Bousfield lattices, quotients, ring maps, and non-Noetherian rings

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Overview

- The bigger picture and background
- Bousfield lattice substructure
- Quotients of lattices and lattices of quotients
- Ring maps, and specific non-Noetherian examples

Let T be a tensor-triangulated category generated by the tensor unit $\mathbb{1}$.

For example, take $\mathsf{T} = D(R)$, where R is a commutative ring, $D(R)$ is the derived category of unbounded chain complexes.

The tensor product $X \wedge Y := X \otimes_R^L Y$, and R is the unit. Every $X \in D(R)$ can be built from R using triangles, retracts, and coproducts.

Other examples: the stable homotopy category, the stable module category $StMod(kG)$ when G is a finite p-group and $char(k) = p$.

Let $\mathsf{S} \subseteq \mathsf{T}$ be a full subcategory.

Definition

1. S is a *thick subcategory* if it is closed under triangles and retracts (so $X \coprod Y \in \mathsf{S}$ implies $X, Y \in \mathsf{S}$).
2. $\mathsf{th}(X)$ is the smallest thick subcategory containing X.
3. S is a *localizing subcategory* if it is closed under triangles and coproducts (and hence retracts).
4. $\mathsf{loc}(X)$ is the smallest localizing subcategory containing X.

Henceforth, let T be a tensor-triangulated category with $\mathsf{T} = \mathsf{loc}(\mathbb{1})$. The thick subcategory $\mathsf{th}(\mathbb{1})$ is the *finite objects*.

BIG GOAL:

Classify the thick subcategories of finite objects in T, and the localizing subcategories of T.

Done for:

- Thick subcats of finites: the stable homotopy category, $D(R)$, $StMod(kG)$.
- Localizing subcats: $D(R)$ when R is Noetherian, $StMod(kG)$.

Motto: "Localizing subcategories are hard."

Question.

Is there a set of localizing subcategories?

The answer is yes for:

- the stable homotopy category [Okhawa 1989]
- a Brown category [Dwyer-Palmieri 2001]
- a well-generated category [Iyengar-Krause 2011]
  – e.g. whenever $\mathsf{T} = \mathsf{loc}(\mathbb{1})$.
- any category with a combinatorial model structure [Casacuberta-Gutiérrez-Rosický 2012]

Question.

If there is a set, then what?

Definition

The *Bousfield class* of an object $X \in \mathsf{T}$ is $\langle X \rangle = \{W \mid W \wedge X = 0\}$.

Every Bousfield class is a localizing subcategory [show], so there is a set of them.

Bousfield classes are given a partial ordering by reverse inclusion, so we say $\langle X \rangle \leq \langle Y \rangle$ when

$$W \wedge Y = 0 \text{ implies } W \wedge X = 0.$$

Then $\langle 0 \rangle$ is the minimum, and $\langle \mathbb{1} \rangle$ is the maximum [show].

Note that $\langle X \rangle = \langle 0 \rangle$ if and only if $X = 0$.

There is a join operation given by

$$\bigvee_{i\in I}\langle X_i\rangle := \left\langle \coprod_{i\in I} X_i \right\rangle.$$

There is a meet given by

$$\langle X\rangle \curlywedge \langle Y\rangle := \bigvee_{\langle W\rangle\le\langle X\rangle \text{ and } \langle W\rangle\le\langle Y\rangle} \langle W\rangle.$$

Thus the collection of Bousfield classes forms a complete lattice, called the *Bousfield lattice* $\mathsf{BL}(\mathsf{T})$. We'll do stuff with it.

Question.

But is every localizing subcategory a Bousfield class?

This was conjectured for the stable homotopy category in [Hovey-Palmieri 1999].

The answer is yes for $D(R)$ when R is Noetherian, and for $StMod(kG)$ (and slightly more generally).

In October [Stevenson 2012] found a non-Noetherian ring S such that the answer is no for $D(S)$.

Theme for Bousfield lattice and localizing subcategories:

When R is Noetherian, $D(R)$ is very nice... *too* nice.
The stable homotopy category is very hard... and *complicated*.

Some are working to extend the niceness as much as possible. Some are trying to simplify the category of spectra using localizations. Some are looking at non-Noetherian rings. In particular, we know a lot about $D(\Lambda)$ for

$$\Lambda = \frac{k[x_1, x_2, x_3, \ldots]}{(x_1^{n_1}, x_2^{n_2}, x_3^{n_3} \ldots)},$$

where k is a countable field, $n_i \geq 2$, and $deg(x_i) = 2^i$.

IN THIS TALK

Let S, T be tensor-triangulated categories generated by their tensor units. Sometimes a functor $F : \mathsf{S} \to \mathsf{T}$ will induce a map of lattices

$$\mathsf{BL}(\mathsf{S}) \to \mathsf{BL}(\mathsf{T}), \text{ where } \langle X \rangle \mapsto \langle FX \rangle.$$

What can this tell us about $\mathsf{BL}(\mathsf{S})$ and $\mathsf{BL}(\mathsf{T})$?

Specifically:

- the Verdier quotient functor $\pi : \mathsf{T} \to \mathsf{T}/\langle Z \rangle$
- a ring map between two commutative rings $f : R \to S$ induces $f_\bullet : D(R) \to D(S)$ via extension of scalars.

BOUSFIELD LATTICE SUBSTRUCTURE

The tensor (smash) product gives another operation on Bousfield classes:

$$\langle X \rangle \wedge \langle Y \rangle := \langle X \wedge Y \rangle.$$

In general $\langle X \rangle \wedge \langle Y \rangle \leq \langle X \rangle \curlywedge \langle Y \rangle$.

Definition

$$\mathsf{DL}(\mathsf{T}) = \{\langle X \rangle \text{ such that } \langle X \wedge X \rangle = \langle X \rangle\}.$$

This is a sublattice of $\mathsf{BL}(\mathsf{T})$.

Proposition. (Bousfield)

In DL the meet agrees with the smash. Hence DL is a distributive lattice.

Definition

1. $\langle X \rangle \in \mathsf{BL}$ is *complemented* if there exists $\langle X^c \rangle$ such that
$$\langle X \rangle \wedge \langle X^c \rangle = \langle 0 \rangle \text{ and } \langle X \rangle \vee \langle X^c \rangle = \langle \mathbb{1} \rangle.$$
2. $\mathsf{BA}(\mathsf{T}) = \{\text{complemented } \langle X \rangle\} \subseteq \mathsf{BL}(\mathsf{T})$.

Note that complements, if they exist, are unique. BA is a Boolean algebra, and
$$\mathsf{BA} \subseteq \mathsf{DL} \subseteq \mathsf{BL}.$$

What we know:

- If R is Noetherian, then in $D(R)$ we have $\mathsf{BA} = \mathsf{DL} = \mathsf{BL}$.
- In $D(\Lambda)$, $I\Lambda = \mathrm{Hom}_k^*(\Lambda, k)$ has $I\Lambda \wedge I\Lambda = 0$, so $\langle I\Lambda \rangle \notin \mathsf{DL}$.

  Furthermore, $\mathsf{BA} = \{\langle 0 \rangle, \langle \Lambda \rangle\}$.
- In the stable homotopy category, the Brown-Comenetz dual IS^0 of the sphere has $IS^0 \wedge IS^0 = 0$, so $\mathsf{DL} \subsetneq \mathsf{BL}$.

  Every finite spectrum $\langle F \rangle \in \mathsf{BA}$. But, for example, $\langle H\mathbb{Z} \rangle \in \mathsf{DL} \backslash \mathsf{BA}$.

Definition

For any $\langle X\rangle \in \mathsf{BL}$, define

$$a\langle X\rangle = \bigvee_{\langle X\wedge Y\rangle=\langle 0\rangle} \langle Y\rangle.$$

Note that $\langle X\rangle \wedge a\langle X\rangle = \langle 0\rangle$ and $\langle X\rangle \vee a\langle X\rangle \leq \langle \mathbb{1}\rangle$.

Lemma. (Bousfield)

1. If $\langle X\rangle$ is complemented, then $\langle X^c\rangle = a\langle X\rangle$.
2. $\langle Y\rangle \leq a\langle X\rangle$ if and only if $\langle Y\rangle \wedge \langle X\rangle = \langle 0\rangle$.[show]
3. $\langle X\rangle \leq \langle Y\rangle$ if and only if $a\langle X\rangle \geq a\langle Y\rangle$.
4. $a^2\langle X\rangle = \langle X\rangle$.

Definition

We say $X \in \mathsf{T}$ is *square-zero* if it is nonzero but $X \wedge X = 0$.

Recall $\langle 0 \rangle \leq \langle X \wedge X \rangle \leq \langle X \rangle$ for all X.

Proposition. (W.)

There are no square-zero objects in T if and only if $\mathsf{BA} = \mathsf{DL} = \mathsf{BL}$.

Proof. [show]

Quotients

Given a localizing subcategory S of a tensor-triangulated category T, the Verdier quotient T/S is tensor-triangulated, and the quotient functor $\pi : \mathsf{T} \to \mathsf{T}/\mathsf{S}$ is exact.

Question.

Does π induce a map on Bousfield lattices?

$$\mathsf{BL}(\mathsf{T}) \to \mathsf{BL}(\mathsf{T}/\mathsf{S}), \text{ where } \langle X \rangle \mapsto \langle \pi X \rangle.$$

[Aside: $\pi(\langle X \rangle)$ is not usually triangulated.]
Does $\langle X \rangle = \langle Y \rangle$ imply $\langle \pi X \rangle = \langle \pi Y \rangle$?
Does $\langle X \rangle \leq \langle Y \rangle$ imply $\langle \pi X \rangle \leq \langle \pi Y \rangle$?
In general, no. However:

Proposition. (W.)

Suppose $\mathsf{S} = \langle Z \rangle$ for some $\langle Z \rangle$. Then $\pi : \mathsf{T} \to \mathsf{T}/\langle Z \rangle$ induces an order-preserving map of lattices $\mathsf{BL}(\mathsf{T}) \to \mathsf{BL}(\mathsf{T}/\langle Z \rangle)$.

Definition

$$\langle X\rangle\!\downarrow = \{\langle Y\rangle \mid \langle Y\rangle \leq \langle X\rangle\} \subseteq \mathsf{BL}.$$

$$\langle X\rangle\!\uparrow = \{\langle Y\rangle \mid \langle Y\rangle \geq \langle X\rangle\} \subseteq \mathsf{BL}.$$

We can form a quotient lattice $\mathsf{BL}/\langle X\rangle\!\downarrow$ of equivalence classes of Bousfield classes, where $[\langle Y\rangle] = [\langle Z\rangle]$ if and only if $\langle Y\rangle \vee \langle X\rangle = \langle Z\rangle \vee \langle X\rangle$.

Likewise $[\langle Y\rangle] \leq [\langle Z\rangle]$ in $\mathsf{BL}/\langle X\rangle\!\downarrow$ if and only if $\langle Y\rangle \vee \langle X\rangle \leq \langle Z\rangle \vee \langle X\rangle$ in BL.

There is an isomorphism of lattices $\mathsf{BL}/\langle X\rangle\!\downarrow \xrightarrow{\sim} \langle X\rangle\!\uparrow$, given by $[\langle Y\rangle] \mapsto \langle Y\rangle \vee \langle X\rangle$.

The picture is this:

$$\begin{tikzcd}
\mathsf{BL}(\mathsf{T}) \arrow[r, "\pi"] \arrow[d] & \mathsf{BL}(\mathsf{T}/\langle Z\rangle) \\
\mathsf{BL}(\mathsf{T})/(a\langle Z\rangle\downarrow) \arrow[ur, dashed, "\overline{\pi}"'] &
\end{tikzcd}$$

Proposition. (W.)

The map $\overline{\pi} : [\langle X\rangle] \mapsto \langle \pi X\rangle$ is a well-defined, order-preserving epimorphism of lattices, such that if $\overline{\pi}[\langle X\rangle] = \langle 0\rangle$ then $[\langle X\rangle] = [\langle 0\rangle]$.

Proposition. (W.)

If $\langle Z\rangle$ is complemented, then this is an isomorphism

$$\langle Z^c\rangle\!\uparrow \,\cong \mathsf{BL}(\mathsf{T})/\langle Z^c\rangle\!\downarrow \xrightarrow{\sim} \mathsf{BL}(\mathsf{T}/\langle Z\rangle).$$

Corollary. (W.)

If there are no square-zero objects in T, then $\overline{\pi}$ is an isomorphism for all $\langle Z\rangle$.

Proposition. (W.)

If $\langle Z\rangle \in \mathsf{DL}\backslash\mathsf{BA}$, then $\overline{\pi}$ is NOT an isomorphism of lattices.

This is the case with the stable homotopy category, if we take $\langle Z\rangle = \langle H\mathbb{F}_p\rangle$. Then, in fact, we have $IS^0 \in \langle H\mathbb{F}_p\rangle \vee a\langle H\mathbb{F}_p\rangle < \langle \mathbb{1}\rangle$. This is also the case in $D(\Lambda)$, if we take $\langle Z\rangle = \langle k\rangle$. Then we have $I\Lambda \in \langle k\rangle \vee a\langle k\rangle < \langle \mathbb{1}\rangle$.

RING MAPS

Let $f : R \to S$ be a ring map between commutative rings. This induces a map on modules, via extension of scalars.

$$f_* : \mathsf{Mod}\text{-}R \to \mathsf{Mod}\text{-}S, \text{ by } M \mapsto M \otimes_R S.$$

This induces a functor $f_* : Ch(R) \to Ch(S)$, which is left adjoint to $f^* : Ch(S) \to Ch(R)$ induced by the forgetful functor.

By abstract nonsense, there is a pair of adjoint functors on the derived categories

$$f_\bullet = L(f_*) : D(R) \rightleftarrows D(S) : R(f^*) = f^\bullet.$$

Lemma.

1. $f_\bullet R = S$
2. $f_\bullet$ is exact, and commutes with coproducts
3. $f_\bullet(X \wedge Y) = f_\bullet X \wedge f_\bullet Y$
4. (Every object is fibrant, so) $f^\bullet(Z) = f^*(Z)$, and $f^\bullet$ is exact and commutes with products and coproducts.

In general, $f^\bullet(X \wedge Y) \neq f^\bullet X \wedge f^\bullet Y$.

Proposition. (W.)

$f_\bullet$ and $f^\bullet$ induce order-preserving maps on Bousfield lattices

$$f_\bullet : \mathsf{BL}_R \to \mathsf{BL}_S, \text{ where } \langle X\rangle \mapsto \langle f_\bullet X\rangle, \text{ and}$$

$$f^\bullet : \mathsf{BL}_S \to \mathsf{BL}_R, \text{ where } \langle Y\rangle \mapsto \langle f^\bullet Y\rangle.$$

Lemma.

For all $A \in D(R)$ and $B \in D(S)$ we have

$$f_\bullet A \wedge B = 0 \text{ if and only if } A \wedge f^\bullet B = 0.$$

So $\langle f_\bullet X \rangle = \langle 0 \rangle$ iff $X \wedge f^\bullet S = 0$ iff $\langle X \rangle \leq a\langle f^\bullet S \rangle$.

Definition

$$\langle M_f \rangle = \bigvee_{\langle f_\bullet X \rangle = \langle 0 \rangle} \langle X \rangle = \bigvee_{\langle X \wedge f^\bullet S \rangle = \langle 0 \rangle} \langle X \rangle = a\langle f^\bullet S \rangle.$$

Then $\langle f_\bullet X \rangle = \langle 0 \rangle$ if and only if $\langle X \rangle \leq \langle M_f \rangle$, i.e. $Ker\, f_\bullet = \langle M_f \rangle \downarrow$.

Lemma. (W.)

The following are equivalent:

1. $f_\bullet f^\bullet \langle X\rangle = \langle X\rangle$ for all $\langle X\rangle$
2. $f^\bullet Y \wedge f^\bullet X = 0$ if and only if $f^\bullet(Y \wedge X) = 0$
3. $f^\bullet\langle Y \wedge X\rangle = \langle f^\bullet Y\rangle \wedge \langle f^\bullet X\rangle$ for all Y and X.

Proposition. (W.)

Assume $\langle f_\bullet f^\bullet X\rangle = \langle X\rangle$ for all $\langle X\rangle$.

1. $f_\bullet$ sends DL_R onto DL_S and the map $f^\bullet$ injects DL_S into DL_R.
2. $f_\bullet$ sends BA_R onto BA_S, and if $\langle f^\bullet S\rangle \vee \langle M_f\rangle = \langle R\rangle$ then $f^\bullet$ injects BA_S into BA_R.

Recall $Ker\, f_\bullet = \langle M_f\rangle\downarrow = (a\langle f^\bullet S\rangle)\downarrow$.

The picture is:

$$\begin{tikzcd}
\mathsf{BL}(R) \arrow[rr, "f_\bullet"] \arrow[d] & & \mathsf{BL}(S) \\
\mathsf{BL}(R)/\langle M_f\rangle\downarrow \arrow[urr, "\exists"] \arrow[r, "(*)"'] & \mathsf{BL}(D(R)/\langle f^\bullet S\rangle) \arrow[ur, "(\dagger)"'] &
\end{tikzcd}$$

The map $(*)$ is an isomorphism when $\langle f^\bullet S\rangle \vee \langle M_f\rangle = \langle R\rangle$.

The map $(\dagger)$ exists and is an isomorphism when $f_\bullet f^\bullet\langle X\rangle = \langle X\rangle$ for all X.

EXAMPLE WITH SOME NON-NOETHERIAN RINGS

Definition

Fix a prime p and choose $n_i \geq 2$. Let $deg(x_i) = 2^i$ and define the following.

1. $\Lambda_{\mathbb{Z}_{(p)}} = \frac{\mathbb{Z}_{(p)}[x_1,x_2,x_3,\ldots]}{(x_1^{n_1},x_2^{n_2},x_3^{n_3}\ldots)}$,
   $\Lambda_{\mathbb{F}_p} = \frac{\mathbb{F}_p[x_1,x_2,x_3,\ldots]}{(x_1^{n_1},x_2^{n_2},x_3^{n_3}\ldots)}$,
   $\Lambda_{\mathbb{Q}} = \frac{\mathbb{Q}[x_1,x_2,x_3,\ldots]}{(x_1^{n_1},x_2^{n_2},x_3^{n_3}\ldots)}$.
2. Let $g : \Lambda_{\mathbb{Z}_{(p)}} \to \Lambda_{\mathbb{Z}_{(p)}}/p\Lambda_{\mathbb{Z}_{(p)}} = \Lambda_{\mathbb{F}_p}$ be the projection map.
3. Let $h : \Lambda_{\mathbb{Z}_{(p)}} \to \Lambda_{\mathbb{Q}}$ be the inclusion map.
4. Let $g_\bullet : D(\Lambda_{\mathbb{Z}_{(p)}}) \leftrightarrows D(\Lambda_{\mathbb{F}_p}) : g^\bullet$ and
   $h_\bullet : D(\Lambda_{\mathbb{Z}_{(p)}}) \leftrightarrows D(\Lambda_{\mathbb{Q}}) : h^\bullet$ be the induced adjoint pairs.

It turns out that $\langle g^\bullet \Lambda_{\mathbb{F}_p} \rangle$ and $\langle h^\bullet \Lambda_{\mathbb{Q}} \rangle$ are a complemented pair in $\mathsf{BL}(\Lambda_{\mathbb{Z}_{(p)}})$. Recall that $\mathsf{BL}(\Lambda_{\mathbb{F}_p})$ and $\mathsf{BL}(\Lambda_{\mathbb{Q}})$ have no nontrivial complemented classes.

Furthermore, $\langle g_\bullet g^\bullet X \rangle = \langle X \rangle$ for all $X \in D(\Lambda_{\mathbb{F}_p})$. Thus we get the following picture.

$$\begin{tikzcd}
\mathsf{BL}(\Lambda_{\mathbb{Z}_{(p)}}) \arrow[rr, "g_\bullet"] \arrow[d] & & \mathsf{BL}(\Lambda_{\mathbb{F}_p}) \\
\mathsf{BL}(\Lambda_{\mathbb{Z}_{(p)}})/\langle h^\bullet \Lambda_{\mathbb{Q}} \rangle{\downarrow} \arrow[r, "\cong"'] \arrow[urr] & \mathsf{BL}(D(\Lambda_{\mathbb{Z}_{(p)}})/\langle g^\bullet \Lambda_{\mathbb{F}_p} \rangle) \arrow[ur, "\cong"'] &
\end{tikzcd}$$

On the other hand, $\langle h_\bullet h^\bullet Y \rangle = \langle Y \rangle$ is not true for all $Y \in D(\Lambda_{\mathbb{Q}})$.

Theorem. (W.)

$$\mathsf{BL}(\Lambda_{\mathbb{Z}_{(p)}}) \xrightarrow{\sim} \langle g^\bullet \Lambda_{\mathbb{F}_p}\rangle\!\downarrow \times \langle h^\bullet \Lambda_{\mathbb{Q}}\rangle\!\downarrow, \text{ via}$$
$$\langle X\rangle \mapsto (\langle X \wedge g^\bullet \Lambda_{\mathbb{F}_p}\rangle, \langle X \wedge h^\bullet \Lambda_{\mathbb{Q}}\rangle).$$

From the work we did with quotients, we get the following.

Proposition. (W.)

$$\langle g^\bullet \Lambda_{\mathbb{F}_p}\rangle\!\downarrow \cong \mathsf{BL}(\Lambda_{\mathbb{F}_p}).$$

In fact, the inclusion functors induce the following isomorphisms.

Lemma. (W.)

$$\mathsf{BL}(\mathsf{loc}(g^\bullet \Lambda_{\mathbb{F}_p})) \xrightarrow{\sim} \langle g^\bullet \Lambda_{\mathbb{F}_p}\rangle\!\downarrow \subseteq \mathsf{BL}(\Lambda_{\mathbb{Z}_{(p)}}) \text{ and}$$
$$\mathsf{BL}(\mathsf{loc}(h^\bullet \Lambda_{\mathbb{Q}})) \xrightarrow{\sim} \langle h^\bullet \Lambda_{\mathbb{Q}}\rangle\!\downarrow \subseteq \mathsf{BL}(\Lambda_{\mathbb{Z}_{(p)}}).$$

Putting all this together, we get a complete splitting of the Bousfield lattice of $D(\Lambda_{\mathbb{Z}_{(p)}})$ and its sublattices.

Theorem. (W.)

$$\mathsf{BL}(\Lambda_{\mathbb{Z}_{(p)}}) \cong \mathsf{BL}(\Lambda_{\mathbb{F}_p}) \times \mathsf{BL}(\mathsf{loc}(h^\bullet \Lambda_{\mathbb{Q}})),$$
$$\mathsf{DL}(\Lambda_{\mathbb{Z}_{(p)}}) \cong \mathsf{DL}(\Lambda_{\mathbb{F}_p}) \times \mathsf{DL}(\mathsf{loc}(h^\bullet \Lambda_{\mathbb{Q}})),$$
$$\mathsf{BA}(\Lambda_{\mathbb{Z}_{(p)}}) \cong \mathsf{BA}(\Lambda_{\mathbb{F}_p}) \times \mathsf{BA}(\mathsf{loc}(h^\bullet \Lambda_{\mathbb{Q}})).$$

Corollary. (W.)

The cardinality of $\mathsf{BL}(\Lambda_{\mathbb{Z}_{(p)}})$ is $2^{2^{\aleph_0}}$.

... Thank you.