A First Approach to Representation Theory

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1. Introduction

This paper is an undergraduate's attempt at tackling a branch of mathematics called *representation theory.* From what I gathered, right now there isn't that much information about representation theory written with undergraduates in mind, and therefore I tried to write this as accessible as I could, in an attempt to minimize that gap.

The main idea of this branch of mathematics is to reduce problems in abstract algebra (that is, algebraic structures) to problems in linear algebra, which is very easy to work on. The way this works is by mapping each element in an algebraic structure (group, ring, module, *etc*) into a matrix, without losing the properties of these elements. The most common example of the power of representation theory is its use in Fourier analysis.

Here in particular we will talk about *finite group representation theory.* When applied to groups, representation theory works by mapping each element of a group into an invertible matrix (which intuitively makes sense, since every element in a group is invertible).

In the first section, a lot of concepts are introduced which might be new for the reader. We then aim to prove one of the most important results in representation theory, known as *Schur's lemma.* In the following section, we introduce a very powerful tool to study representations. The importance of the *character* might not be so obvious from the theory alone, but in practice it will prove itself quite useful. After introducing the *character* we will want to decompose the representations, and this will be done through *projections.* This projections will generate a couple of very interesting results. After all this we see some actual examples. This section will be focused on the *symmetric groups*, which are the most interesting cases at this level, and they also have a lot of practical use, since they represent the actual symmetries or permutations of objects.

I wouldn't recommend reading all the theory and then the examples, as you are very likely to miss important things if you don't have an example in mind. Instead, I'd advise the reader to alternate between the theory and the examples, and try to fill in the missing details in the examples (like calculations) in order to assure your understanding of the theory.

2. Irreducible Representations and Schur's Lemma

Definition 2.1. The general linear group $\mathrm{GL}(\mathbf{V})$ over a vector space V is the group of invertible matrices of dimension $(\dim V \times \dim V)$, over V.

Definition 2.2. A *representation* of a finite group G on a finite-dimensional complex vector space V is a homomorphism $\rho : G \to \mathrm{GL}(V)$. For the purpose of simplification, we will call V itself a *representation* of G. We will also suppress the symbol ρ and write $g.v$ or gv for $\rho(g)(v)$.

Definition 2.3. Let $\rho : G \to \mathrm{GL}(V)$ be a representation of G on V. Then the *degree* of ρ is the dimension of V.

Definition 2.4. An *invariant* subspace $W \subset V$ of a linear mapping $T : V \to V$ is a subspace such that $T(W) \subset W$.

Definition 2.5. A *left action* of a group G in V is a function $T : G \times V \to V$, $(g, v) \mapsto g.v$ such that $g.(h.v) = (g.h).v$ and $e.v = v$, for any $g, h \in G, v \in V$ with e the identity of G.

Definition 2.6. A *subrepresentation* of a representation V is a vector subspace W of V which is invariant under the left action of G. A representation V is called *irreducible* if the only *subrepresentations* of V are 0 and V.

Since our objective is to study the representations of groups, the concept of *irreducible representation* is very important to us. Ultimately, we will want to classify all the irreducible representations, because (as is later shown) all other representations are built of them.

Definition 2.7. If V and W are representations of a group G, $V \otimes W$ is also a representation of G, defined by $g(v \otimes w) = g.v \otimes g.w$.

Definition 2.8. The *dual* of a vectorial space V is $\mathbf{V}^* = \mathrm{Hom}(V, \mathbb{C})$.

If V is a representation of G, V^* is also a representation of G, but it is not defined in the most obvious way.

Given $\rho : G \to \mathrm{GL}(V)$, we need to define $\rho^* : G \to \mathrm{GL}(V^*)$ such that

$$\langle \rho^*(g)(v^*), \rho(g)(v) \rangle \;=\; \langle v^*, v \rangle$$

is valid for any $g \in G, v \in V$ and $v^* \in V^*$ (and $\langle \varphi, v \rangle = \varphi(v)$).

This forces us to define the dual representation the following way:

Definition 2.9. The *dual representation* $\rho^*(g)$ of $g \in G$ is the transpose of $\rho(g^{-1})$.

We also need to define another representation, which works for every group, and will be of some use to us:

Definition 2.10. The *regular representation* R of a group G corresponds to the left action of G on itself.

Proposition 2.11. *Let W be a subrepresentation of V of a finite group G. Then there is another subrepresentation W' of V such that $V = W \oplus W'$.*

Proof Let $\langle -, - \rangle$ be a hermitian product in V, such that $\forall g \in G\ \forall v \in V, w \in W, \langle g.v, g.w \rangle = \langle v, w \rangle$. Then, if $W \subset V$ is a subrepresentation of V, $W^{\perp}$ is also a subrepresentation of V, as shown:

$$g.v \in W^{\perp} \Leftrightarrow \langle g.v, w \rangle = 0, \forall w \in W \Leftrightarrow \langle v, g^{-1}.w \rangle = 0, \forall w \in W$$

, since $g^{-1}.w \in W$, and therefore we have described the representation $W^{\perp}$ using W.

We now need to show that such hermitian product exists. The main idea here is to take a hermitian product $H_0 : V \times V \to \mathbb{C}$, and average it throughout G. So we define

$$\langle v, w \rangle := \frac{1}{|G|} \sum_{g \in G} H_0(g.v, g.w)$$

This is in fact an hermitian product, and it does accomplish what we needed. $\square$

From this proposition, we observe that any representation is in fact a direct sum of subrepresentations. By induction on the dimension of the representation, we conclude that any representation is a direct sum of irreducible representations. Therefore, if we know all the irreducible representations of a group, we know all the possible representations of that same group.

We now need to find ways of decomposing representations into direct sums of irreducible representations, so that we only need to study the irreducible representations.

Now is the time to introduce the focus of this section.

Lemma 2.12 (Schur's lemma)**.** *Let V and W be irreducible representations of G and $\phi : V \to W$ a G-module homomorphism. Then the following is true:*

- *Either ϕ is an isomorphism or ϕ is the zero homomorphism*
- *If $V = W$, then $\phi = \lambda.\mathrm{Id}$, for some $\lambda \in \mathbb{C}$.*

Proof

- Since ϕ is a G-module homomorphism, both *Im* ϕ and *Ker* ϕ are invariant subspaces. But, since V and W are irreducible representations, there aren't subrepresentations of any of them, and we have that either *Im* $\phi = 0$ (and *Ker* $\phi = V$) or *Im* $\phi = W$ (and *Ker* $\phi = 0$). In the first case, ϕ is an isomorphism and in the second $\phi = 0$.
- Since $\mathbb{C}$ is algebraically closed, ϕ must have an eigenvalue λ. That is, $\phi - \lambda.\mathrm{Id}$ has a nonzero kernel, and therefore by the first part of the lemma we have $\phi - \lambda.\mathrm{Id} = 0 \Leftrightarrow \phi = \lambda.\mathrm{Id}$.

□

From Schur's lemma we gather that every irreducible representation is unique, up to isomorphism. That is good news for us, as the opposite would make the study of irreducible representations a lot harder.

And now, from 2.11 and Schur's lemma we get a proposition very similar to the Fundamental Theorem of Algebra. This is pretty much the result we have been aiming for since the beginning.

Proposition 2.13. *For any representation V of a finite group G, there is a decomposition*

$$V = V_1^{\oplus a_1} \oplus \cdots \oplus V_k^{\oplus a_k}$$

where V_i are distinct irreducible representations. Moreover, the irreducible representations V_i are unique, and so are their multiplicities a_i.

Proof As we have already seen, any representation has a decomposition into irreducible representations. Let $W = W_1^{\oplus b_1} \oplus \cdots \oplus W_t^{\oplus b_t}$ another representation of G and $\varphi : V \to W$ a map of representations. From Schur's lemma, ϕ must map $V_j^{\oplus a_j}$ into the factor $W_i^{\oplus b_i}$ for which $V_j \cong W_i$, and therefore $a_j = b_i$.

□

3. Characters

We now know that every representation is a direct sum of irreducible representations, and we also know that any representation has a unique decomposition into irreducible representations. Our goal now is to find how to decompose this representations.

This section discusses a great tool for understanding the possible representations of a group. This tool is called *character theory*.

Definition 3.1. Let V be a representation of G. Then, the *character* χ of V is defined by

$$\chi_V(g) = \mathrm{Tr}(g_{|V})$$

the trace of g on V.

We now note that

$$\chi_V(hgh^{-1}) = \chi_V(g)$$

which means that the character is constant on the conjugacy classes of G. We also define the character of a representation as a vector of characters of the different conjugacy classes. That is

$$\chi_V = (\chi_V(g_1), \cdots, \chi_V(g_n))$$

where $g_1, \cdots, g_n$ are representatives of different conjugacy classes in G. The character is then what we call a *class function*: it is constant in the conjugacy classes.

Now, we have a way to differentiate the representations. But we'll need to go into more detail to find out why this tool is so useful.

It is also worth noting that $\chi_V(1) = \mathrm{Tr}(1_{|V}) = \mathrm{dim}V$.

Proposition 3.2. *Let V and W be representations of G. Then,*

i) $\chi_{V\oplus W} = \chi_V + \chi_W$
ii) $\chi_{V\otimes W} = \chi_V . \chi_W$
iii) $\chi_{V^*} = \overline{\chi}_V$
iv) $\chi_{\Lambda^2(V)}(g) = \frac{1}{2}(\chi_V(g)^2 - \chi_V(g^2))$
v) $\chi_{Sym^2(V)}(g) = \frac{1}{2}(\chi_V(g)^2 + \chi_V(g^2))$

Proof Let $\rho_V : G \to \mathrm{GL}(V)$ and $\rho_W : G \to \mathrm{GL}(W)$ be the representations corresponding to V and W.

i) Let $\rho_{V\oplus W} : G \to \mathrm{GL}(V \oplus W)$ be the representation corresponding to $V \oplus W$. Take $g \in G$. Let $A = \rho_V(g)$ and $B = \rho_W(g)$. Then, $\rho_{V\oplus W}(g) = A \oplus B$, and therefore $\mathrm{Tr}(A \oplus B) = \mathrm{Tr}(A) + \mathrm{Tr}(B) = \chi_V(g) + \chi_W(g)$. Since g could be any element in G, $\chi_{V\oplus W} = \chi_V + \chi_W$.

ii) Take $g \in G$. Then,

$$\begin{aligned}\chi_{V\otimes W}(g) &= \mathrm{Tr}(\rho_V(g) \otimes \rho_W(g)) \\ &= \sum_i \sum_j \rho_V(g)_{i,i} \cdot \rho_W(g)_{j,j} \\ &= \sum_i \rho_V(g)_{i,i} \sum_j \rho_W(g)_{j,j} \\ &= \chi_V(g). \chi_W(g)\end{aligned}$$

where $\rho_V(g)_{i,j}$ is the entry i, j in the matrix $\rho_V(g)$. Again, since g could be any element in G, $\chi_{V\otimes W} = \chi_V . \chi_W$.

iii) Take $g \in G$ and let $\lambda_1, \cdots, \lambda_n$ be the eigenvalues of $\rho_V(g)$. Notice that $\mathrm{Tr}(A)$ is the sum of the eigenvalues of A. Then, since $\rho_V^*(g) = {}^t(\rho_V(g^{-1}))$, the eigenvalues of $\rho_V^*(g)$ are $\lambda_1^{-1}, \cdots, \lambda_n^{-1}$. Now we only need to verify that $\overline{\lambda_i} = \lambda_i^{-1}$. Indeed this is true, since $\lambda_1, \cdots, \lambda_n$ are nth roots of the unity, and therefore their inverse is their complex conjugate.

iv) Again, we can choose a basis for V composed by the eigenvalues of $\rho_V(g)$. Then, in this basis,

$$\begin{aligned}
\chi_{\Lambda^2(V)}(g) &= \sum_{i<j} \lambda_i.\lambda_j \\
&= \frac{1}{2} \sum_{i<j} 2\lambda_i.\lambda_j \\
&= \frac{1}{2}((\lambda_1^2 + \cdots + \lambda_n^2 + \sum_{i<j} 2\lambda_i\lambda_j) - (\lambda_1^2 + \cdots + \lambda_n^2)) \\
&= \frac{1}{2}((\lambda_1 + \cdots + \lambda_n)^2 - (\lambda_1^2 + \cdots + \lambda_n^2)) \\
&= \frac{1}{2}(\chi_V(g)^2 - \chi_V(g^2))
\end{aligned}$$

v) Since $V \otimes V = Sym^2(V) \oplus \Lambda^2(V)$, we have

$$\begin{aligned}
\chi_{Sym^2(V)}(g) &= \chi_V(g)^2 - \frac{1}{2}(\chi_V(g)^2 - \chi_V(g^2)) \\
&= \frac{1}{2}\chi_V(g)^2 + \frac{1}{2}\chi_V(g^2) \\
&= \frac{1}{2}(\chi_V(g)^2 + \chi_V(g^2))
\end{aligned}$$

□

Now, if we go back to our problem of decomposing a representation V into irreducible representations $V = V_1^{a_1} \oplus \cdots \oplus V_n^{a_n}$, we now know that $\chi_V = a_1.\chi_{V_1} + \cdots + a_n.\chi_{V_n}$.

4. Projection Formulas and its Consequences

We now have some pretty powerful tools to study representations of groups in general. However, we still have no explicit way to find the decomposition of a representation into the direct sum of irreducible representations.

In order to do just that, we need to find a way to project the irreducible representations into the representation V we want to study, so that we know whether this particular irreducible representation is in the decomposition of V, and its multiplicity.

Taking the idea of the proof used in 2.11, we average out the endomorphisms in V, and thus we get

$$\varphi(v) = \frac{1}{|G|} \sum_{g \in G} g.v \tag{4.1}$$

where $\rho : G \to \mathrm{GL}(V)$ is the representation corresponding to V.

Definition 4.2. $\mathbf{V^G} = \{v \in V : g.v = v, \forall g \in G\}$

Proposition 4.3. *The map φ defined in* (4.1) *is a projection of V into V^G.*

Proof Take $v \in V$. Then $\varphi(v) = \frac{1}{|G|}\sum_{g\in G} g.v$. Now take $h \in G$, and multiply by $\varphi(v)$.

$$h.\varphi(v) = h\frac{1}{|G|}\sum_{g\in G} g.v = \frac{1}{|G|}\sum_{g\in G} h.g.v = \frac{1}{|G|}\sum_{g\in G} g.v = \varphi(v)$$

This means that $Im\ \varphi \subset V^G$. Going with the same line of thought, we compute

$$\varphi(\varphi(v)) = \frac{1}{|G|}\sum_{g\in G} g.(\varphi(v)) = \frac{1}{|G|}\sum_{g\in G}\varphi(v) = \frac{|G|}{|G|}\varphi(v) = \varphi(v)$$

since $\varphi(v) \in V^G$. □

We now have a way of explicitly finding the multiplicities of the irreducible subrepresentations of a given representation.

In particular, if we want to find the multiplicity of the trivial representation (all elements are represented by the identity), we can just compute the dimension of V^G, that is, the trace of the projection φ.

$$\dim V^G = \mathrm{Tr}(\varphi) = \frac{1}{|G|}\sum_{g\in G}\mathrm{Tr}(g) = \frac{1}{|G|}\sum_{g\in G}\chi_V(g)$$

This obviously means that for any irreducible representation V (other than the trivial), $\sum_{g\in G}\chi_V(g) = 0$.

However, we can still do better than this. Let

$$\mathrm{Hom}(V,W)^G = \{G\text{-module homomorphisms from } V \text{ to } W\}$$

If V is irreducible, then (by Schur's lemma), $\dim V^G$ is the multiplicity of V in W. If both V and W are irreducible, we get

$$\dim \mathrm{Hom}(V,W)^G = \begin{cases} 1 & \text{if } V = W \\ 0 & \text{if } V \neq W \end{cases}$$

Now, we know that $\mathrm{Hom}(V,W) = V^* \otimes W$, since

$$\mathrm{Hom}(V,W) = \mathrm{Hom}(V,\mathbb{C}) \otimes \mathrm{Hom}(\mathbb{C},W) = V^* \otimes W$$

and thus $\chi_{\mathrm{Hom}(V,W)}(g) = \overline{\chi_V(g)}.\chi_W(g)$. Then

$$\dim \mathrm{Hom}(V,W)^G = \frac{1}{|G|}\sum_{g\in G}\overline{\chi_V(g)}.\chi_W(g) = \begin{cases} 1 & \text{if } V = W \\ 0 & \text{if } V \neq W \end{cases}$$

We are now ready to define an inner product on the class functions of $\mathbb{C}$

$$(\alpha,\beta) = \frac{1}{|G|}\sum_{g\in G}\overline{\alpha(g)}.\beta(g) \tag{4.4}$$

Theorem 4.5. *The characters of the irreducible representations of G are orthonormal (with respect to the inner product* (4.4)*).*

Proof We have proved this already unintentionally, from the equation

$$\frac{1}{|G|}\sum_{g\in G}\overline{\chi_V(g)}.\chi_W(g)=\begin{cases}1 & \text{if } V=W\\ 0 & \text{if } V\neq W\end{cases}$$

we get that (χ_V,χ_W) (with $V\neq W$ irreducible representations) is indeed 0. $\square$

Corollary 4.6. *The number of irreducible representation of G is smaller or equal than the number of conjugation classes.*

Proof Assume there are more irreducible representations of G than the number of its conjugacy classes. Then, $\{\chi_{V_i}\}_i$ (with V_i irreducible representations) is linearly dependent (contradiction with 4.5). $\square$

Corollary 4.7. *Any representation is determined by its character.*

Proof Let $V=V_1^{\oplus a_1}\oplus\cdots\oplus V_n^{\oplus a_n}$, with V_i irreducible representations. Then $\chi_V=\sum_i a_i.\chi_{V_i}$. Since the χ_{V_i} are linearly independent, we thus conclude that in fact a representation is determined by its character. $\square$

Corollary 4.8. *A representation V is irreducible if and only if* $(\chi_V,\chi_V)=1$

Proof The implication from the left to the right has already been proven, when introducing Theorem 4.5. However, both implications can be proven very easily:

Let $V=V_1^{\oplus a_1}\oplus\cdots\oplus V_n^{\oplus a_n}$, with V_i irreducible representations. Then, $(\chi_V,\chi_V)=\sum_i a_i^2$, which is 1 if and only if $a_i=1\,\forall_i$ and $n=1$ (since a_i are positive integers). $\square$

Corollary 4.9. *The multiplicity a_i of V_i in V is the inner product $a_i=(\chi_V,\chi_{V_i})$.*

Proof

$$(\chi_V,\chi_{V_i})=\sum_j a_j(\chi_{V_j},\chi_{V_i})=a_i$$

since $(\chi_{V_j},\chi_{V_i})=0,\ \forall_{i\neq j}$ and $(\chi_{V_i},\chi_{V_i})=1$. $\square$

We can apply some of this results to the regular representation R, in order to get some general results. The character of R is

$$\chi_R(g)=\begin{cases}0 & \text{if } g\neq e\\ |G| & \text{if } g=e\end{cases}$$

where e is the identity. Therefore R isn't irreducible (unless $G=e$), and $R=V_1^{\oplus a_1}\oplus\cdots\oplus V_n^{\oplus a_n}$, with V_i irreducible representations. Then

$$a_i=(\chi_R,\chi_{V_i})=\frac{1}{|G|}\chi_{V_i}(e).|G|=\dim V_i$$

thus getting the following result:

Corollary 4.10. *Any irreducible representation V_i appears in the regular representation dim V_i times.*

As a consequence, we get an interesting result which will actually be useful to us later:

$$|G| = \dim R = \sum_i a_i.\dim V_i = \sum_i (\dim V_i)^2 \tag{4.11}$$

Proposition 4.12. *The number of irreducible representations of G is equal to the number of conjugacy classes of G.*

Proof Take $\alpha : G \to \mathbb{C}$ a class function and $(\alpha, \chi_V) = 0$ for all irreducible representation V. We must then show that $\alpha = 0$.

Consider the endomorphism $\varphi_{\alpha,V} = \sum_{g\in G} \alpha(g).g$, with $\varphi_{\alpha,V} : V \to V$. First, we show that this endomorphism is a homomorphism of G-modules:

$$\begin{aligned}
\varphi_{\alpha,V}(hv) &= \sum \alpha(g).g(hv) \\
&= \sum \alpha(hgh^{-1}).hgh^{-1}(hv) \\
&= h(\sum \alpha(hgh^{-1}).g(v)) \\
\text{(replacing } hgh^{-1} \text{ by } g) \quad & \\
&= h(\sum \alpha(g).g(v)) \\
(\alpha \text{ is a class function}) \quad & \\
&= h(\varphi_{\alpha,V}(v))
\end{aligned}$$

Now, by Schur's lemma, $\varphi_{\alpha,V} = \lambda.\,\mathrm{Id}$. Then

$$\begin{aligned}
\lambda &= \frac{1}{\dim V}.\mathrm{Tr}(\varphi_{\alpha,V}) \\
&= \frac{1}{\dim V}.\sum \alpha(g)\chi_V(g) \\
&= \frac{|G|}{\dim V}.\overline{(\alpha, \chi_{V^*})} \\
&= 0
\end{aligned}$$

and so $\varphi_{\alpha,V} = 0$. In particular, this is true for the regular representation $V = R$, and in the representation R the elements of G are linearly independent. This means that $\alpha(g) = 0, \forall_{g\in G}$.

□

5. Examples

5.1. Abelian Groups.

Our first example will be abelian groups, which are actually pretty simple.

Proposition 5.1. *Let $\rho : G \to \mathrm{GL}(V)$ be a representation of G. Then ρ is irreducible if and only if* $\dim V = 1$.

Proof Let $\dim V = 1$. Then ρ is irreducible, because V has no invariant subspaces (other than 0 and itself).

Now suppose ρ is irreducible. We must see that $\dim V = 1$. Since $\rho(g)\rho(h) = \rho(gh) = \rho(hg) = \rho(h)\rho(g)$, by Schur's lemma $\forall h \in G$,

$\rho(h) = \lambda.\mathrm{Id}$, for some $\lambda \in \mathbb{C}$. This means that any subspace of V is invariant, and therefore $\dim V = 1$. $\square$

And now all we need to note is that the irreducible representations of a finite abelian group G are simply the homomorphisms $\rho : G \to \mathbb{C}^*$ from G to the invertible elements of $\mathbb{C}$.

5.2. **Symmetric group S_3.**

We begin with the two simplest representations of S_3 (which also work for any symmetric group): the *trivial representation* U, where every element is sent into the identity; and the *alternating representation* U', defined by $gv = \mathrm{sgn}(g)v$, with $\mathrm{sgn}(g)$ being the sign of the permutation g.

By proposition 4.12, we know that there is one more irreducible representation (since S_3 has 3 conjugacy classes). We now have to find it. Let's consider the *natural representation*, which is the representation in which S_3 acts in $\mathbb{C}^3$ by permuting the coordinates. That is, if we take the permutation $(132) \in S_3$ and the vector $(a, b, c) \in \mathbb{C}^3$, then $(132)(a, b, c) = (b, c, a)$. This representation has dimension 3.

But now we notice that this *natural representation* has an invariant subspace, generated by $(1, 1, 1)$ (any permutation applied to the vector (a, a, a) is still (a, a, a)). Now $V = \{(a, b, c) \in \mathbb{C}^3 : a+b+c = 0\}$ (the orthogonal complement of the space generated by $(1, 1, 1)$) is a representation of dimension 2. We will see that this representation, called the *standard representation* of S_3 is irreducible.

To prove that this representation is indeed irreducible, we will use character theory. We must compute the characters of U and U' first. We have 3 conjugacy classes in S_3:

- the identity, represented by 1 - this class has 1 element;
- the transpositions, represented by (12) - this class has 3 elements;
- the 3-cycles, represented by (123) - this class has 2 elements.

The character of U χ_U is $(1, 1, 1)$, since in this representation every $g \in G$ is represented by the identity.

The character of U' $\chi_{U'}$ is $(1, -1, 1)$, since $gv = \mathrm{sgn}(g)v$, and $\mathrm{sgn}(1) = 1$; $\mathrm{sgn}(12) = -1$ and $\mathrm{sgn}(123) = 1$.

We now construct a character table, for a summary of the information we have.

n. of elements	1	3	2
S_3	1	(12)	(123)
U	1	1	1
U'	1	-1	1
V	?	?	?

We have almost everything we need. To compute χ_V we can use the fact that $\mathbb{C}^3 = U \oplus V$ and therefore $\chi_{\mathbb{C}^3} = \chi_U + \chi_V$.

$$\chi_{\mathbb{C}^3} = (\mathrm{Tr}(\mathrm{Id}), \mathrm{Tr}(\rho(12)), \mathrm{Tr}(\rho(123))) = (3, 1, 0)$$

This might not be so obvious, but if you think of the permutation matrices, you notice that the amount of times the number 1 appears in the main diagonal is the same as the amount of elements the permutation leaves fixed.

Now we just need to compute

$$\chi_{\mathbb{C}^3} = \chi_U + \chi_V \iff (3,1,0) = (1,1,1) + \chi_V \iff \chi_V = (2,0,-1)$$

and now we can verify that V is indeed irreducible:

$$(\chi_V, \chi_V) = \frac{1}{6}\sum_{g\in G} \chi_V(g)^2 = \frac{1}{6}(1*2^2 + 3*0^2 + 2*(-1)^2) = \frac{1}{6}(6) = 1$$

and as we have seen in 4.8, V is irreducible.
We have thus completed our character table for S_3:

	1	3	2
S_3	1	(12)	(123)
U	1	1	1
U'	1	-1	1
V	2	0	-1

We can verify that the characters of this representations are orthogonal:

$$(\chi_U, \chi_{U'}) = \frac{1}{6}\sum_{g\in G} \chi_U(g).\chi_{U'}(g) = \frac{1}{6}(1*1*1 + 3*1*(-1) + 2*1*1) = 0$$

$$(\chi_U, \chi_V) = \frac{1}{6}\sum_{g\in G} \chi_U(g).\chi_V(g) = \frac{1}{6}(1*1*2 + 3*1*0 + 2*1*(-1)) = 0$$

$$(\chi_{U'}, \chi_V) = \frac{1}{6}\sum_{g\in G} \chi_{U'}(g).\chi_V(g) = \frac{1}{6}(1*1*2 + 3*(-1)*0 + 2*1*(-1)) = 0$$

As we had seen, for any representation W of S_3, $W = U^{\oplus a} \oplus U'^{\oplus b} \oplus V^{\oplus c}$ and $\chi_W = a.\chi_U + b.\chi_{U'} + c.\chi_V$. Since the characters of this representations are independent, we can conclude that W is determined up to isomorphism by its character, confirming corollary 4.7.

5.3. Symmetric Group S_4.

In S_4 we have 5 conjugacy classes:

- the identity 1;
- 6 transpositions;
- 8 3-cycles;
- 6 4-cycles;
- 3 products of two disjoint transpositions (represented by (12)(34)).

We can start constructing our character table, with the representations we already have:

	1	6	8	6	3
S_3	1	(12)	(123)	(1234)	(12)(34)
U	1	1	1	1	1
U'	1	-1	1	-1	1
V	3	1	0	-1	-1

Again, by 4.12 we know that we are 2 irreducible representations short. From 4.11 we know that $\sum_i (\dim V_i)^2 = |S_4| = 24$. From the 3 representations we have, we get $1^2 + 1^2 + 3^2 + x^2 + y^2 = 24$, and so $x^2 + y^2 = 13$, which means that $x = 3$ and $y = 2$.

The representation with dimension 3 is relatively intuitive to locate: we just need to tensor V with U', and so $\chi_{V\otimes U'} = \chi_V.\chi_{U'} = (3, -1, 0, 1, -1)$. To verify that this representation is irreducible, we compute $(\chi_{V\otimes U'}, \chi_{V\otimes U'})$:

$$(\chi_{V\otimes U'}, \chi_{V\otimes U'}) = \frac{1}{24}(1{*}3^2{+}6{*}1^2{+}8{*}0^2{+}6{*}(-1)^2{+}3{*}(-1)^2) = \frac{1}{24}(24) = 1$$

We are missing a representation W of dimension 2. We have already everything we need, since $(\chi_{V_i}, \chi_W) = 0$ for all V_i irreducible and $(\chi_W, \chi_W) = 1$. On top of that, we also know that $\chi_W(e) = 2$, since we know the dimension of W. We solve

$$\begin{cases} 1 \cdot 2 + 6 \cdot 1 \cdot \chi_W(12) + 8 \cdot 1 \cdot \chi_W(123) + \\ \qquad +6 \cdot 1 \cdot \chi_W(1234) + 3 \cdot 1 \cdot \chi_W((12)(34)) = 0 \\ 1 \cdot 2 + 6 \cdot (-1) \cdot \chi_W(12) + 8 \cdot 1 \cdot \chi_W(123) + \\ \qquad +6 \cdot (-1) \cdot \chi_W(1234) + 3 \cdot 1 \cdot \chi_W((12)(34)) = 0 \\ 3 \cdot 2 + 6 \cdot 1 \cdot \chi_W(12) + 8 \cdot 0 \cdot \chi_W(123) + \\ \qquad +6 \cdot (-1) \cdot \chi_W(1234) + 3 \cdot (-1) \cdot \chi_W((12)(34)) = 0 \\ 3 \cdot 2 + 6 \cdot (-1) \cdot \chi_W(12) + 8 \cdot 0 \cdot \chi_W(123) + \\ \qquad +6 \cdot 1 \cdot \chi_W(1234) + 3 \cdot (-1) \cdot \chi_W((12)(34)) = 0 \\ 2^2 + 6 \cdot (\chi_W(12))^2 + 8 \cdot (\chi_W(123))^2 + \\ \qquad +6 \cdot (\chi_W(1234))^2 + 3 \cdot (\chi_W((12)(34)))^2 = 24 \end{cases}$$

and we get $\chi_W = (2, 0, -1, 0, 2)$. This is obviously irreducible, since one of the conditions in this equations was precisely that, and it is linearly independent of all other irreducible representations.

We have the entire character table:

	1	6	8	6	3
S_3	1	(12)	(123)	(1234)	(12)(34)
U	1	1	1	1	1
U'	1	-1	1	-1	1
V	3	1	0	-1	-1
$V \otimes U'$	3	-1	0	1	-1
W	2	0	-1	0	2

5.4. **Alternating Group A_4.**

The alternating group is the subgroup of S_4 with only the even permutations. The conjugacy classes of A_4 are:

- the identity;
- 4 3-cycles represented by (123);
- 4 3-cycles represented by (132);
- 3 products of two disjoint transpositions, represented by (12)(34).

We can consider the representations of S_4 and simply restrict them to A_4 and see where it leads us.

There are 4 irreducible representations, and the sum of the squares of their dimensions is 12.

$$\chi_{U_{|A_4}} = (1, 1, 1, 1) = \chi_{U'_{|A_4}}$$

This means that $U_{|A_4}$ and $U'_{|A_4}$ are isomorphic.

$$\chi_{V_{|A_4}} = (3, 0, 0, -1) = \chi_{(V\otimes U')_{|A_4}} \Rightarrow (\chi_{V_{|A_4}}, \chi_{V_{|A_4}}) = 1$$

And so $V_{|A_4}$ is irreducible, with dimension 3. We know that we are missing 2 representations and their dimensions must be 1, since $1^2+3^2+1^2+1^2 = 12$.

We continue restricting the irreducible representations of S_4 to A_4:

$$\chi_{W_{|A_4}} = (2, -1, -1, 2) \Rightarrow (\chi_{W_{|A_4}}, \chi_{W_{|A_4}}) = 2$$

This means that $W_{|A_4} = V_1 \oplus V_2$ with V_i irreducible representation. We have to find these V_i, which we will do using the projection formulas.

$$(\chi_{W_{|A_4}}, \chi_{U_{|A_4}}) = \frac{1}{12}(1*2*1 + 4*(-1)*1 + 4*(-1)*1 + 3*2*1) = 0$$

$$(\chi_{W_{|A_4}}, \chi_{V_{|A_4}}) = \frac{1}{12}(1*2*3 + 4*(-1)*0 + 4*(-1)*0 + 3*2*(-1)) = 0$$

We can conclude that neither $U_{|A_4}$ nor $V_{|A_4}$ are subrepresentations of $W_{|A_4}$.

We still have 2 representations left and at least one of them is a subrepresentation of $W_{|A_4}$. However, we can note that

$$A_4/\{1, (12)(34), (13)(24), (14)(23)\} = \mathbb{Z}_3$$

Since $\mathbb{Z}_3 = \{0, 1, 2\}$ is an abelian group, we know its irreducible representations $\rho : \mathbb{Z}_3 \to \mathbb{C}$ (with $\omega = e^{2\pi i/3}$):

- the trivial representation, which sends all elements into 1;
- a representation which sends $0 \to 1$, $1 \to \omega$, $2 \to \omega^2$;
- a representation which sends $0 \to 1$, $1 \to \omega^2$, $2 \to \omega$;

Let this 3 representations be U, U' and U'' respectively, and we want to use them for A_4. We know that

$$\chi_U((12)(34)) = \chi_{U'}((12)(34)) = \chi_V((12)(34)) = 1$$

and now we have our character table:

	1	4	4	3
A_3	1	(123)	(132)	(12)(34)
U	1	1	1	1
U'	1	ω	ω^2	1
U''	1	ω^2	ω	1
V	3	0	0	-1

5.5. **Symmetric Group S_5.**

Again we must find the conjugacy classes of S_5. They are:

- the identity;
- 10 transpositions;
- 20 3-cycles;
- 30 4-cycles;
- 24 5-cycles;
- 15 products of two transpositions;
- 20 products of a transposition with a 3-cycle.

This gives us $5! = 120$ elements. We compute the characters of the trivial, alternate, standard and the tensor of the standard with the alternate representation (which we will from now on denote as V'), and we get the table:

	1	10	20	30	24	15	20
S_5	1	(12)	(123)	(1234)	(12345)	(12)(34)	(12)(345)
U	1	1	1	1	1	1	1
U'	1	-1	1	-1	1	1	-1
V	4	2	1	0	-1	0	-1
V'	4	-2	1	0	-1	0	1

Using the same procedures as before, we know that there are 3 more representations. We must now find the others. A good place to start looking would be the tensor product of two of the representations we already have.

We take $V \otimes V$. But $V \otimes V = \Lambda^2 V \oplus \mathrm{Sym}^2 V$, so now we have two possibilities. Let's compute the first, using $\chi_{\Lambda^2(V)}(g) = \frac{1}{2}(\chi_V(g)^2 - \chi_V(g^2))$ from 3.2.

$$\chi_{\Lambda^2(V)} = \frac{1}{2}((4,2,1,0,-1,0,-1)^2 - (4,4,1,0,-1,4,1)) = (6,0,0,0,1,-2,0)$$

To compute $\chi_V(g^2)$ we only need to note that:

- $1^2 = 1$;
- $(12)^2 = 1$;
- $(123)^2 = (132)$;
- $(1234)^2 = (13)(24)$;
- $(12345)^2 = (13524)$;
- $((12)(34))^2 = 1$;
- $((12)(345))^2 = (354)$.

We must see that this representation is irreducible:

$$(\chi_{\Lambda^2(V)}, \chi_{\Lambda^2(V)}) = \frac{1}{120}(36 + 24 * 1 + 15 * 4) = 1$$

There are only two representations left. We can find out their dimensions: $120 = 1^2 + 1^2 + 4^2 + 4^2 + 6^2 + a^2 + b^2 \iff a^2 + b^2 = 50$. So either one of them has dimension 1 (and the other 7) or they both have dimension 5. There can't be any more representations with dimension 1, because then the image of S_5 by this representation would be abelian (because $\mathbb{C}^*$ is abelian). Since the only normal subgroups of S_5 are $\{1\}$, A_5 and S_5, the only abelian quotients of S_5 are $S_5/A_5 = \{\pm 1\}$ and $S_5/S_5 = \{1\}$.

So we have a representation W, with character

$$\chi_W = (5, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6)$$

Now we try to tensor it with the alternate representation, and get $W \otimes U' = W'$ with character

$$\chi_{W'} = (5, -\alpha_1, \alpha_2, -\alpha_3, \alpha_4, \alpha_5, -\alpha_6)$$

Right now we have no guarantee that $W \neq W'$, but checking the orthogonality conditions, we conclude that it is true. Since $W = W' \Rightarrow \alpha_1 = \alpha_3 = \alpha_6 = 0$, we get:

$$\begin{pmatrix} \chi_U \\ \chi_{U'} \\ \chi_V \\ \chi_{V'} \\ \chi_{\Lambda^2 V} \end{pmatrix} \cdot \begin{pmatrix} 1 * 5 \\ 10 * 0 \\ 20 * \alpha_2 \\ 30 * 0 \\ 24 * \alpha_4 \\ 15 * \alpha_5 \\ 20 * 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 120 \end{pmatrix}$$

Now we know that $\alpha_1 \cdot \alpha_3 \cdot \alpha_6 \neq 0$ and so $W \neq W'$. To find χ_W and χ'_W we solve a system of linear equations very similar to 5.3, thus getting our final character table of S_5:

	1	10	20	30	24	15	20
S_5	1	(12)	(123)	(1234)	(12345)	(12)(34)	(12)(345)
U	1	1	1	1	1	1	1
U'	1	-1	1	-1	1	1	-1
V	4	2	1	0	-1	0	-1
V'	4	-2	1	0	-1	0	1
$\Lambda^2 V$	6	0	0	0	1	-2	0
W	5	1	-1	-1	0	1	1
W'	5	-1	-1	1	0	1	-1

References

[1] William Fulton & Joe Harris, *Representation Theory: A First Course*, New York, Springer-Verlag, 1991.