Representation Theory of Symmetric Groups

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2nd semester of 2012/2013

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1 Introduction

Representation theory is very much a 20^{th} century subject. In the 19^{th} century, when groups were dealt with, they were generally understood as subsets, closed under composition and inverse, of the permutations of a set or of the automorphisms $GL(V)$ of a vector space V. The notion of an abstract group was only given in the 20^{th} century, making it possible to make a distinction between the properties of the abstract group and the properties of the particular realization as a subgroup of a permutation group or of $GL(V)$.

What would have been called in the 19^{th} century simply "group theory" is now factored into two parts. First, there is the study of the structure of abstract groups (e.g., the classification of simple groups). Second is the companion question: given a group G, how can we describe all the ways in which G may be embedded in (or just mapped to) linear group $GL(V)$? This, of course, is the subject matter of representation theory.

Given this point of view, it makes sense when first introducing representation theory to do so in a context where the nature of the groups G in question is itself simple, and relatively well understood. In fact, throughout the entire article, only representations of finite groups are mentioned.

In section 2, we start defining what a representation is, then give some basic examples and end by establishing the Schur's lemma and concluding that every representation is a direct sum of irreducible ones.

In section 3, we talk about a very useful tool in representation theory: character theory. Using character theory, we can prove that every finite group has a finite number of irreducible representations. In fact, that number is equal to the number of conjugacy classes in the group.

In section 4, we define what an induced representation is, a tool that will be necessary to prove Frobenius' formula for the characters of irreducible representations of symmetric groups.

In section 5, we give a brief introduction to the concept of group algebra, which will be a key concept in studying the representations of the symmetric groups, since irreducible representations will be identified as minimal left ideals of the group algebra.

In section 6, we finally reach our goal: determining all irreducible representations of the symmetric groups. We even get an explicit bijection between the set of irreducible representations and the set of conjugacy classes of the symmetric group.

Last but not least, in section 7, using symmetric polynomials and induced representations, we prove Frobenius' formula for the characters of irreducible representations of symmetric groups, from which we can get a simpler formula for the dimension of an irreducible representation.

2 Basic notions

Definition 1. *A representation of a finite group G on a finite dimensional complex vector space V is a group homomorphism $\rho : G \to GL(V)$ of G to the group of automorphisms of V.*

The map ρ gives V the structure of a G-module.

When there is little ambiguity about the map ρ, we sometimes call V itself a representation of G and we often suppress the symbol ρ, writing $g \cdot v$ or gv for $\rho(g)(v)$. The dimension of V is also called the degree of ρ.

Definition 2. *The unit or trivial representation of G is the representation $\rho : G \to GL(\mathbb{C})$ such that $\rho(g) = 1$ for every $g \in G$.*

Definition 3. *If G has a subgroup H with index 2, then we can define the alternating representation associated to the pair (G, H) as the representation $\rho : G \to GL(\mathbb{C})$ such that $\rho(g) = 1$ if $g \in H$ and $\rho(g) = -1$ otherwise.*

The particular case of this definition that will sometimes be referred to in this article is the case when $G = S_d$ is the symmetric group and $H = A_d$ is the alternating group.

Definition 4. *If X is any finite set and G acts on the left on X, there is an associated permutation representation. Let V be the vector space with basis $\{e_x : x \in X\}$ and let G act on V by*

$$g \cdot \sum_{x \in X} a_x e_x = \sum_{x \in X} a_x e_{gx}.$$

Definition 5. *The regular representation, denoted R_G or just R, is the permutation representation corresponding to the left action of G on itself.*

Definition 6. *A sub-representation of a representation V is a vector subspace W of V which is invariant under G. In other words, W is a G-submodule of V.*

If V and W are representations, then the direct sum $V \oplus W$ and the tensor product $V \otimes W$ are also representations, the latter via $g(v \otimes w) = gv \otimes gw$.

For a representation V, the n-th tensor power $V^{\otimes n}$ is again a representation of G by this rule, and the exterior powers $\Lambda^n(V)$ and symmetric powers $\mathrm{Sym}^n(V)$ are sub-representations of it. The dual $V^* = \mathrm{Hom}(V, \mathbb{C})$ of V is also a representation, though not in the most obvious way: we want the two representations of G to respect the natural pairing (denoted $\langle\ ,\ \rangle$) between V^* and V, so that if $\rho : G \to GL(V)$ is a representation and $\rho^* : G \to GL(V^*)$ is the dual, we should have

$$\langle \rho^*(g)(v^*), \rho(g)(v) \rangle = \langle v^*, v \rangle$$

for all $g \in G$, $v \in V$ and $v^* \in V^*$. This in turn forces us to define the dual representation by

$$\rho^*(g)(v^*) = v^* \circ \rho(g^{-1}), \text{ for every } g \in G.$$

Having defined the dual of a representation and the tensor product of two representations, then $\mathrm{Hom}(V, W)$ is also a representation, via the identification $\mathrm{Hom}(V, W) = V^* \otimes W$. Unravelling this, if we view an element of $\mathrm{Hom}(V, W)$ as a linear map $\varphi : V \to W$, we have

$$(g\varphi)(v) = g\varphi(g^{-1}v), \text{ for every } g \in G \text{ and } v \in V. \tag{1}$$

In other words, the definition is such that the diagram

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d, "g"'] & W \arrow[d, "g"] \\ V \arrow[r, "g\varphi"'] & W \end{tikzcd}$$

commutes. Note that the dual representation is, in turn, a special case of this: when $W = \mathbb{C}$ is the trivial representation, that is, $gw = w, \forall w \in \mathbb{C}$, this makes V^* into a G-module, with $(g\varphi)(v) = \varphi(g^{-1}v)$.

Definition 7. *A representation V is called irreducible if it has no proper non-zero invariant subspaces.*

Example 8. *We consider the symmetric group S_d. We have two one-dimensional representations: the trivial representation and the alternating representation, which we denote by U and U' respectively. Since S_d is a permutation group, we have a natural permutation representation, in which G acts on $\mathbb{C}^d$ by permuting the coordinates. Explicitly, if $\{e_1, \cdots, e_d\}$ is the standard basis, then $g \cdot e_i = e_{g(i)}$, or, equivalently, $g \cdot (z_1, \cdots, z_d) = \Big(z_{g^{-1}(1)}, \cdots, z_{g^{-1}(d)}\Big)$.*

This representation, like any permutation representation, is not irreducible: the line spanned by the sum $\sum_{i=1}^{d} e_i$ of the basis vectors is invariant, with complement subspace $V = \{(z_1, \cdots, z_d) : z_1 + \cdots + z_d = 0\}$.

This $(d-1)$-dimensional representation V is easily seen to be irreducible. Moreover, we have $\mathbb{C}^d \cong U \oplus V$.

Definition 9. *The representation V defined in the previous example is called the standard representation of S_d.*

Definition 10. *A G-module homomorphism φ between two representations V and W of G is a vector space map $\varphi : V \to W$ such that $\varphi \circ g = g \circ \varphi$ for every $g \in G$.*

$$\begin{array}{ccc} V & \xrightarrow{\varphi} & W \\ g\big\downarrow & & \big\downarrow g \\ V & \xrightarrow[\varphi]{} & W \end{array}$$

We will also call φ a G-linear map, particularly when we want to distinguish it from an arbitrary linear map between the vector spaces V and W. As expected, if φ is bijective, we say that V and W are isomorphic representations.

Proposition 11. *Both the kernel and the image of φ are sub-representations of V and W, respectively.*

We have seen that the representations of G can be built up out of other representations by taking the direct sum. We should focus, then, on representations that are "atomic" with respect to this operation, that is, that cannot be expressed as a direct sum of others. Happily, a representation is atomic in this sense if and only if it is irreducible, and every representation is the direct sum of irreducible ones, in a suitable sense uniquely so.

Proposition 12. *If W is a sub-representation of a representation V of a finite group G, then there is a complementary invariant subspace W' of V, so that $V = W \oplus W'$.*

We present two proofs.

Proof. We introduce an inner product H on V which is preserved by each $g \in G$ the following way: if H_0 is any inner product on V, one gets H by averaging over G:

$$H(v, w) = \sum_{g \in G} H_0(gv, gw)$$

We choose W' to be the orthogonal complement of W with respect to the inner product H. □

Proof. Choose an arbitrary space U complementary to W. Let $\pi_0 : V \to W$ be the projection given by the direct sum decomposition $V = W \oplus U$, and average the map over G:

$$\pi(v) = \sum_{g \in G} g(\pi_0(g^{-1}v))$$

This map is a G-linear map from V onto W, which is multiplication by $|G|$ on W, therefore its kernel is a subspace of V invariant under G and complementary to W. □

Corollary 13. *Any representation is a direct sum of irreducible representations.*

The extent to which the decomposition of an arbitrary representation into a direct sum of irreducible ones is unique is one of the consequences of the following:

Lemma 14 (Schur)**.** *If V and W are irreducible representations of G and $\varphi : V \to W$ is a G-module homomorphism, then:*

- *Either φ is an isomorphism or $\varphi = 0$.*
- *If $V = W$, then $\varphi = \lambda I$ for some $\lambda \in \mathbb{C}$, where I is the identity.*

We can summarize the previous results in:

Proposition 15. *For any representation V on G, there is a decomposition $V = V_1^{\oplus a_1} \oplus \cdots \oplus V_k^{\oplus a_k}$, where the V_i are distinct irreducible representations. The decomposition of V into a direct sum of the k factors is unique, as are the V_i that occur and their multiplicities a_i.*

In both [1] and [2], it is shown that every irreducible representation of an abelian group has degree one. Moreover, one can also find in [2] as interesting method to determine the irreducible representations of the dihedral group. However, there is a remarkably effective tool for understanding the representations of a finite group G, called character theory.

3 Character theory

Definition 16. *If V is a representation of G, its character χ_V is the complex-valued function on the group defined by $\chi_V(g) = \mathrm{Tr}(g_{|V})$, the trace of g on V.*

In particular, we have $\chi_V(hgh^{-1}) = \chi_V(g)$, so that χ_V is constant on the conjugacy classes of G, and such a function is called a class function. Note also that $\chi_V(1) = \dim V$ and $\chi_V(g^{-1}) = \overline{\chi_V(g)}$ for every $g \in G$.

Proposition 17. *Let V and W be representations of G. Then $\chi_{V\oplus W} = \chi_V + \chi_W$, $\chi_{V\otimes W} = \chi_V \cdot \chi_W$, $\chi_{V^*} = \overline{\chi_V}$ and $\chi_{\Lambda^2 V}(g) = \frac{\chi_V(g)^2 - \chi_V(g^2)}{2}$.*

Proposition 18 (The original fixed-point formula)**.** *If V is the permutation representation associated to the action of a group G on a finite set X, then, for every $g \in G$, $\chi_V(g)$ is the number of elements of X fixed by g.*

As we said before, the character of a representation of a group G is a function on the set of conjugacy classes in G. This suggests expressing the basic information about the irreducible representations of a group G in the form of a character table. This is a table with the conjugacy classes $[g]$ of G listed across the top, usually given by a representative g, with the number of elements in each conjugacy class over it, the irreducible representations V of G listed on the left and, in the appropriate box, the value of the character χ_V on the conjugacy class $[g]$.

Example 19. *The symmetric group S_3 has three irreducible representations: the trivial representation, the alternating representation and the standard representation. There are no more irreducible representations since S_3 has three conjugacy classes (we will see later that the number of irreducible representations of a finite group is always equal to the number of conjugacy classes). As one can see in [1, Section 1.3], that can also be proved using only what we learned in section 2, and the character table of S_3 is:*

	1	3	2
S_3	1	(12)	(123)
trivial U	1	1	1
alternating U'	1	-1	1
standard V	2	0	-1

Now we start by giving an explicit formula for the projection of a representation onto the direct sum of the trivial factors in it. This formula has tremendous consequences.

Definition 20. *For any representation V of a group G, we set*

$$V^G := \{v \in V : gv = v, \forall g \in G\}.$$

We ask for a way of finding V^G explicitly. We observed before that for any representation V of G and any $g \in G$, the endomorphism $g : V \to V$ is, in general, not a G-module homomorphism. On the other hand, if we take the average of all these endomorphisms, that is, we set

$$\varphi = \frac{1}{|G|} \sum_{g \in G} g \in \operatorname{End}(V),$$

then the endomorphism φ will be G-linear. In fact:

Proposition 21. *The map φ is a projection of V onto V^G.*

We thus have a way of finding explicitly the direct sum of the trivial sub-representations of a given representation, although the formula can be hard to use if it does not simplify. If we just want to know the number m of copies of the trivial representation appearing in the decomposition of V, we can do this numerically, since this number will just be the trace of the projection φ. We have

$$m = \dim V^G = \operatorname{Tr}(\varphi) = \frac{1}{|G|} \sum_{g \in G} \operatorname{Tr}(g) = \frac{1}{|G|} \sum_{g \in G} \chi_V(g). \tag{2}$$

In particular, we observe that for an irreducible representation other than the trivial one, the sum over all $g \in G$ of the values of the character is zero.

We can do much more with this idea. If V and W are representations of G, then $\operatorname{Hom}(V, W)$ is a representation of G and $\operatorname{Hom}(V, W)^G$ is the set of G-module homomorphisms between V and W. If V is irreducible then, by Schur's lemma, $\dim \operatorname{Hom}(V, W)^G$ is the multiplicity of V in W. Similarly, if W is irreducible, $\dim \operatorname{Hom}(V, W)^G$ is the multiplicity of W in V, and in the case where both V and W are irreducible, we have

$$\dim \operatorname{Hom}(V, W)^G = \begin{cases} 1, & \text{if} \quad V \cong W \\ 0, & \text{if} \quad V \ncong W \end{cases}$$

But now the character $\chi_{\operatorname{Hom}(V,W)}$ of the representation $\operatorname{Hom}(V, W) = V^* \otimes W$ is given by $\chi_{\operatorname{Hom}(V,W)}(g) = \overline{\chi_V(g)} \cdot \chi_W(g)$.

We can now apply formula (2) in this case to obtain the striking

$$\frac{1}{|G|} \sum_{g \in G} \overline{\chi_V(g)} \cdot \chi_W(g) = \begin{cases} 1, & \text{if} \quad V \cong W \\ 0, & \text{if} \quad V \ncong W \end{cases} \tag{3}$$

To express this, let $\mathbb{C}_{\text{class}}(G)$ be the set of class functions on G an define an Hermitian inner product on $\mathbb{C}_{\text{class}}(G)$ by

$$\langle \alpha, \beta \rangle = \frac{1}{|G|} \sum_{g \in G} \overline{\alpha(g)} \cdot \beta(g) \tag{4}$$

Formula (3) then amounts to:

Theorem 22. *The characters of the irreducible representations are orthogonal with respect to the inner product* (4)*.*

Example 23. *The orthonormality of the three irreducible representations of S_3 can be read from its character table. The numbers over each conjugacy class indicate how many times to count entries in that column.*

Corollary 24. *The number of irreducible representations of G is less than or equal to the number of conjugacy classes.*

We will soon show that there are no nonzero class functions orthogonal to the characters, so equality holds in the previous corollary.

Corollary 25. *Any representation is determined by its character.*

Corollary 26. *A representation V is irreducible if and only if $\langle \chi_V, \chi_V \rangle = 1$.*

Corollary 27. *The multiplicity a_i of V_i in V is the inner product of χ_V with χ_{V_i}, that is, $a_i = \langle \chi_V, \chi_{V_i} \rangle = 1$.*

We obtain some further corollaries by applying all of this to the regular representation R of G. By proposition 18 we know the character of R, it is simply

$$\chi_R(g) = \begin{cases} 0 & \text{if} \quad g \neq e \\ |G| & \text{if} \quad g = e \end{cases}.$$

Corollary 28. *Any irreducible representation V of G appears in the regular representation* $\dim V$ *times.*

In particular, this proves again that there are only finitely many irreducible representations. As a numerical consequence of this we have the formula

$$|G| = \dim R = \sum_i (\dim V_i)^2. \tag{5}$$

Also, applying this to the value of the character of the regular representation on an element $g \in G$ other than the identity, we have

$$\sum_i (\dim V_i)\chi_{V_i}(g) = 0.$$

These two formulas amount to the Fourier inversion for finite groups. For example, if all but one of the characters is known, they give a formula for the unknown character.

Now we give a more general formula for the projection of a general representation V onto the direct sum of the factors in V isomorphic to a given irreducible representation W. The main idea for this is a generalization of the "averaging" of endomorphisms $g : V \to V$ used in the beginning of this section, the point being that instead of simply averaging all the g we can ask the question: what linear combinations of the endomorphisms $g : V \to V$ are G-linear endomorphisms? The answer is given by:

Proposition 29. *Let $\alpha : G \to \mathbb{C}$ be any function on the group G, and for any representation V of G set*

$$\varphi_{\alpha,V} = \sum_{g \in G} \alpha(g) \cdot g : V \to V.$$

Then $\varphi_{\alpha,V}$ is a homomorphism of G-modules for all V if and only if α is a class function.

As an immediate consequence of this proposition, we have:

Proposition 30. *The number of irreducible representations of G is equal to the number of conjugacy classes of G. Equivalently, their characters $\{\chi_V\}$ form an orthonormal basis for $\mathbb{C}_{\text{class}}(G)$.*

Proof. Suppose $\alpha : G \to \mathbb{C}$ is a class function and $\langle \alpha, \chi_V \rangle = 0$ for every irreducible representation V. We must show that $\alpha = 0$. Consider the endomorphism

$$\varphi_{\alpha,V} = \sum_{g \in G} \alpha(g) \cdot g : V \to V$$

as defined above. By Schur's lemma, $\varphi_{\alpha,V} = \lambda \cdot \mathrm{Id}$; and if $n = \dim V$, then

$$\lambda = \frac{1}{n} \cdot \mathrm{Tr}(\varphi_{\alpha,V}) = \frac{1}{n} \sum_{g \in G} \alpha(g) \chi_V(g) = \frac{|G|}{n} \overline{\langle \alpha, \chi_{V^*} \rangle} = 0.$$

Thus, $\varphi_{\alpha,V} = 0$, or $\sum_{g \in G} \alpha(g) \cdot g = 0$ on any representation V of G, whether it is irreducible or not. In particular, this will be true for the regular representation $V = R$. But in R the elements $\{g \in G\}$, thought as elements of $\mathrm{End}(R)$, are linearly independent. In particular, the elements $\{g(e)\}$ are all independent. Thus $\alpha(g) = 0$ for all g, as required. $\square$

Both [1] and [2] study some particular cases as the symmetric group S_d and the alternating group A_d, with $d \leq 5$, using character theory. Despite the fact that studying this cases is important for whoever wants to study representation theory (and in particular put character theory in practice), they do not appear in this article since it is mainly focused on determining the irreducible representations of a general symmetric group.

4 Induced representations

Definition 31. *If $H \subseteq G$ is a subgroup, any representation V of G restricts to a representation of H, denoted $\operatorname{Res}_H^G V$ (or simply $\operatorname{Res} V$ when there is no ambiguity).*

In this section, we describe an important construction which produces representations of G from representations of H. Suppose V is a representation of G, and $W \subseteq V$ is a subspace which is H-invariant. For any $g \in G$, the subspace $g \cdot W$ depends only on the left coset gH of g modulo H. For a coset σ in G/H, we write $\sigma \cdot W$ for this subspace of V.

Definition 32. *Given two representations V and W of G and H respectively, where H is a subgroup of G, we say that V is induced by W if the following equality holds:*

$$V = \bigoplus_{\sigma \in G/H} \sigma \cdot W$$

In this case we write $V = \operatorname{Ind}_H^G W$ or simply $V = \operatorname{Ind} W$.

Example 33. *The permutation representation associated to the left action of G on G/H is induced from the trivial one dimensional representation W of H. Here V has basis $\{e_\sigma : \sigma \in G/H\}$ and $W = \mathbb{C} \cdot e_H$, with H being the trivial coset.*

Example 34. *The regular representation of G is induced from the regular representation of H. Here V has basis $\{e_g : g \in G\}$, whereas W has basis $\{e_H : h \in H\}$.*

We claim that, given a representation W of H, such V exists and is unique up to isomorphism. It is not hard to do this by hand. Choose a representative $g_\sigma \in G$ for each coset $\sigma \in G/H$, with e representing the trivial coset H. To see the uniqueness, note that each element of V has a unique expression $v = \sum g_\sigma w_\sigma$ for elements $w_\sigma \in W$. Given $g \in G$, write $g \cdot g_\sigma = g_\tau \cdot h$ for some $\tau \in G/H$ and $h \in H$. Then we must have $g \cdot (g_\sigma w_\sigma) = (g \cdot g_\sigma) w_\sigma = (g_\tau \cdot h) w_\sigma = g_\tau \cdot (h w_\sigma)$.

This proves the uniqueness and tells us how to construct $V = \operatorname{Ind}(W)$ from W. Take a copy W^σ of W for each left coset $\sigma \in G/H$. For $w \in W$, let $g_\sigma w$ denote the element in W^σ corresponding to w in W. Let $V = \bigoplus_{\sigma \in G/H} W^\sigma$, so every element of V has a unique expression $v = \sum g_\sigma w_\sigma$ for elements w_σ in W. Given $g \in G$, define $g \cdot (g_\sigma w_\sigma) = g_\tau (h w_\sigma)$ if $g \cdot g_\sigma = g_\tau \cdot h$. To show that this defines an action of G on V, we must verify that $g' \cdot (g \cdot (g_\sigma w_\sigma)) = (g' \cdot g) \cdot (g_\sigma w_\sigma)$, for another element g' in G. Now if $g' \cdot g_\tau = g_\rho \cdot h'$, then $g' \cdot (g \cdot (g_\sigma w_\sigma)) = g' \cdot (g_\tau (h w_\sigma)) = g_\rho (h'(h w_\sigma))$. Since $(g' \cdot g) \cdot g_\sigma = g' \cdot (g \cdot g_\sigma) = g' \cdot g_\tau \cdot h = g_\rho \cdot h' \cdot h$, we have $(g' \cdot g) \cdot (g_\sigma w_\sigma) = g_\rho((h' \cdot h) w_\sigma) = g_\rho(h' \cdot (h w_\sigma))$, as required.

Example 35. *If* $W = \bigoplus W_i$, *then* $\operatorname{Ind} W = \bigoplus \operatorname{Ind} W_i$.

The existence of induced representations also follows from the two previous examples since any representation W is a direct sum of summands of the regular representation.

To compute the character of $V = \operatorname{Ind} W$, note that $g \in G$ maps σW to $g\sigma W$, so the trace is calculated from those cosets σ with $g\sigma = \sigma$, that is, such that $s^{-1}gs \in H$ for $s \in \sigma$. Therefore,

$$\chi_{\operatorname{Ind} W}(g) = \sum_{g\sigma=\sigma} \chi_W(s^{-1}gs) \quad (s \in \sigma \text{ arbitrary}).$$

Proposition 36. *If C is a conjugacy class of G, and $C \cap H$ decomposes into conjugacy classes $D_1, \cdots, D_r$ of H, the value of the character of* $\operatorname{Ind} W$ *on C is*

$$\chi_{\operatorname{Ind} W}(C) = \frac{|G|}{|H|} \sum_{i=1}^{r} \frac{|D_i|}{|C|} \chi_W(D_i).$$

Proof. For each $1 \le i \le r$,

$$\sum_{g\sigma=\sigma, s^{-1}gs \in D_i} \chi_W(s^{-1}gs) = \chi_W(D_i) \cdot \left| \left\{ \sigma : g\sigma = \sigma, s^{-1}gs \in D_i \right\} \right|.$$

Therefore it is enough to prove that

$$\left| \left\{ \sigma : s^{-1}gs \in D_i \right\} \right| = \frac{|G| \cdot |D_i|}{|H| \cdot |C|}.$$

Given $g \in C$, define $\phi : G \to C$ such that $\phi(s) = s^{-1}gs, \forall s \in G$. We start by proving that every element of C has the same number of pre-images. Let $g' \in C$. Then there exists $s_0 \in G$ such that $s_0^{-1}gs_0 = g'$. It is trivial to verify that right multiplication by s_0 is a bijection between $\phi^{-1}(g)$ and $\phi^{-1}(g')$.

Since every element of C has the same number of pre-images, then every element of C has precisely $|G| \,/\, |C|$ pre-images and therefore $D_i \subseteq C$ has $|G| \cdot |D_i| \,/\, |C|$ pre-images and thus

$$\left| \left\{ \sigma : s^{-1}gs \in D_i \right\} \right| = \frac{|\phi^{-1}(D_i)|}{|H|} = \frac{|G| \cdot |D_i|}{|H| \cdot |C|}.$$

□

Corollary 37. *If W is the trivial representation of H, then*

$$\chi_{\operatorname{Ind} W}(C) = \frac{[G:H]}{|C|} |C \cap H| .$$

5 The group algebra

There is an important notion that we have already dealt with implicitly but not explicitly, the group algebra $\mathbb{C}G$ associated to a finite group G. This is an object that for all intents and purposes can completely replace the group G itself, for any statement about the representations of G has an exact equivalent statement about the group algebra. Indeed, to a large extent the choice of language is a matter of taste.

The underlying vector space of the group algebra of G is the vector space with basis $\{e_g\}$ corresponding to the elements of the group G, that is, the underlying vector space of the regular representation. We define the algebra structure on this vector space simply by $e_g \cdot e_h = e_{gh}$.

By a representation of the algebra $\mathbb{C}G$ on a vector space V we mean simply an algebra homomorphism between $\mathbb{C}G$ and $\mathrm{End}(V)$, so that a representation V of $\mathbb{C}G$ is the same thing as a left $\mathbb{C}G$-module. In particular, the left $\mathbb{C}G$-module given by $\mathbb{C}G$ itself corresponds to the regular representation and any irreducible representation is isomorphic to a minimal left ideal in $\mathbb{C}G$.

Proposition 38. *Let $I \neq 0$ be a left ideal of a ring R. If I is a direct summand of R, then $I^2 \neq 0$.*

Proof. Since I is a direct summand of R, there exists a left ideal J of R such that $I \oplus J = R$. In particular, there exist $i \in I$ and $j \in J$ such that $1 = i + j$. Multiplying both sides by i on the left, one gets $i = i^2 + ij$. Now $I^2 \neq 0$, for otherwise $i = ij \in I \cap J = 0$, $1 = j \in J$, $J = R$ and hence $I = 0$. □

This proposition will be used to determine all irreducible representations in the next section. We include the following result even though we will not need it later as it puts into a general context the result we will prove in the next section.

Proposition 39. *If I is a minimal left ideal of a ring R, then either $I^2 = 0$ or I is generated by some non-zero idempotent.*

Proof. Suppose $I^2 \neq 0$. Then there exists $a \in I$ such that $Ia \neq 0$, and since Ia is a left ideal contained in I, $Ia = I$. In particular, there exists $e \in I$ such that $a = ea$. Let $J = \{x \in I : xa = 0\}$. Then J is a left ideal contained in I, and since $e \notin J$, $J = 0$. On other hand, $e^2 - e \in J$, therefore $e^2 = e$. Now Re is a left ideal contained in I, and since $e \neq 0$, one gets $Re \neq 0$, so $I = Re$. □

In the case when R is the group algebra of the symmetric group, we will find such idempotents explicitly in the next section.

6 Irreducible representations of S_d

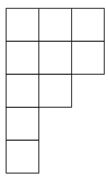
We finally arrive at what we wanted to study: the irreducible representations of S_d. We learned that a group has as many irreducible representations as conjugacy classes, so our first question must be: what are the conjugacy classes of S_d? And we get a nice result:

Theorem 40. *In any symmetric group S_d, the conjugacy classes correspond naturally to the partitions of d, that is, expressions of d as a sum of positive integers $a_1, \cdots, a_k$, where the correspondence associates to such a partition the conjugacy class of a permutation consisting on disjoint cycles of length $a_1, \cdots, a_k$.*

The number of irreducible representations of S_d is the number of conjugacy classes, which is the number of partitions of d, that is, the number of ways to write $d = \lambda_1 + \cdots + \lambda_k$, with $\lambda_1 \geq \cdots \geq \lambda_k \geq 1$.

To a partition $\lambda = (\lambda_1, \cdots, \lambda_k)$ is associated a Young diagram (sometimes called a Young frame or Ferrers diagram) with k rows lined up on the left and λ_i boxes in the i-th row. The *conjugate partition* $\lambda' = (\lambda'_1, \cdots, \lambda'_r)$ to the partition λ is defined by interchanging rows and columns in the Young diagram, that is, reflecting the diagram on the diagonal.

For example, the diagram below is that of the partition $(3, 3, 2, 1, 1)$ whose conjugate is that of the partition $(5, 3, 2)$.



Young diagrams can be used to describe projection operators for the regular representation, which will then give the irreducible representations of S_d. For a given Young diagram, number the boxes consecutively as shown:

$$\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & 5 & \\ \hline 6 & 7 & \\ \hline 8 & & \\ \hline \end{array}$$

More generally, define a tableau on a given Young diagram to be a numbering of the boxes by the integers $1, \cdots, d$. Given a tableau, say the canonical one shown, define two subgroups of the symmetric group $P = P_\lambda = \{g \in S_d : g \text{ preserves each row}\}$ and $Q = Q_\lambda = \{g \in S_d : g \text{ preserves each column}\}$.

In the group algebra $\mathbb{C}S_d$, we introduce two elements corresponding to these subgroups: we set

$$a_\lambda = \sum_{g \in P} e_g \quad \text{and} \quad b_\lambda = \sum_{g \in Q} \operatorname{sgn}(g) \cdot e_g.$$

To see what a_λ and b_λ do, observe that if V is any vector space and S_d acts on the d-th tensor power $V^{\otimes d}$ by permuting the factors, the image of the element $a_\lambda \in \mathbb{C}S_d$ in $\operatorname{End}(V^{\otimes d})$ is just the subspace $\operatorname{Im}(a_\lambda) = \operatorname{Sym}^{\lambda_1} V \otimes \cdots \otimes \operatorname{Sym}^{\lambda_k} \subseteq V^{\otimes d}$, where the inclusion on the right is obtained by grouping the factors of $V^{\otimes d}$ according to the rows of the Young tableau. Similarly, the image of b_λ in this tensor power is $\operatorname{Im}(b_\lambda) = \Lambda^{\mu_1} V \otimes \cdots \otimes \Lambda^{\mu_l} \subseteq V^{\otimes d}$, where μ is the conjugate partition to λ.

Finally, we set $c_\lambda = a_\lambda \cdot b_\lambda \in \mathbb{C}S_d$; this is called a Young symmetrizer. For example, when $\lambda = (d)$, $c_{(d)} = a_{(d)} = \sum_{g \in S_d} e_g$, and the image of $c_{(d)}$ on $V^{\otimes d}$ is $\operatorname{Sym}^d V$. When $\lambda = (1, \cdots, 1)$, $c_{(1,\cdots,1)} = b_{(1,\cdots,1)} = \sum_{g \in S_d} \operatorname{sgn}(g) e_g$, and the image of $c_{(1,\cdots,1)}$ on $V^{\otimes d}$ is $\Lambda^d V$.

Theorem 41. *Some scalar multiple of c_λ is idempotent, that is, $c_\lambda^2 = n_\lambda c_\lambda$, and the image of c_λ (by right multiplication on $\mathbb{C}S_d$) is an irreducible representation V_λ of S_d. Moreover, $n_\lambda = d!/\dim V_\lambda$ and every irreducible representation of S_d can be obtained in this way from a unique partition.*

Example 42. *For any positive integer d, the trivial representation corresponds to the partition $d = d$ while the alternating representation corresponds to the partition $d = 1 + \cdots + 1$. The standard representation V corresponds to the partition $d = (d-1) + 1$. Moreover, as one can see in [1, Section 3.2], each exterior power $\Lambda^k V$ is irreducible for $0 \leq k \leq d-1$, and as one can see in [1, Section 4.2, exercise 4.6], $\Lambda^k V$ corresponds to the partition $d = (d-k) + 1 + \cdots + 1$.*

We are not very far from proving this theorem. Let $A = \mathbb{C}S_d$ be the group ring of S_d. For a partition λ of d, let P and Q be the corresponding subgroups preserving the rows and columns of a Young tableau T corresponding to λ. Let $a = a_\lambda$, $b = b_\lambda$ and let $c = c_\lambda = ab$ be the corresponding Young symmetrizer, so $V_\lambda = Ac_\lambda$ is the corresponding representation.

Note that $P \cap Q = \{1\}$, so an element of S_d can be written in at most one way as a product $p \cdot q$ for $p \in P$ and $q \in Q$. Thus, c is the sum $\sum \pm e_g$ over all g that can be written as $p \cdot q$, with coefficient ± 1 being $\operatorname{sgn}(q)$. In particular, the coefficient of e_1 in c is 1.

Lemma 43. *1. For $p \in P$, $p \cdot a = a \cdot p = a$.*

2. For $q \in Q$, $(\mathrm{sgn}(q)q) \cdot b = b \cdot (\mathrm{sgn}(q)q) = b$.

3. For all $p \in P$ and $q \in Q$, $p \cdot c \cdot (\mathrm{sgn}(q)q) = c$, and, up to multiplication by a scalar, c is the only such element in A.

Proof. Only the second part of last assertion is not obvious. If $\sum n_g e_g$ satisfies 3, then $n_{pgq} = \mathrm{sgn}(q) n_g$ for all $g \in G$, $p \in P$ and $q \in Q$. In particular, $n_{pq} = \mathrm{sgn}(q) n_1$. Thus, it suffices to verify that $n_g = 0$ if $g \notin PQ$. For such g it suffices to find a transposition t such that $p = t \in P$ and $q = g^{-1} t g \in Q$, for then $g = pgq$, so $n_g = -n_g$. If $T' = gT$ is the tableau obtained by replacing each entry i of T by $g(i)$, the claim is that there are two distinct integers that appear in the same row of T and in the same column of T': t is then the transposition of these two integers. We must verify that if there were no such pair of integers, then one could write $g = p \cdot q$ for some $p \in P$ and $q \in Q$. To do this, first take $p_1 \in P$ and $q_1' \in Q' = gQg^{-1}$ so that $p_1 T$ and $q_1' T'$ have the same first row. Repeating on the rest of the tableau, one gets $p \in P$ and $q' \in Q'$ so that $pT = q'T'$. Then $pT = q'gT$, so $p = q'g$ and therefore $g = pq$, where $q = g^{-1} q'^{-1} g \in Q$, as required. □

We order partitions lexicographically: $\lambda > \mu$ if the first non-vanishing $\lambda_i - \mu_i$ is positive.

Lemma 44. *1. If $\lambda > \mu$, then for all $x \in A$, $a_\lambda \cdot x \cdot b_\mu = 0$. In particular, if $\lambda > \mu$, then $c_\lambda \cdot c_\mu = 0$.*

2. For all $x \in A$, $c_\lambda \cdot x \cdot c_\lambda$ is a scalar multiple of c_λ. In particular, $c_\lambda \cdot c_\lambda = n_\lambda c_\lambda$ for some $n_\lambda \in \mathbb{C}$.

Proof. For 1, we may take $x = g \in S_d$. Since $g \cdot b_\mu \cdot g^{-1}$ is the element constructed from gT', where T' is the tableau used to construct b_μ, it suffices to show that $a_\lambda \cdot b_\mu = 0$. One verifies that $\lambda > \mu$ implies that there are two integers in the same row of T and the same column of T'. If t is the transposition of these integers, then $a_\lambda = a_\lambda \cdot t$, $t \cdot b_\mu = -b_\mu$, so $a_\lambda \cdot b_\mu = a_\lambda \cdot t \cdot t \cdot b_\mu = -a_\lambda \cdot b_\mu$, as required. Part 2 follows from part 3 of the previous lemma. □

Corollary 45. *If $\lambda < \mu$, then $c_\lambda \cdot A \cdot c_\mu = 0$ still holds.*

Proof. Use the anti-automorphism * of A induced by the map $g \mapsto g^{-1}$ for $g \in S_d$, noting that $a_\lambda, b_\lambda, a_\mu, b_\mu$ are its fixed points, so $(c_\lambda \cdot x \cdot c_\mu)^* = (a_\lambda \cdot b_\lambda \cdot x \cdot a_\mu \cdot b_\mu)^* = b_\mu^* a_\mu^* x^* b_\lambda^* a_\lambda^* = b_\mu a_\mu x^* b_\lambda a_\lambda = 0$ since $a_\mu x^* b_\lambda = 0$. □

Lemma 46. *1. Each V_λ is an irreducible representation of S_d.*

2. If $\lambda \neq \mu$, then V_λ and V_μ are not isomorphic.

Proof. For 1 note that $c_\lambda V_\lambda \subseteq \mathbb{C}c_\lambda$ by the previous lemma. If $W \subseteq V_\lambda$ is a sub-representation, then $c_\lambda W$ is either $\mathbb{C}c_\lambda$ or 0. If $c_\lambda W = \mathbb{C}c_\lambda$, then $c_\lambda \in c_\lambda W \subseteq W$, so $V_\lambda = A \cdot c_\lambda \subseteq W$. If $c_\lambda W = 0$, then $W \cdot W \subseteq A \cdot c_\lambda W = 0$, and $W = 0$ thanks to proposition 38. In particular, $c_\lambda V_\lambda \neq 0$.

For 2, we may assume $\lambda > \mu$. Then $c_\lambda V_\lambda \neq 0$ but $c_\lambda V_\mu = c_\lambda \cdot Ac_\mu = 0$, so V_λ and V_μ are not isomorphic A-modules. □

Proposition 47. *For every partition λ, $c_\lambda^2 \neq 0$.*

Proof. Let F be right multiplication by c_λ on A. The coefficient of e_g in $e_g \cdot c_\lambda$ is 1, so the trace of F is $|S_d| = d!$. On other hand, let $J = \{x \in V_\lambda : xc_\lambda = 0\}$. If $c_\lambda^2 = 0$, then $J \neq 0$ is a sub-representation of V_λ and so $J = V_\lambda$. Since $\mathrm{Im}(F) \subseteq V_\lambda$, the trace of F must be 0, a contradiction. This also shows that $J = 0$. □

Lemma 48. *For every partition λ, $c_\lambda \cdot c_\lambda = n_\lambda c_\lambda$, with $n_\lambda = d!/\dim V_\lambda$.*

Proof. Let F be as in the previous proof. As we saw there, $V_\lambda \cap \ker(F) = 0$. Moreover, every other irreducible representation V_μ is contained in $\ker(F)$, so V and $\ker(F)$ are complementary subspaces of A. Since F is multiplication by n_λ on V_λ, its trace must be $n_\lambda \dim V_\lambda$. As seen in the previous proof, the trace of F is also equal to $d!$, so $n_\lambda = d!/\dim V_\lambda$. □

We finally reached our goal: determining every irreducible representation of S_d. But we will not end with this, since we would also like to know how to compute the character of any irreducible representation of S_d.

7 Frobenius' formula

We end by discussing Frobenius' formula for the character χ_λ of V_λ, which in particular gives a formula for the dimension of V_λ.

Introduce independent variables $x_1, \cdots, x_k$, with k at least as large as the number of rows in the Young diagram of λ. Define the power sums $P_j(x)$, $1 \leq j \leq d$, and the discriminant $\Delta(x)$ by $P_j(x) = x_1^j + \cdots + x_k^j$ and $\Delta(x) = \prod_{i<j}(x_i - x_j)$, respectively.

Definition 49. *If $f(x) = f(x_1, \cdots, x_k)$ is a formal power series, and $(l_1, \cdots, l_k)$ is a k-tuple of non negative integers, then $[f(x)]_{(l_1,\cdots,l_k)}$ is the coefficient of $x_1^{l_1} \cdots x_k^{l_k}$ in f.*

Definition 50. *Given a partition λ of d, where $\lambda_1 \geq \cdots \geq \lambda_k \geq 0$, set $l_i = \lambda_i + k - i$.*

Note that from the decreasing sequence (λ_i) we obtain a strictly decreasing sequence (l_i).

Definition 51. *For a d-tuple $\mathbf{i} = (i_1, \cdots, i_d)$ of non-negative integers with $\sum \alpha i_\alpha = d$, $C_\mathbf{i} \subseteq S_d$ is the conjugacy class consisting of elements made up of i_1 1-cycles, ..., i_d d-cycles.*

Theorem 52 (Frobenius' formula)**.** *Given a d-tuple $\mathbf{i} = (i_1, \cdots, i_d)$ as in the previous definition and a partition λ of d, the character of V_λ at $C_\mathbf{i}$ is given by*

$$\chi_\lambda(C_\mathbf{i}) = \left[\Delta(x) \cdot \prod_{j=1}^{d} P_j(x)^{i_j}\right]_{(l_1,\cdots,l_k)}$$

Example 53. *If $\lambda = (d-1, 1)$, that is, $V = V_\lambda$ is the standard representation of S_d, then $l = (d, 1)$ and*

$$\chi_\lambda(C_\mathbf{i}) = \left[(x_1 - x_2) \prod_{j=1}^{d} (x_1^j + x_2^j)^{i_j}\right]_{(d-1,1)} = i_1 - 1$$

One can arrive at the same result using only character theory: consider the permutation representation $\mathbb{C}^d$ of S_d and take into account that $\mathbb{C}^d \cong U \oplus V$ and that $\chi_{\mathbb{C}^d}(C_i)$ is the number of fixed points of any permutation in C_i, that is, $\chi_{\mathbb{C}^d}(C_i) = i_1$.

In order to prove Frobenius' formula, we will need to use induced representations and some basic facts about symmetric polynomials.

Definition 54. *Given a partition λ of d, the subgroup S_λ, often called a Young subgroup, is $S_\lambda = S_{\lambda_1} \times \cdots \times S_{\lambda_k} \subseteq S_d$.*

Definition 55. *Given a partition λ of d, U_λ is the representation of S_d induced from the trivial representation of S_λ. Equivalently, $U_\lambda = A \cdot a_\lambda$.*

Let ψ_λ be the character of U_λ. The main key to prove Frobenius' formula is the relation between the representations U_λ and V_λ, that is, between the characters ψ_λ and χ_λ. Start by noting that V_λ is a summand of U_λ, since right multiplication by b_λ is a surjection between $U_\lambda = Aa_\lambda$ and $V_\lambda = Aa_\lambda b_\lambda$.

Example 56. *We have $U_{(d-1,1)} \cong V_{(d-1,1)} \oplus V_{(d)}$, expressing the fact that the permutation representation $\mathbb{C}^d$ of S_d is the sum of the standard representation and the trivial representation.*

As we will find out, every U_λ contains V_λ with multiplicity one, and contains only other V_μ for $\mu > \lambda$.

It is not hard to see that the number of elements in $C_\mathbf{i}$ is given by

$$|C_\mathbf{i}| = \frac{d!}{1^{i_1} i_1! \cdot \ldots \cdot d^{i_d} i_d!}. \tag{6}$$

Recall that U_λ is an induced representation. By corollary 37,

$$\begin{aligned}\psi_\lambda(C_\mathbf{i}) &= \frac{1}{|C_\mathbf{i}|}\,[S_d : S_\lambda] \cdot |C_\mathbf{i} \cap S_\lambda| = \\ &= \frac{1^{i_1} i_1! \cdot \cdots \cdot d^{i_d} i_d!}{d!} \cdot \frac{d!}{\lambda_1! \cdot \ldots \cdot \lambda_k!} \cdot \sum \prod_{p=1}^{k} \frac{\lambda_p!}{1^{r_{p1}} r_{p1}! \cdot \ldots \cdot d^{r_{pd}} r_{pd}!},\end{aligned}$$

where the sum is over all the collections $\{r_{pq} : 1 \leq p \leq k, 1 \leq q \leq d\}$ of non-negative integers satisfying $i_q = r_{1q} + \cdots + r_{kq}$ and $\lambda_p = r_{p1} + \cdots + dr_{pd}$.

Simplifying,

$$\psi_\lambda(C_\mathbf{i}) = \sum \prod_{q=1}^{d} \frac{i_q!}{r_{1q}! \cdot \ldots \cdot r_{kq}!},$$

where the sum is over the same collections $\{r_{pq}\}$.

This sum is equal to the coefficient of the monomial $x_1^{\lambda_1} \cdot \ldots \cdot x_k^{\lambda_k}$ in the symmetric polynomial

$$P^{(\mathbf{i})} = (x_1 + \cdots + x_k)^{i_1} \cdot \ldots \cdot (x_1^d + \cdots + x_k^d)^{i_d}.$$

Definition 57. *Given a partition λ of d and a homogeneous symmetric polynomial P of degree d in k variables, $\psi_\lambda(P) = [P]_\lambda$ and $\omega_\lambda(P) = [\Delta \cdot P]_l$, where $l = (\lambda_1 + k - 1, \lambda_2 + k - 2, \cdots, \lambda_k)$.*

As we just saw, $\psi_\lambda(C_\mathbf{i}) = \psi_\lambda(P^{(\mathbf{i})})$, so the use of the notation ψ_λ in this definition makes sense. Similarly, we can also take advantage of the existing notation and let $\omega_\lambda(\mathbf{i}) = \omega_\lambda(P^{(\mathbf{i})})$.

Definition 58. *Given any two partitions μ and λ of d, the integer $K_{\mu\lambda}$ is the number of ways to fill the boxes of the Young diagram for μ with λ_1 1's, λ_2 2's, up to λ_k k's, in such a way that the entries in each row are non-decreasing, and those in each column are strictly increasing. Such a tableau is called a semi-standard tableau on μ of type λ.*

In particular, $K_{\lambda\lambda} = 1$ and $K_{\mu\lambda} = 0$ for $\mu < \lambda$.

Lemma 59. *For any symmetric polynomial P of degree d in k variables,*

$$\psi_\lambda(P) = \sum_\mu K_{\mu\lambda} \cdot \omega_\mu(P).$$

Proof. See [1, Appendix A.1]. □

In particular, for $P = P^{(\mathbf{i})}$, one gets

$$\psi_\lambda(C_\mathbf{i}) = \sum_\mu K_{\mu\lambda}\omega_\mu(\mathbf{i}) = \omega_\lambda(\mathbf{i}) + \sum_{\mu>\lambda} K_{\mu\lambda}\omega_\mu(\mathbf{i}). \tag{7}$$

Lemma 60. *For partitions λ and μ of d,*

$$\sum_\mathbf{i} \frac{1}{1^{i_1} i_1! \cdot \ldots \cdot d^{i_d} i_d!}\omega_\lambda(\mathbf{i})\omega_\mu(\mathbf{i}) = \begin{cases} 1, & \text{if} \lambda = \mu \\ 0, & \text{otherwise.} \end{cases}$$

Proof. See [1, Appendix A.1]. □

The result of this lemma can be written, using (6), in the form

$$\frac{1}{d!}\sum_\mathbf{i} |C_\mathbf{i}|\, \omega_\lambda(\mathbf{i})\omega_\mu(\mathbf{i}) = \delta_{\lambda\mu}. \tag{8}$$

This means that the functions ω_λ, regarded as class functions on S_d, satisfy the same orthogonality relations as the irreducible characters of S_d. Moreover, we can deduce from these equations that ω_λ is the character χ_λ of V_λ, that is, we will prove the following proposition:

Proposition 61. *For any conjugacy class $C_\mathbf{i}$ of S_d, $\chi_\lambda(C_\mathbf{i}) = \omega_\lambda(\mathbf{i})$.*

Proof. We have seen that the representation U_λ, whose character is ψ_λ, contains the irreducible representation V_λ, whose character is χ_λ. This implies that

$$\psi_\lambda = \sum_\mu n_{\lambda\mu}\chi_\mu, \tag{9}$$

with $n_{\lambda\lambda} \geq 1$ and all $n_{\lambda\mu} \geq 0$. On the other hand, since ω_λ is a class function on S_d and the χ_μ form a basis for $\mathbb{C}_{\text{class}}(S_d)$, we can write

$$\omega_\lambda = \sum_\mu m_{\lambda\mu}\chi_\mu,\ m_{\lambda\mu} \in \mathbb{C}.$$

Fix λ and assume inductively that $\chi_\mu = \omega_\mu$ for all $\mu > \lambda$, so by (7),

$$\psi_\lambda = \omega_\lambda + \sum_{\mu > \lambda} K_{\mu\lambda}\chi_\mu = \sum_\mu m_{\lambda\mu}\chi_\mu + \sum_{\mu>\lambda} K_{\mu\lambda}\chi_\mu.$$

Comparing this with (9), and using the linear independence of characters, one gets $m_{\lambda\lambda} = n_{\lambda\lambda}$, so $m_{\lambda\lambda} \in \mathbb{R}$ and $m_{\lambda\lambda} \geq 1$.

The ω_λ, like the χ_λ, are orthonormal by (8), so

$$1 = \langle \omega_\lambda, \omega_\lambda \rangle = \sum_\mu |m_{\lambda\mu}|^2 ,$$

and since $m_{\lambda\lambda} \geq 1$, it follows that $\omega_\lambda = \chi_\lambda$. □

We have finally proved Frobenius' formula. We can use it to compute the dimension of V_λ, as it is shown in [1, Section 4.1], obtaining

$$\dim V_\lambda = \frac{d!}{l_1! \cdot \ldots \cdot l_k!} \prod_{i<j} (l_i - l_j),$$

with $l_i = \lambda_i + k - i$.

There is also another way of expressing the dimension of V_λ.

Definition 62. *The hook length of a box in a Young diagram is the number of squares directly below or directly to the right of the box, including the box once.*

Using Frobenius' formula and inducting from the diagram obtained by omitting the first column (noting that the hook lengths of the boxes in the first column are the numbers $l_1, \cdots, l_k$), one gets:

Proposition 63 (Hook length's formula)**.**

$$\dim V_\lambda = \frac{d!}{\prod(\text{Hook lengths})}$$

Example 64. *If $\lambda = (d-k, 1, \cdots, 1)$, then the product of the hook lengths of the corresponding Young diagram is*

$$1 \cdot \ldots \cdot k \cdot d \cdot (d-k-1) \cdot \ldots \cdot 1 = k!(d-k-1)!d.$$

By the hook length's formula,

$$\dim V_\lambda = \frac{d!}{k!(d-k-1)!d} = \binom{d-1}{k}.$$

Recall that this result is also a consequence of the fact that $V_\lambda = \Lambda^k V$.

References

[1] William Fulton and Joe Harris, *Representation Theory - A First Course*, Springer-Verlag, 1991.

[2] Jean-Pierre Serre, *Linear Representations of Finite Groups*, Springer-Verlag, 1977.