a deformation retraction of $M \times [0,\infty)$ onto $\psi_0(W)$, which implies that the inclusion $W \hookrightarrow X$ is in fact an homotopy equivalence. Since the maps $M \hookrightarrow X$ and $W \hookrightarrow X$ are homotopy equivalences, it follows that $M \hookrightarrow W$ is also an homotopy equivalence. By a similar argument, the inclusion $N \hookrightarrow W$ is also an homotopy equivalence, and the result follows. □

3.2. **Monoidal Structure.** Given two manifolds, M and M', their disjoint union has a unique structure of (smooth oriented) manifold such that the inclusion maps are orientation preserving. The disjoint union of two cobordisms, $W : M \to N$ and $W' : M' \to N'$ is again a cobordism, $W \coprod W' : M \coprod M' \to N \coprod N'$.

The notion of disjoint union of cobordisms is well behaved on the equivalence cobordism classes. If $W : M \to N$ is equivalent to $V : M \to N$, and $W' : M' \to N'$ equivalent to $V' : M' \to N'$, with diffeomorphisms, defining the equivalences, g, h, respectively, then by the universal property of disjoint union of manifolds there exists a diffeomorphism $f = g \coprod h : W \coprod W' \to V \coprod V'$, such that the restriction of f to each component *equals* g and h, respectively. Also, f is a diffeomorphism that makes the diagram presented in Definition 3.4. commute, showing that these cobordisms are in fact equivalent. Therefore, we can consider a disjoint union of equivalence classes of cobordisms.

The twist cobordism. By the *Cylinder Construction*, the twist map $M \coprod M' \xrightarrow{\sim} M' \coprod M$ induces a cobordism class $T_{M,M'} : M \coprod M' \to M' \coprod M$, which defines an isomorphism in Cob_n, we call it the *twist cobordism.* That is, $(\mathrm{Cob}_n, \coprod, \emptyset, T)$ is a symmetric monoidal structure, where the symmetry natural isomorphism is determined by twist cobordisms.
It is important to notice that the twist cobordism does not define the same equivalence class as the disjoint union of two identity cobordisms. In fact, the existence of such an equivalence between these two cobordism would imply that should exist a bijection $\psi : \{0,1\} \to \{0,1\}$ such that the following diagram commute



Where $\varphi_1(x) = 0,\ \varphi_1(y) = 1$, $\varphi_2(x) = 1\ \varphi_2(y) = 0$ and $\lambda_1(x) = 1,\ \lambda_1(y) = 0,\ \lambda_2(x) = 0,\ \lambda_2(y) = 0$. Clearly this is impossible. More generally, not all disconnected cobordisms are disjoint unions of connected cobordisms.

Fact 3.15. *The category* Cob_n *admits a symmetric monoidal structure, by letting the tensor product be given by disjoint union of manifolds on objects and by disjoint union of cobordisms classes on morphisms, with the empty n-manifold as a unit and the symmetry given by twist cobordisms.*

3.3. **The Category Cob_2.** We will study the case of $n = 2$. Note that the objects of Cob_2 are closed 1 manifolds, that is, a finite disjoint union of manifolds diffeomorphic to S^1. In order to simplify our study we will be concerned with a skeleton of Cob_2. The objects of such skeleton will be indexed by nonnegative integers $\mathbb{Z}_{\geq 0}$, where the integer n corresponds to a disjoint union of n copies os S^1, that we denote by $\mathbf{n}$. Moreover the disjoint union of $\mathbf{n}$ and $\mathbf{m}$ is precisely $\mathbf{n{+}m}$, and the symmetry is given by equalities. This skeleton codifies the relevant structure of Cob_2, therefore, by abuse of notation, we denote it by Cob_2. Until the end of this section, whenever we refer to Cob_2 we are considering such skeleton.
The category Cob_2 is finitely generated as a symmetric monoidal category.

Definition 3.16 (**Normal form of a connected surface**). *Given a* ***connected*** *oriented surface S, with m in-boundaries, genus g, and n out-boundaries, its normal form is a decomposition of it into three connected oriented surfaces. One such surface, the* ***in-part****, with m in-boundaries and one out-boundary, and genus 0, a surface with one in-boundary and one out-boundary, and with genus g,* ***middle part****, and the other, the* ***out-part****, with one in-boundary, n out-boundaries and genus 0.*

Fact 3.17. *Every connected oriented surface admits a normal form.*

Let S be a surface, and assume that the in-boundary of S has $m > 0$ components. The in-part is constructed by gluing $m - 1$ copies of pairs of pants in series, such that the upper component of the in-boundary of each pair of pants is glued to a cylinder and the other component glued to the out-boundary of other pair of pants. If $m = 0$, we take instead a disc for the in-part.
The out-part is described similarly, if the number of components of the out-boundary of S is $n > 0$, we take $n - 1$ copies of copair of pants and make the same construction, again with cylinders on top. If $n = 0$, we consider the out-part to be a disc.
The middle-part is the surface obtained by gluing g copairs of pants to g pairs of pants, having one in-boundary and one out-boundary.
The Classification Theorem of Surfaces imply that the surface obtained, by gluing the in-part to the middle-pat, and the resulting surface to the out-part, is difeomorphic to S.

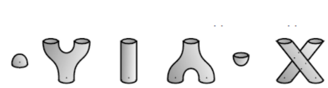
Corollary 3.18. *Every connected two dimensional cobordism can be obtained by gluing the following cobordisms:*

The nonconnected case can be treated similarly using *permutation cobordisms*, i.e., cobordisms given by the Cylinder Construction applied to a diffeomorphism $\coprod_{i=1}^{n} S^1 \xrightarrow{\sim} \coprod_{i=1}^{n} S^1$.

Proposition 3.19. *Every cobordism in Cob_2 can be obtained as a gluing of a permutation cobordism, a disjoint union of connected cobordisms, and a permutation cobordism.*

Proof. Let $W : \mathbf{m} \to \mathbf{n}$ be a two dimensional cobordism. The connected case was discussed previously. First, assume that W has exactly two connected component W_0 and W_1. Then the in-boundary of W_0, M_0, is a proper submanifold of $\mathbf{m}$. Suppose that it is composed by the $i_1, \ldots, i_k$, $k \leq m$, components of $\mathbf{m}$. Thus the in-boundary of W_1, M_1, consists precisely of the complement of M_0 in $\mathbf{m}$. Consider a diffeomorphism $\mathbf{m} \to \mathbf{m}$, that consists in a permutation of the components of $\mathbf{m}$, such that the components $i_1, \ldots, i_k$ are placed in the first k components of $\mathbf{m}$. The permutation cobordism, P_1, induced by this diffeomorphism is such that the in-boundary of WP_1 is the disjoint union of the in-boundary of M_0 and the in-boundary of M_1. We can also proceed similarly for the out-boundary case. Thus, we obtain a permutation cobordism P_2 such that P_2WP_1 is in fact a cobordism, V, from $\mathbf{m}$ to $\mathbf{n}$ that can be decomposed as disjoint union. Note also that the a permutation cobordism is invertible, by the Cylinder Construction, therefore $W = P_2^{-1}VP_1^{-1}$, and the result follows. □

Corollary 3.20. *The monoidal category Cob_2 is generated under composition and disjoint union by morphisms corresponding to the following cobordisms:*



These cobordisms are called, respectively: The cocup cobordism, the pair of pants cobordism, the cylinder cobordism, the copair of pants cobordism, the cocup cobordism and the twist cobordism.

Proof. It follows follows from 3.17. and by the fact that the S_n is generated by transpositions. □

We can give a finite set of relations for the morphisms, under the monoidal strucute, in Cob_2. Such relations will be of great importance to our study, since allow us to relate the monoidal structure of Cob_2 with the algebraic structure of a *Frobenius algebra*.

- **Identity Relation**, we have shown that cylinders are the identity morphisms, this means that if we attach to a cobordism a cylinder we will obtain an equivalent cobordism;
- **Unit and Counit Relations**, it is convenient to express such a relation by the following figure: Here,

FIGURE 1. The unit and counit relations, resp.

  and in the next figures, we consider the in-bounday of the cobordisms on top and the out-boundary on the bottom.
- **Associativity and Coassociativity Relations**, we express such a relation by the following figure:

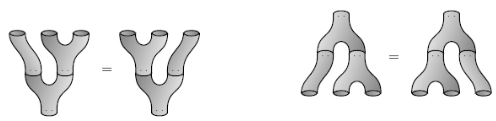


FIGURE 2. The associative and coassociative relations, resp.

- **Commutativity and Cocommutativity Relations**, such relations are expressed by:

FIGURE 3. The commutativity and cocommutativity relations, resp.

- **Frobenius Relation**, the Frobenius relation can be stated as:

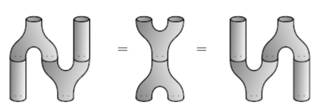


FIGURE 4. The Frobenius relation.

- **Relations for the Twist Cobordism** There are also relations for the twist cobordism. These relations are important to the additional symmetry structure of Cob_2. The reader should consult [1] to find a description of these.

Proposition 3.21. *The above relations are a complete set of relations on the generators of* Cob_2.

The reader can find a proof of this theorem in [1].

4. The Category of Frobenius Algebras

Frobenius algebras will play an importat role in our study, since we can codify the information in a two dimensional TQFT by the algebraic structure of a particular Frobenius algebra. Therefore, in this section we present the main definitions and results concerning Frobenius algebras.

Definition 4.1. *A linear map $\beta : V \otimes W \to k$, is said to be a nondegenerate pairing with respect to V if there exists a linear map $\gamma : k \to W \otimes V$, called a copairing, such that:*

$$V \xrightarrow{\sim} V \otimes k \xrightarrow{id_v \otimes \gamma} V \otimes (W \otimes V) \xrightarrow{\sim} (V \otimes W) \otimes V \xrightarrow{\beta \otimes id_V} k \otimes V \xrightarrow{\sim} V$$

is the identity map. Analogously, we can define nondegenerate pairing with respect to W.

We say that β is a nondegenerate pairing, if it is a pairing with respect to V and a nondegenerate pairing with respect to W, and, in this case, the two associated copairings must be equal, as stated in the following lemma.

Lemma 4.2. *If β is a nondegenerate pairing, then the associated copairings are equal. Also, in this case, β admits a* ***unique*** *associated copairing.*

Theorem 4.3. *The pairing $\beta : V \otimes W \to k$ is nondegenerate if and only if both V, W are finite dimensional and the induced maps $V \to W^*$, $W \to V^*$ are injective.*

It is possible to find a proof of these reults in [1].

Definition 4.4 (**Associative Nondegenerate Pairing**)**.** *Let A be a k-algebra, M a right A-module and N a left A-module. An associative pairing $\beta : M \otimes N \to k$ is such that the following diagram commutes:*

$$\begin{array}{ccccc} & & M \otimes A \otimes N & & \\ & \swarrow^{\alpha \otimes id_N} & & \searrow^{id_M \otimes \alpha} & \\ M \otimes N & & & & M \otimes N \\ & \searrow_{\beta} & & \swarrow_{\beta} & \\ & & k & & \end{array} ,$$

That is, the pairing is associative whenever $x \otimes y \mapsto \langle xa|y\rangle = \langle x|ay\rangle$, for all $x \in M,\ a \in A,\ y \in N$.

Remark 2. A linear functional $\tau : V \to k$ defines an associative nondegenerate pairing $\beta : V \otimes V \to k$, given by $x \otimes y \mapsto \tau(xy)$.

Definition 4.5 (**Frobenius Algebra**)**.** *A Frobenius algebra A is a k-algebra together with a linear functional* $\mathrm{tr} : A \to k$*, such that the corresponding induced associative pairing, $\beta : A \otimes A \to k$, is nondegenerate. The map* tr *is called a Frobenius form or a trace map.*

Example 4.6. The field $\mathbb{C}$ is a Frobenius algebra over $\mathbb{R}$, by considering the Frobenius form

$$\mathbb{C} \to \mathbb{R}$$

$$a + ib \mapsto a.$$

Example 4.7. The ring $\mathrm{Mat}_n(k)$ of all n-by-n matrices over k admits a Frobenius algebra structure, by considering the Frobenius form the usual trace map

$$\mathrm{Mat}_n \to k$$

$$(a_{ij}) \mapsto \sum_i a_{ii}.$$

In fact, the associated pairing is nondegenerate, since the each element of the canonical basis of $\mathrm{Mat}_n(k)$, E_{ij}, with only the entry $e_{ij} = 1$ different than zero, admits E_{ji} as a dual basis element, under such a pairing.

Example 4.8. Let $G = \{g_0, \dots, g_n\}$ be a finite group, written multiplicatively, such that g_0 is the identity of the group and k be a field. The group algebra $k[G]$ has a natural structure of Frobenius algebra, by considering the functional $\tau : k[G] \to k$ given by $g_0 \mapsto 1$ and $g_i \mapsto 0$, for $i \neq 0$. The induced pairing β, which is given by $g \otimes h \mapsto \tau(gh)$, is nondegenerate, since $\beta(g \otimes h) = 1$ if and only if $g = h^{-1}$, and therefore each basis element of $k[G]$ admits a dual basis element under this pairing.

Example 4.9. Given a compact oriented manifold of dimension n, M, then de Rham cohomology $H(X) = \oplus_{i=0}^n H^i(M)$, where $H^i(M)$ is the i-th cohomology group, is a ring under the wedge product. Integration on X provides a linear map $H(X) = \oplus_{i=0}^n H^i(M) \to \mathbb{R}$, and the associated linear map $H(X) \otimes H(X) \to \mathrm{R}$ is nondegenerate, by the Poincaré Duality Theorem.

4.1. **Coalgebra structure.** A Frobenius algebra admits a natural coalgebra structure, whose counit map is the Frobenius form. Given a Frobenius algebra we denote its multiplication map by μ and its unit map by η. Given a fixed Frobenius algebra, A, we note that a linear map $\phi : A^m \to A^n$ can be expressed by a cobordism with m in-boundaries and n out-boundaries. The tensor product of two maps can be written as the disjoint union of the corresponding cobordisms. The composition of two maps can be expressed as the gluing of the corresponding cobordisms. For example, the multipication map $\mu : A \otimes A \to A$, can be expressed as a pair of pants. Therefore, each commutative diagram, expressing an algebraic property of Frobenius algebra, can be expressed as an equality between cobordisms.

Definition 4.10. *The three point function ϕ is defined by $\phi = \beta \circ (\mu \otimes id_A)$.*

Since β is an associative pairing we have $\phi = \beta \circ (\mu \otimes id_A) = \beta \circ (id_A \otimes \mu)$. We are now able to construct a comultiplication map δ, noting that since β is nondegenerate, it admits a copairing $\gamma : k \to A \otimes A$.

Definition 4.11. *We define a comultiplication map, δ, on a Frobenius algebra A to be given by $\delta = (id_A \otimes \phi \otimes id_a) \circ (\gamma \otimes id_A \otimes \gamma)$.*

We can forget about this definition after proving that this comultiplication allow us to define a natural coalgebra structure on A whose counit is the Frobenius form. By a direct calculation we can obtain again the multiplication map μ form δ, noting that the following diagram is commutative:

$$\begin{array}{ccccc} A \otimes A & \xrightarrow{\delta \otimes id_A} & A \otimes A \otimes A & \xleftarrow{id_A \otimes \delta} & A \otimes A \\ & {\scriptstyle \mu} \searrow & \downarrow {\scriptstyle id_A \otimes \beta} & \swarrow {\scriptstyle \mu} & \\ & & A & & \end{array}$$

Lemma 4.12. *The Frobenius form* tr *is a counit for δ, and such a comultiplication δ is associative.*

For a proof of this Lemma, the reader should consult [1]. Therefore, we see that a Frobenius algebra admits a natural coalgebra structure, for the constructed comultiplication δ. Next we make the translation of the Frobenius Relation, in the context of Frobenius algebras.

Lemma 4.13. *A Frobenius algebra with the coalgebra structure defined above is such that the following diagramas are commutative:*

$$\begin{array}{ccc} A \otimes A & \xrightarrow{id_A \otimes \delta} & A \otimes A \otimes A \\ \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle id_A \otimes \mu} \\ A & \xrightarrow{\delta} & A \otimes A \end{array} \qquad \begin{array}{ccc} A \otimes A & \xrightarrow{\delta \otimes id_A} & A \otimes A \otimes A \\ \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle id_A \otimes \mu} \\ A & \xrightarrow{\delta} & A \otimes A \end{array}$$

We call to the relation expressed by these diagramas the Frobenius condition.

Lemma 4.14. *The following relations hold:*

$$\gamma = \delta \circ \mathrm{tr} \qquad (id_A \otimes \mathrm{tr}) \circ \gamma = \eta = (\mathrm{tr} \otimes id_A) \circ \gamma.$$

Proposition 4.15. *Given a Frobenius algebra A, there exists a unique coassociative comultiplication, δ, whose counit if* tr*, satisfying the Frobenius condition.*

The Frobenius condition characterises Frobenius algebras among vector spaces equipped with a unitary multiplication.

Proposition 4.16. *Let A be a vector space with a multiplication $\mu : A \otimes A \to A$, a unit map $\eta : k \to A$, a comultiplication $\delta : k \to A \otimes A$ and a counit $\mathrm{tr} : A \to k$ such that the Frobenius condition holds. Then:*

- *i.* *A is a finite dimensional vector space;*
- *ii.* *The multiplication μ is associative, i.e., A is a finite dimensional k-algebra;*
- *iii.* *The comultiplication is coassociative;*
- *iv.* *The counit* tr *is a Frobenius form, which means that A is a Frobenius algebra.*

For a more detailed discussion of these results, the reader should consult [1].

Definition 4.17. *A Frobenius algebra A is commutative if the following diagram commutes:*

$$\begin{array}{ccc} A \otimes A & \xrightarrow{\ \sigma\ } & A \otimes A \\ & \searrow^{\mu} & \downarrow{\scriptstyle \mu} \\ & & A \end{array}$$

Where $\sigma : A \otimes A \xrightarrow{\sim} A \otimes A$ is the natural isomorphism given on elementary tensors by $\sigma(x \otimes y) = y \otimes x$.

We can also have state the cocommutativity of comultiplication similarly, we just need to reverse the arrows in the above diagram.

Proposition 4.18. *The comultiplication of a Frobenius algebra is cocommutative if and only if the multiplication is commutative.*

The proof of such result can be found in [1].

4.2. **Category of Frobenius algebras.** We define the category FA_k whose objects are k-Frobenius algebras and the morphisms are algebra homomorphisms which are simultaneously coalgebra homomorphisms, (such homomorphisms preserve the Frobenius form).
We note that given two Frobenius algebras A and A', we have that $A \otimes A'$ has a canonical structure of Frobenius algebra. In fact, $A \otimes A'$ is an associative algebra, considering the component-wise multiplication,

$$(A \otimes A') \otimes (A \otimes A') \to A \otimes A'$$

$$(x \otimes x') \otimes (y \otimes y') = xy \otimes x'y'$$

Also the tensor product of coalgebras has again a canonical structure of a coalgebra. Therefore, $A \otimes A'$ has structure of an algebra and of a coalgebra. In fact, the Frobenius condition holds for $A \otimes A'$, and by Proposition 4.15, $A \otimes A'$ has a natural structure of Frobenius algebra.
We denote by cFA_k the full subcategory of all commutative Frobenius algebras. Note that, the tensor product of two commutative Frobenius algebras is again commutative, since the multiplication on the tensor product is just component-wise multiplication.

5. Atiyah's Definition of TQFTs

Atiyah proposed a definition in terms of monoidal categories of a Topological Quantum Field Theory. We state his definition, recalling that we are working in the strict case.

Definition 5.1. *An n-dimensional **Topological Quantum Field Theory** is a symmetric monoidal functor*

$$Z : (\mathrm{Cob}_n, \coprod, \emptyset, \tau) \to (Vect_k, \otimes, k, \sigma).$$

That is, to each closed manifold of dimension $n-1$ we associate a k-vector space, and to each cobordism a linear map. But we have more, since the functor is monoidal, we associate to a disjoint union of $n-1$ manifolds the tensor product of the corresponding vector spaces. This definition encodes much more than it seems at a first look.

Example 5.2. Let $Z : (\mathrm{Cob}_2, \coprod, \emptyset, \tau) \to (Vect_k, \otimes, k, \sigma)$ be a two dimensional TQFT, and let M be a closed 1 manifold. There is a cobordism, which we call the pairing cobordism, $W : M \coprod \bar{M} \to \emptyset$, where $\bar{M}$ is the manifold obtained from M by reversing its orientation, which can be obtained from the cylinder $M \times [0,1]$, by changing the orientation of its out-boundary. This cobordism can also be obtained by gluing a pair of pants and a counit cobordisms, since they define diffeomorphic manifolds. We can also define a copairing cobordism, which is obtained as the pairing cobordism, but we instead change the orientation of the in-boundary of the above cylinder. Therefore, $Z(M \coprod \bar{M}) = V_1 \otimes V_2$, where $Z(M) = V_1$, $Z(\bar{M}) = V_2$, and $Z(W : M \coprod \bar{M} \to \emptyset) = \beta : V_1 \otimes V_2 \to k$. Reasoning similarly, there is also a cobordism $W' : \emptyset \to M \coprod \bar{M}$, and it is mapped by Z to the linear map $\gamma : k \to V_1 \otimes V_2$. Taking into account that $(M \times [0,1] \coprod W')(W \coprod M \times [0,1])$ is equivalent to the cylinder $M \times [0,1]$, which is called the **Snake relation**, we have that the composition

$$V_1 \xrightarrow{id_{V_1} \otimes \gamma} V_1 \otimes V_2 \otimes V_1 \xrightarrow{\beta \otimes id_{V_1}} V_1$$

is the identity map, and the same is true if we interchange V_1 with V_2. This means, that $\beta : V_1 \otimes V_2 \to k$ is a nondegenerate pairing. The existence of such nondegenerate pairing implies that both V_1, V_2 are finite dimensional and $V_1^* \simeq V_2$.

Fact 5.3. *Let Z be a two dimensional TQFT. Then, $Z(S^1)$ is a Frobenius algebra over k.*

Let Z be a two dimensional TQFT. Note that a monoidal functor respects the relations presented for the set of generators of Cob_2. Let $A = Z(S^1)$, (since every closed 1 manifold is diffeomorphic to a finite disjoint union of S^1, we have that $Z(M) \simeq A^{\otimes n}$, for $M \in \mathrm{Cob}_2$), and $W : S^1 \coprod S^1 \to S^1$ be a pair of pants, then $Z(W : S^1 \coprod S^1 \to S^1)$ is a linear map $A \otimes A \to A$, and the *associative relation* shows that this defines an associative multiplication on A, also there is a unit map for this multiplication, given by the linear map obtained by the evaluation $Z(V : \emptyset \to S^1)$. Therefore, A is an algebra in Vect_k, and taking into account the *commutative relation*, A is commutative. There is also a *trace map*, that is a linear map $A \to k$ given by the evaluation of Z on the cobordism $T : S^1 \to \emptyset$. The nondegenerate pairing, mentioned in the above example, is induced by such a trace map. This follows from our previous observation, in example 5.2., that the pairing cobordism can be obtained from a cylinder, with reversing orientation on its out-boundary, or from a gluing from a pair of pants with a counit cobordism.

Thus, the image of S^1, by a TQFT, is a Frobenius algebra. Evaluating Z on a copair of pants we get the comultiplication on A, and in fact such a comultiplication corresponds to the one defined in the previous section for a Frobenius algebra. This can also be seen from the fact, the cobordism that represents the construction of δ is equivalent to a copair of pants.

Example 5.4. TQFTs are also important because they provide invariants for manifolds. Let S be a closed surface and $A = Z(S^1)$, in the previous example we have seen that A admits a structure of Frobenius algebra. We can associate to S an invariant, its genus g.

- If $g = 0$, then S is diffeomorphic to S^2, this means that we can decompose S into two manifolds with boundary S_0 and S_1, diffeomorphic to closed discs, such that their intersection is diffeomorphic to S^1. Therefore $S = S_0 S_1$, and evaluating Z on S we have $Z(S) = Z(S_0 S_1) = Z(S_0) Z(S_1)$, that is $Z(S)$ is obtained by the composition of the unit and trace maps and the identity of k is mapped to $tr(1) \in k$.
- If $g = 1$, then $S \simeq T^2$, and it is possible to decompose S into two manifolds, which can be seen as cobordisms $S_0 : \emptyset \to S^1 \coprod S^1$ and $S_1 : S^1 \coprod S^1 \to \emptyset$, respectively, i.e., the gluing of a pairing cobordism with a copairing cobordism. Therefore, S is mapped by Z to the composition of the nondegenerate pairing with the corresponding copairing. We know that every such pairing can be *identified*, (by a change of basis), to the evaluating pairing

$$A \otimes A^* \xrightarrow{ev} k$$

It is immediate to check, that the copairing associated to the evaluating pairing is given by

$$1_k \mapsto \sum e_i \otimes e_i^*$$

where $\{e_i\}$ is a basis for A, and $\{e_i^*\}$ its dual basis, thus we have that S is mapped by Z to the linear map given by

$$1_k \mapsto \sum e_i \otimes e_i^* \mapsto \dim A$$

Example 5.5. We give an explicit example of a 2-dimensional TQFT. Let G be a finite abelian group and $Z : (\mathrm{Cob}_2, \coprod, \emptyset) \to (\mathbf{Vect}_k, \otimes, k)$ be a two dimensional TQFT, such that it sends S^1 to $k[G]$, the pair of pants to the usual multiplication on $k[G]$, the cup cobordism to the unit map for this multiplication, (i.e., $1_k \mapsto 1_{k[G]}$), and the counit map to the trace map defined no Example 4.8.. We begin by calculate the image of the pairing cobordism, that we denote by β. We have that β is a nondegenerate pairing such that $\beta = \tau \circ \mu$, and therefore, on basis elements:

$$g \otimes h \mapsto gh \mapsto \tau(gh)$$

And so, we conclude that β is 1_k on the elementary tensors of the form $g \otimes g^{-1}$ and 0, otherwise.
We need to determine the associated copairing γ. Since γ is such that

$$k[G] \to k[G] \otimes k \xrightarrow{id_{k[G]} \otimes \gamma} k[G] \otimes k[G] \otimes k[G] \xrightarrow{\beta \otimes id_{k[G]}} k[G]$$

equals the identity on $k[G]$. Which means that we have

$$g \mapsto g \otimes 1_k \mapsto g \otimes \gamma(1_k) \mapsto g$$

If we let γ be given by $1_k \mapsto \sum_{h \in G} h \otimes h^{-1}$, we obtain

$$g \otimes \gamma(1_k) \mapsto g \otimes \sum_{h \in G} h \otimes h^{-1} = \sum_{h \in G} g \otimes h \otimes h^{-1} \mapsto \sum \beta(g \otimes h) \otimes h^{-1} \mapsto (g^{-1})^{-1} = g$$

and, by uniqueness of the associated copairing, we have that γ is as required.
We calculate now the the image of the copair of pants, δ, under Z. Note that, since the copairing cobordism is equivalent to the cobordism obtained by gluing a disjoint union of a pairing cobordism with a cylinder cobordism with a disjoint union of an identity cobordism and a pair of pants, and the functor Z is monoidal, we have that

$$\delta = k[G] \to k[G] \otimes k \xrightarrow{id_{k[G]} \otimes \gamma} k[G] \otimes k[G] \otimes k[G] \xrightarrow{\mu \otimes id_{k[G]}} k[G] \otimes k[G]$$

$$g \to g \otimes 1 \mapsto g \otimes \sum_{h \in G} h \otimes h^{-1} = \sum_{h \in G} g \otimes h \otimes h^{-1} \mapsto \sum_{h \in G} gh \otimes h^{-1}$$

which is exactly the comultiplication given in Example 2.8..
Let S be a surface of genus 1, without boundary. Note that, S is equivalent to a gluing of a pairing cobordism with a copairing cobordism. Hence, $Z(S) = \beta \circ \gamma$, which is given by

$$1_k \mapsto \sum_{h \in G} h \otimes h^{-1} \mapsto \sum_{h \in G} \beta(h \otimes h^{-1}) = |G|$$

as expected, by the above example. If S has genus $g = 2$, we can decompose it has the gluing of a unit cobordism, a copair of pants, a pair of pants, a copair of pants, a pair of pants and a counit cobordism. Therefore, we have that S is mapped to the linear map given by:

$$1_k \mapsto \sum_{h \in G} h \otimes h^{-1} \mapsto |G| \mapsto |G| \sum_{h \in G} h \otimes h^{-1} \mapsto |G||G| = |G|^2$$

Thus, by an easy inductive argument we have that a surface of genus g, without boundary, is mapped to the linear map given by:

$$1_k \mapsto |G|^g$$

5.1. The category of $\mathbf{TQFT}_k$.

We define the category of n-dimensional Topological Quantum Field Theories over k, that we denote by $n\mathrm{TQFT}_k$ to be the category whose objects are n TQFTs over k and whose morphisms are monoidal natural transformations between them.

5.2. **Main Theorem.** We present a correspondence between two dimensional TQFTs and commutative Frobenius algebras.

Proposition 5.6. *There is a bijective correspondence between* 2 *dimensional TQFTs and commutative Frobenius algebras.*

Proof. By simplicity, we consider again the constructed skeleton for Cob_2. Although, it is straightforward to generalize this proof to the general case. Given a symmetric monoidal functor $Z : \mathrm{Cob}_2 \to \mathbf{Vect}_k$, we just need to know what is the image of S^1, under Z, and a linear map for each generator. We already know that $Z(\mathbf{1})$ is a commutative Frobenius algebra. Also, given a commutative Frobenius algebra A we can can construct a TQFT, Z, by letting $Z(\mathbf{1}) = A$, and by specifying the value of Z on the generators of Cob_2, since Cob_2 is a finitely generated as a monoidal category and Z is a monoidal functor. That is,

i. To the cylinder cobordism, we associate the identity on A.
ii. To a pair of pants, we associate the multiplication map, μ, on A.
iii. To a copair os pants, we associate the comultiplication map, δ.
iv. To a unit cobordism, we associate the unit map, η.
v. To a counit cobordism, we associate the counit map, tr.
vi. To the twist cobordism, we associate the symmetry isomorphism $\sigma : A \otimes A \to A \otimes A$.

In fact, Z is well defined since each relation in Cob_2 can be translated by a relation concerning the algebraic structure A. For example, each cobordism presented in the Frobenius relation, in Cob_2, is mapped to the same linear map by Z, since the Frobenius condition holds. Thus Z is well defined. Also these constructions are inverse to each other, since given a monoidal functor Z, such that $Z(\mathbf{1}) = A$, by defining a new monoidal functor by letting $\mathbf{1} \mapsto A$, by our construction, we recover Z. □

Definition 5.7. *Given a k-commutative Frobenius algebra A, we define the functor* $Z_A \in nTQFT_k$ *as the one constructed in the above proof.*

Theorem 5.8. *There is a natural equivalence of categories* $2TQFT_k \simeq cFA$*.*

Proof. We prove our theorem considering the skeleton presented for Cob_2, in which case, our equivalence is an isormorphism of categories.

We must specify two functors $A : TQFT \to \mathrm{cFA}$, the evaluation functor, and $E : \mathrm{cFA} \to TQFT$. the extension functor, and show that these functors define an equivalence of categories. The functors A, E are defined on objects by $Z \mapsto A(\mathbf{1})$ and $A \mapsto Z_A$, respectively. On arrows A is given by $(\eta : Z \to Z') \mapsto (\eta_{\mathbf{1}} : Z(\mathbf{1}) \to Z'(\mathbf{1}))$ and E by $(\varphi : A \to B) \mapsto (\eta_\varphi : Z_A \to Z_B)$, where η_φ is such that $\eta_\varphi(\mathbf{n}) = \varphi^{\otimes n} : A^{\otimes n} \to B^{\otimes n}$. We prove that η_φ is in fact a natural transformation between Z_A and Z_B. To do so, we just need to prove the commutativity of the diagram:

$$\begin{array}{ccc} A^{\otimes n} & \xrightarrow{\varphi^{\otimes n}} & B^{\otimes n} \\ \downarrow{\scriptstyle Z_A(W)} & & \downarrow{\scriptstyle Z_B(W)} \\ A^{\otimes m} & \xrightarrow{\varphi^{\otimes m}} & B^{\otimes m} \end{array}$$

For each generator W of Cob_2 and $n, m \in \mathbb{N}$. But this is true, since φ is a Frobenius algebra homomorphism, the tensor product of commutative Frobenius algebras is again a Frobenius algebra and each generator W corresponds to a structural operation on A.

It remains to show that, in fact, these two functors define an equivalence of categories. We begin to construct a natural isomorphism $\alpha : A \circ E \to id_{\mathrm{cFA}}$. Let $(\alpha_A : A \circ E(A) = A \to A) = id_A$. Note that, by definition, given a homormophism of Frobenius algebras $\varphi : A \to B$, we have $A \circ E(\varphi) = \varphi$. Thus the required diagrams are immediately satisfied, which means that α is in fact a natural isomorphism.

Note that $E \circ A(Z)$ equals Z, since they coincide on the objects of Cob_2 and on the generators, by the construction of E and by the correspondence between TQFTs and commutative Frobenius algebras. So we define a natural transformation $\beta : E \circ A \to id_{2TQFT_k}$ by defining $\beta_Z : E \circ A(Z) \to Z$ to be the identity natural transformation on Z. It remains to show that given a natural transformation $\xi : Z \to Z'$ in TQFT that $E \circ A(\xi) = \xi$. This follows from the observation that $E \circ A(\xi)$ is completely determined by its value on $\mathbf{1}$, (we evaluate ξ at $\mathbf{1}$ and then extend monoidally), because it is a monoidal natural transformation, and both the extension functor E and the evaluation functor A are monoidal. □

References

[1] Kock, J., "Frobenius Algebras and 2D Topological Quantum Field Theories", *London Mathematical Society,* 2004.
[2] Milnor, J., "Lectures on the h-cobordism Theorem", *Princeton University Press*, Princeton, 1965.
[3] Hirsch, M., ´´ Differential Topology", *Graduate Texts in Mathematics*, Springer-Verlag, 1976.
[4] Saunders, M., "Categories for the Working Mathematician", *Graduate Texts in Mathematics,* Springer-Verlag, 1998.