

Berezin-Toeplitz quantization of the spin-oscillator system

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A **Hamiltonian system** is a $2n$ -dimensional manifold equipped with:

- a symplectic form ω
- a Hamiltonian function $f \in C^\infty(M)$

Quantization of Hamiltonian systems

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These define a Hamiltonian vector field X_f by

$$\iota_{X_f}\omega = df$$

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- linearity
- $Q_1 = \text{Id}$
- $[Q_f, Q_g] = i\hbar Q_{\{f, g\}}$
- irreducibility

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No-go theorems

This is not possible.

Example:

S^1 acts on S^2 :

$$e^{i\theta} \cdot (h, \varphi) = (h, \varphi + \theta)$$

Take $\mu : M \rightarrow \mathbb{R}$ such that

$$\mu(h, \varphi) = h$$

.

Kähler toric manifolds

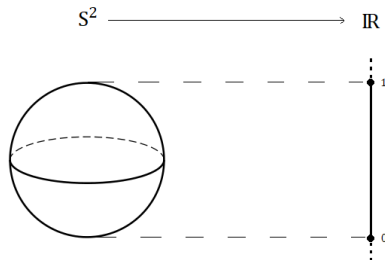
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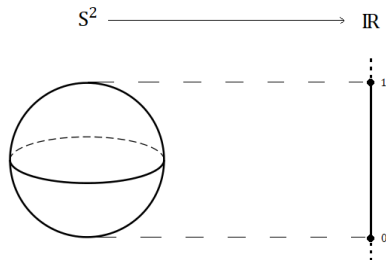
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Theorem (Delzant)

$\{\text{compact symplectic toric manifolds}\} \leftrightarrow \{\text{Delzant polytopes}\}$

$$(M^{2n}, \omega, \mathbb{T}^n, \mu) \mapsto \mu(M) \subset \mathbb{R}^n$$

Kähler toric manifolds

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such that $g(\cdot, \cdot) = \omega(\cdot, J\cdot)$ is a Riemannian metric.

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Theorem (Guillemin, Abreu)

A potential $g : \mu(M^{2n}) \rightarrow \mathbb{R}$ defines a compatible complex structure with local holomorphic coordinates

$$w_k = e^{\frac{\partial g}{\partial x_k} + i\theta_k}, k \in \{1, \dots, n\}$$

For the compact case:

$$\{\text{symplectic potentials}\} \leftrightarrow \{\text{compatible toric Kähler structures}\}$$

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$$\begin{aligned}(M, \omega) &\longmapsto H_J^0(M, L) \\ f &\longmapsto Q_f = P_J(f \cdot)\end{aligned}$$

where $H_J^0(M, L)$ is the space of J -holomorphic sections and P_J projection onto $H_J^0(M, L)$.

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$S^1 \times S^1$ acts on $S^2 \times \mathbb{R}^2$

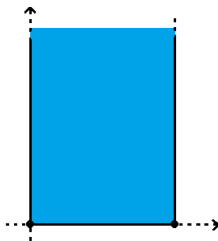
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with moment map $\mu(h, \varphi, r, \alpha) = (x_1, x_2) = \left(h, \frac{r^2}{2}\right)$.



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to get a family

$$g_s = g_0 + sH$$

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For each time s , there is an orthogonal basis for $H_{J_s}^0(M, L)$,

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Therefore

$$Q_f(p_{m_1, m_2}) = P_{J_s}(f \cdot p_{m_1, m_2}) = \sum_{n_1=0}^1 \sum_{n_2=0}^{\infty} \frac{\langle f \cdot p_{m_1, m_2}, p_{n_1, n_2} \rangle}{\|p_{n_1, n_2}\|^2} p_{n_1, n_2}$$

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$$\frac{\sqrt{\frac{s}{\pi}} \int_0^{+\infty} \left(\int_{x-1}^x f(x, x_2) F(x, x_2) dx_2 \right) e^{-s(x-m_1-m_2)^2} dx}{\sqrt{\frac{s}{\pi}} \int_0^{+\infty} \left(\int_{x-1}^x F(x, x_2) dx_2 \right) e^{-s(x-m_1-m_2)^2} dx}$$

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$$\xrightarrow{s \rightarrow +\infty} \frac{\int_{m_1+m_2-1}^{m_1+m_2} f(m_1+m_2, x_2) F(m_1+m_2, x_2) dx_2}{\int_{m_1+m_2-1}^{m_1+m_2} F(m_1+m_2, x_2) dx_2}$$

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$$H(x_1, x_2, \theta_1, \theta_2) = \frac{x_1^2 + x_2^2}{2}$$

The coefficients do **not converge** as $s \rightarrow +\infty$.