

Random flights in 3 dimensions

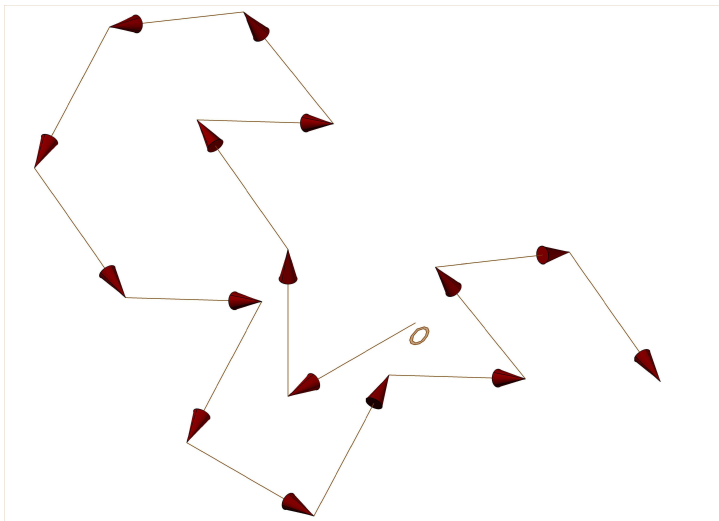
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 - The jumps are mutually independent

Random flights in 3 dimensions

- Because of the isotropy of space
 - The probability to find the particle after n jumps in a small cube doesn't depend on the cube's normed position vector

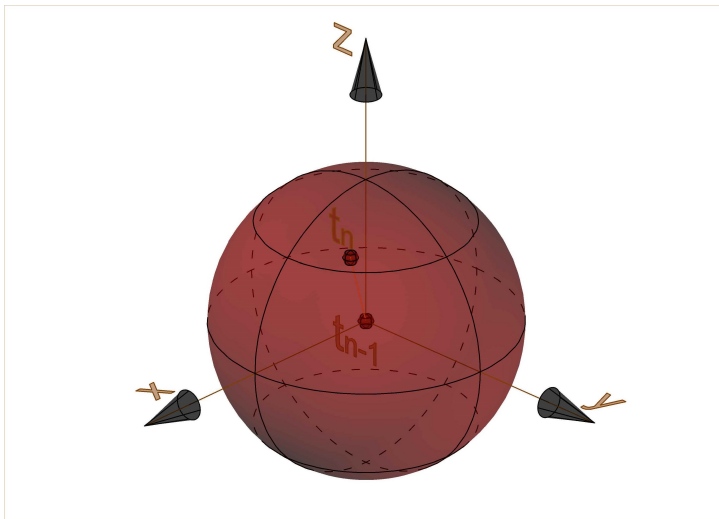
Random flights in 3 dimensions

- Because of the isotropy of space
 - The probability to find the particle after n jumps in a small cube doesn't depend on the cube's normed position vector
 - There's only need to study its distance from the origin

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 - define a local coordinate system:
 - its origin is the position of the particle at the instant t_{n-1}
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 - and define
 - Z_n as the random variable that indicates the z-coordinate of the position of the particle at the instant t_n according to this local coordinate system

Study of Z_n



- Between consecutive instants, the particle jumps exactly one unit
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- in the case $z > 1$

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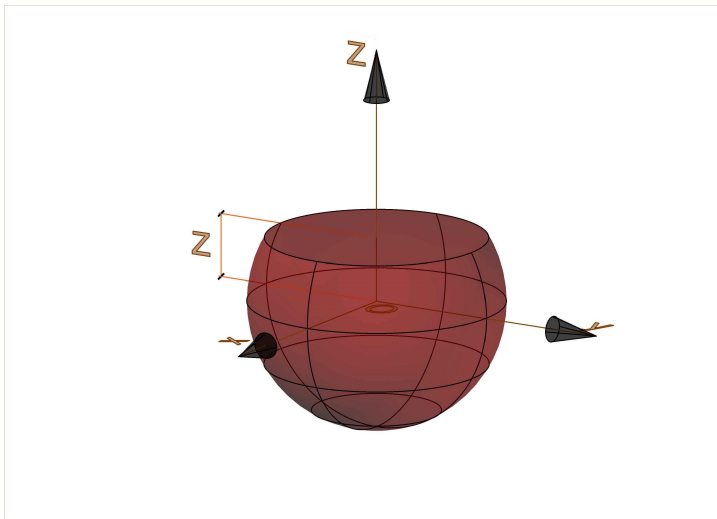
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 - Thus in case $z \in [-1, 1]$

$$P(Z_n \leq z) = P(P_n \in S_z) = \frac{V_{S_z}}{V_{S_1}} = \frac{2\pi(1+z)}{4\pi} = \frac{1+z}{2}$$

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- Distribution function of Z_n :

$$F_{Z_n}(z) = P(Z_n \leq z) = \begin{cases} 0 & \text{if } z < -1 \\ \frac{1+z}{2} & \text{if } z \in [-1, 1] \\ 1 & \text{if } z > 1 \end{cases}$$

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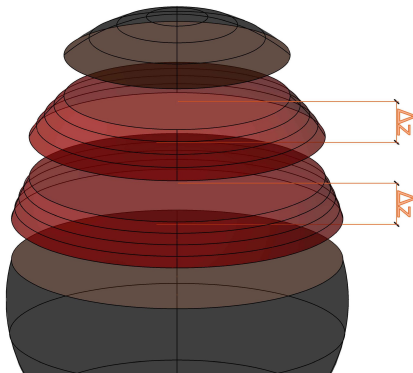
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- Thus

$$Z_n \sim U([-1, 1])$$

Archimedes's Theorem

The area of the portion of the spherical surface delimited by two parallel planes that intersect it only depends on the radius and the distance between the planes



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Recurrence expression for R_n

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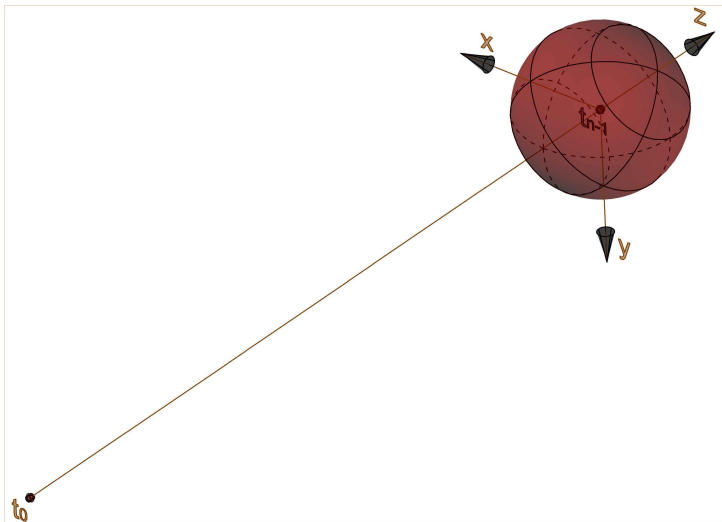
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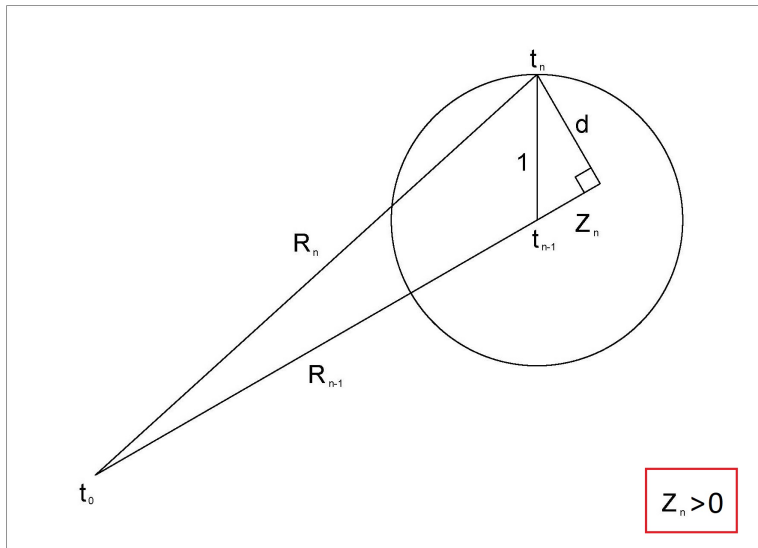
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$$Z_n^2 + d^2 = 1$$

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- We know that $R_0 = 0$ so

$$R_1 = 1, \quad R_2 = \sqrt{2 + 2Z_2}, \quad R_3 = \sqrt{3 + 2Z_2 + 2Z_3\sqrt{2 + 2Z_2}},$$

$$R_4 = \sqrt{4 + 2Z_2 + 2Z_3\sqrt{2 + 2Z_2} + 2Z_4\sqrt{3 + 2Z_2 + 2Z_3\sqrt{2 + 2Z_2}}}, \dots$$

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- Since the directions of the steps are mutually independent
 - according to well known theorems of probability

$$f_{(Z_2, \dots, Z_n)}(z_2, \dots, z_n) = \prod_{j=2}^n f_{Z_j}^*(z_j) = \frac{1}{2^{n-1}} \mathbb{1}_{]-1,1[^{n-1}}(z_2, \dots, z_n)$$

Example: Study of R_3

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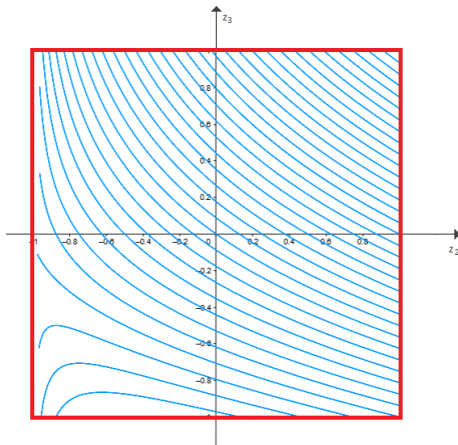
$$F_{R_3}(r) = \iint_{S_3(r)} f_{(Z_2, Z_3)}(z_2, z_3) dz_2 dz_3 = \frac{V(S_3(r))}{4}$$

with

$$S_3(r) = \left\{ (z_2, z_3) \in]-1, 1[^2 : \sqrt{3 + 2z_2 + 2z_3\sqrt{2 + 2z_2}} \leq r \right\}$$

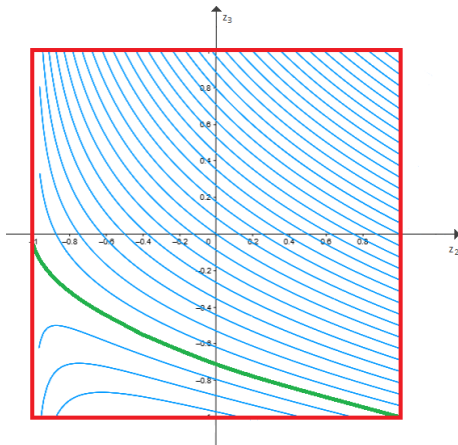
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$$\phi_3(z_2, z_3) = \sqrt{3 + 2z_2 + 2z_3\sqrt{2 + 2z_2}}$$



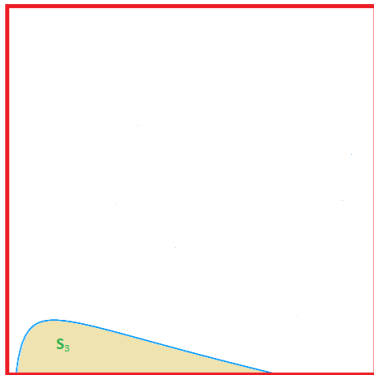
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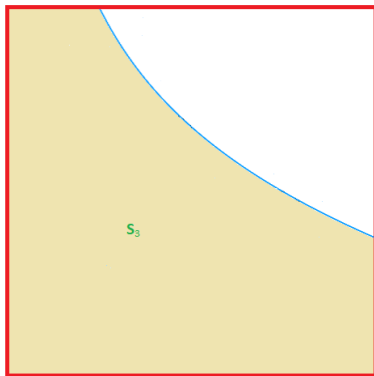
- When $r \in]0, 1[$

$$V(S_3(r)) = \int_{-1}^{-\sqrt{1-r^2}} \left(\frac{(r^2 - 3 + 2z^2) - 2z\sqrt{z^2 + r^2 - 1}}{2} - \frac{(r^2 - 3 + 2z^2) + 2z\sqrt{z^2 + r^2 - 1}}{2} \right) dz$$

$$V(S_3(r)) = \frac{2}{3}r^3$$

Example: Study of R_3

$$r \in [1, 3[$$



Example: Study of R_3

- When $r \in [1, 3[$

$$V(S_3(r)) = \int_{-1}^{\frac{r^2}{4} - \frac{5}{4}} 1 - (-1) dz + \int_{\frac{r^2}{4} - \frac{5}{4}}^1 \left(\frac{r^2 - 3 + 2z^2 - 2z\sqrt{z^2 + r^2 - 1}}{2} - (-1) \right) dz$$

$$V(S_3(r)) = -\frac{r^3}{3} + \frac{3}{2}r^2 - \frac{1}{2}$$

Example: Study of R_3

- Distribution function of R_3 :

$$F_{R_3}(r) = P(R_3 \leq r) = \begin{cases} 0 & \text{if } r \leq 0 \\ \frac{r^3}{6} & \text{if } r \in]0, 1[\\ -\frac{r^3}{12} + \frac{3}{8}r^2 - \frac{1}{8} & \text{if } r \in [1, 3[\\ 1 & \text{if } r \geq 3 \end{cases}$$

- Density function of R_3 :

$$f_{R_3}(r) = \begin{cases} 0 & \text{if } r \notin]0, 3[\\ \frac{r^2}{2} & \text{if } r \in]0, 1[\\ -\frac{r^2}{4} + \frac{3}{4}r & \text{if } r \in [1, 3[\end{cases}$$

- For each $n \in \mathbb{N} \setminus \{1\}$ and each $r \in \mathbb{R}$

$$F_{R_n}(r) = \int_{S_n(r)} f_{(Z_2, \dots, Z_n)}(z_2, \dots, z_n) dz_2 \dots dz_n = \frac{V(S_n(r))}{2^{n-1}}$$

with

$$S_n(r) = \left\{ (z_2, \dots, z_n) \in]-1, 1[^{n-1} : \phi_n(z_2, \dots, z_n) \leq r \right\},$$

where ϕ_n are complicated, inductively defined expressions

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 - a direct computation of $V(S_n(r))$ along these lines is very hard

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where ϕ_n are complicated, inductively defined expressions

- But when $n > 3$
 - a direct computation of $V(S_n(r))$ along these lines is very hard
- Using a multivariate change of variables we can solve the problem for all $n \in \mathbb{N} \setminus \{1\}$

Study of $(R_2, \dots, R_n)$

- For each $n \in \mathbb{N} \setminus \{1\}$ let's define

$$\begin{aligned} \phi: \quad U &\rightarrow V \\ (z_2, \dots, z_n) &\mapsto (\phi_2(z_2, \dots, z_n), \dots, \phi_n(z_2, \dots, z_n)) \end{aligned}$$

with

$$U =]-1, 1[^{n-1}$$

$$V = \{(r_2, \dots, r_n) \in \mathbb{R}^{n-1} : r_2 \in]0, 2[\wedge r_j \in [|r_{j-1} - 1|, r_{j-1} + 1[, \forall j \in \{3, \dots, n\}\}$$

$$\phi_2(z_2, \dots, z_n) = \sqrt{2 + 2z_2}$$

$$\phi_j(z_2, \dots, z_n) = \sqrt{\phi_{j-1}^2(z_2, \dots, z_n) + 2z_j\phi_{j-1}(z_2, \dots, z_n) + 1}, \forall j \in \{3, \dots, n\}$$

Study of $(R_2, \dots, R_n)$

- ϕ is bijective and

$$\begin{array}{ccc} \phi^{-1}: & V & \rightarrow U \\ & (r_2, \dots, r_n) & \mapsto ((\phi^{-1})_2(r_2, \dots, r_n), \dots, (\phi^{-1})_n(r_2, \dots, r_n)) \end{array}$$

with

$$(\phi^{-1})_2(r_2, \dots, r_n) = \frac{r_2^2}{2} - 1$$

$$(\phi^{-1})_j(r_2, \dots, r_n) = \frac{r_j^2}{2r_{j-1}} - \frac{r_{j-1}}{2} - \frac{1}{2r_{j-1}}, \forall j \in \{3, \dots, n\}$$

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- It can be proved that the support of $(R_2, \dots, R_n)$ is V and

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- $f_{(Z_2, \dots, Z_n)}$ is the density of $(Z_2, \dots, Z_n)$, so

$$P((Z_2, \dots, Z_n) \in (\phi^{-1}(B))) = \int_{\phi^{-1}(B)} f_{(Z_2, \dots, Z_n)}(z_2, \dots, z_n) dz_2 \dots dz_n$$

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- According to the theorem of integration by substitution the last expression is equal to

$$\int_B f_{(Z_2, \dots, Z_n)}(\phi^{-1}(r_2, \dots, r_n)) |J(\phi^{-1}(r_2, \dots, r_n))| dr_2 \dots dr_n$$

Study of $(R_2, \dots, R_n)$

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- We can compute from a formula we had earlier, that

$$f_{(Z_2, \dots, Z_n)}(\phi^{-1}(r_2, \dots, r_n)) = \frac{1}{2^{n-1}} \mathbb{1}_V(r_2, \dots, r_n)$$

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Recurrence expression for f_{R_n}

- For each $n \in \mathbb{N} \setminus \{1, 2\}$
 - according the definition of V we have

$$f_{(R_2, \dots, R_n)}(r_2, \dots, r_n) = \frac{r_n}{2^{n-1}} \mathbb{1}_{]0, 2[}(r_2) \dots \mathbb{1}_{|r_{n-1}-1|, r_{n-1}+1[}(r_n)$$

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Recurrence expression for f_{R_n}

- For each $n \in \mathbb{N} \setminus \{1, 2\}$
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- Thus

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Recurrence expression for f_{R_n}

- It follows by a well known theorem on marginal variables that
 - R_n is absolutely continuous and

$$f_{R_n}(r_n) = \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} f_{(R_2, \dots, R_n)}(r_2, \dots, r_n) dr_2 \dots dr_{n-1}$$

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- By interchanging integrations with the function not depending on $r_2, \dots, r_{n-2}$, the last expression is equal to

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- Again according to the theorem on marginal variables

$$f_{R_n}(r_n) = \int_{-\infty}^{+\infty} \frac{r_n}{2r_{n-1}} f_{R_{n-1}}(r_{n-1}) \mathbb{1}_{|r_{n-1}-1|, r_{n-1}+1}[(r_n) dr_{n-1}]$$

- By computing $f_{R_n}(r)$ for some small n , one conjectures that
 - for each $n \in \mathbb{N} \setminus \{1\}$

$$f_{R_n}(r) = \sum_{j=0}^{\lceil \frac{n}{2} \rceil - 1} g_{n,j}(r) \mathbb{1}_{]0, n-2j[}(r)$$

- with

$$g_{n,j}(r) = \frac{(-1)^j \binom{n}{j}}{2^{n-1} (n-2)!} (n-2j-r)^{n-2} r$$

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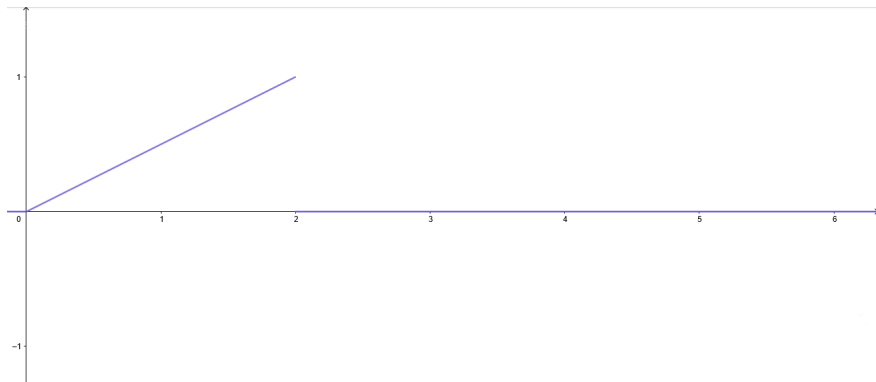
- with

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- In fact, one can prove this formula for general n ($n \in \mathbb{N} \setminus \{1\}$) by induction, Pascal's Rule and manipulation of complicated expressions

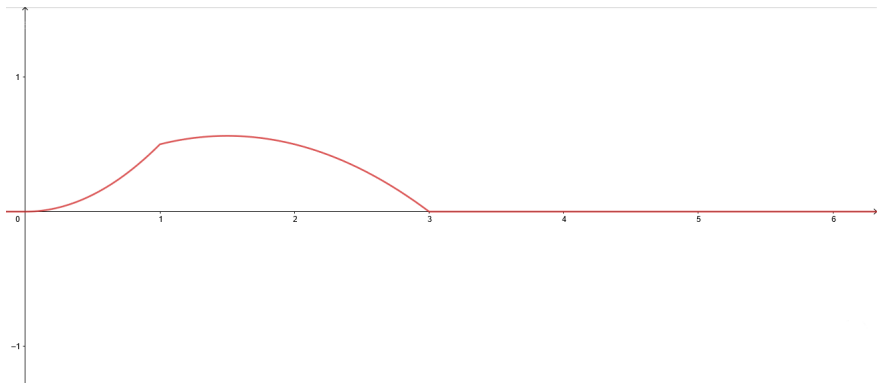
Study of R_n

- Graphics of some densities
 - f_{R_2}



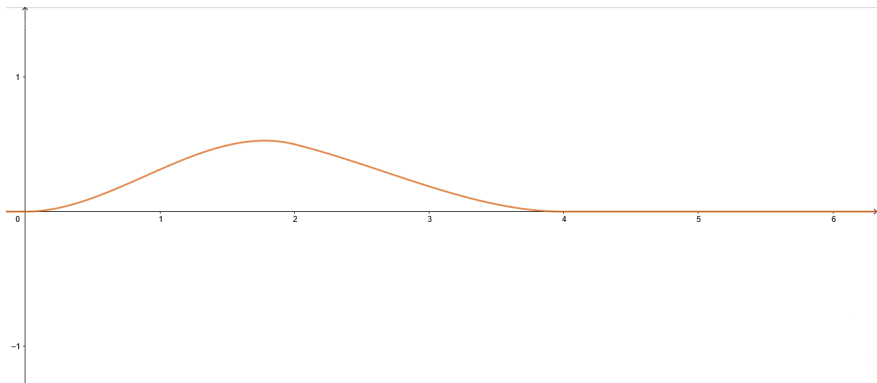
Study of R_n

- Graphics of some densities
 - f_{R_3}



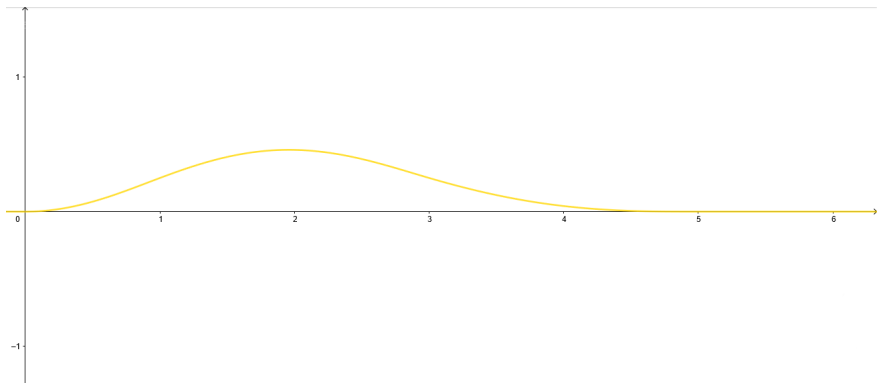
Study of R_n

- Graphics of some densities
 - f_{R_4}

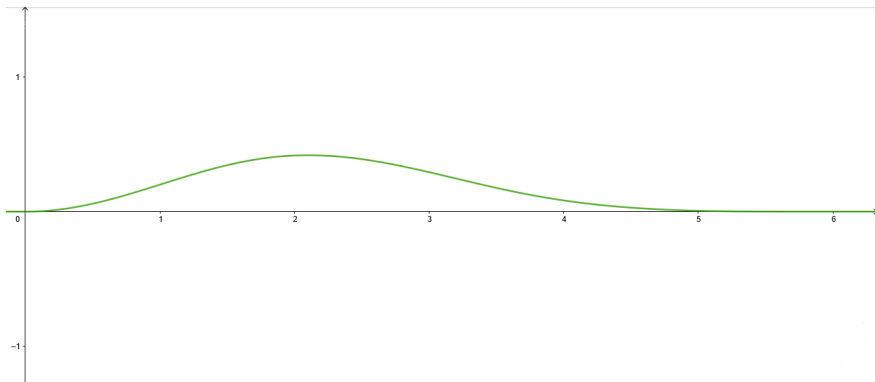


Study of R_n

- Graphics of some densities
 - f_{R_5}

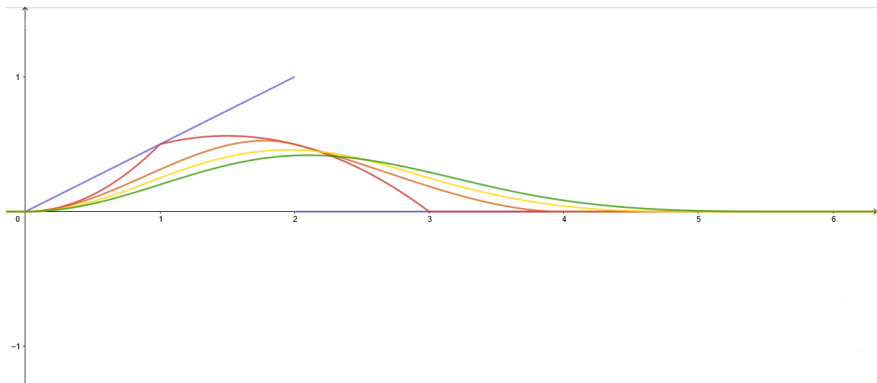


- Graphics of some densities
 - f_{R_6}



Study of R_n

- Graphics of some densities
 - $f_{R_2}, f_{R_3}, f_{R_4}, f_{R_5}, f_{R_6}$



- According to a well known theorem of probability and definitions of the distribution function
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- After an integration by parts the distribution function of R_n is

$$F_{R_n}(r) = P(R_n \leq r) = \sum_{j=0}^{\lceil \frac{n}{2} \rceil - 1} G_{n,j}(r) \mathbb{1}_{]0, n-2j[}(r) + \mathbb{1}_{\mathbb{R}^+}(r)$$

- with

$$G_{n,j}(r) = \frac{(-1)^{j+1} \binom{n}{j}}{2^{n-1} n!} (n-2j-r)^{n-1} [(n-1)r + n-2j]$$

**THANK YOU FOR
YOUR ATTENTION**