

Kähler Geometry and Hamiltonian Flows in Imaginary Time

Pedro Magalhães

3rd year (IST – UL)

September 7, 2019

Definition:

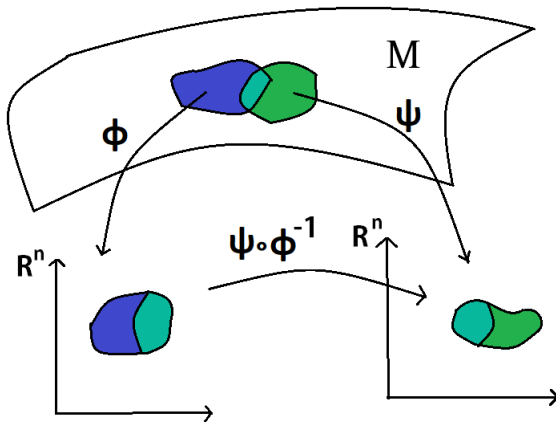
A topological manifold M is a topological space such that:

- M is Hausdorff;
- M is second-countable;
- $\exists n \in \mathbb{N}$ such that M is locally homeomorphic to $\mathbb{R}^n$.

For each $p \in M$, there exists a neighborhood U homeomorphic to an open subset of $\mathbb{R}^n$.

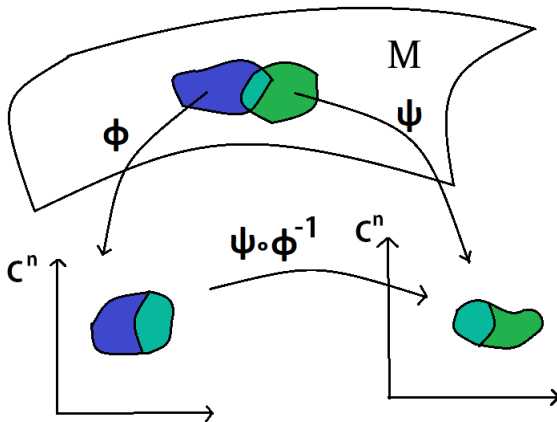
If we denote the homeomorphism by ϕ we call the pair (U, ϕ) a chart.

Smooth Manifolds



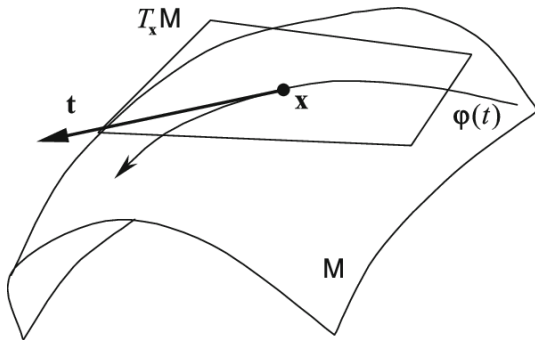
$\psi \circ \phi^{-1} : \phi(U \cap V) \rightarrow \psi(U \cap V)$ and its inverse are smooth.

Complex Manifolds



$\psi \circ \phi^{-1} : \phi(U \cap V) \rightarrow \psi(U \cap V)$ and its inverse are holomorphic.

Tangent Space



Tangent vectors are defined at every point of M .

Flow of a Vector Field

Let X be a smooth vector field. Given $p \in M$, there exists a unique curve $c_p : (-\epsilon, \epsilon) \rightarrow M$ such that:

- $c_p(0) = p$
- $\forall t \in (-\epsilon, \epsilon), \quad \dot{c}_p(t) = X_{c_p(t)}$

We say that c_p is an integral curve of X in p .

Flow of a Vector Field

There also exists a neighborhood of p , U , such that, fixing $t \in (-\epsilon, \epsilon)$, the map

$$\begin{aligned}\psi_t : U &\rightarrow M \\ q &\mapsto c_q(t)\end{aligned}$$

is smooth.

We call ψ_t the flow of X .

Definition:

A Riemannian manifold is a pair (M, g) where M is a smooth manifold and g is a covariant, smooth, symmetric and positive definite 2-tensor field.

In each $p \in M$, g is a metric in the tangent space of M at p . This gives us a way to compute norms of vectors, angles between vectors and lengths of curves.

Definition:

A Symplectic manifold is a pair (M, ω) where M is a smooth manifold and ω is a 2-form such that:

- ω is closed ($d\omega = 0$);
- $\forall p \in M, \omega_p$ is anti-symmetric and non-degenerate.

Given $H : M \rightarrow \mathbb{R}$ smooth,

ω and H induce a unique smooth vector field that we call the Hamiltonian vector field and denote by X_H .

Compatible Structures

M complex manifold $\rightarrow$ complex structure J .

$\forall p \in M$,

- $J_p : T_p M \rightarrow T_p M$ Linear Transformation;
- $J_p^2 = -Id$.

We say that the triple (g, ω, J) is compatible if it satisfies one of the following compatibility relations:

$\forall p \in M, \forall u, v \in T_p M$,

$$g_p(u, v) = \omega_p(u, J_p v)$$

$$\omega_p(u, v) = g_p(J_p u, v)$$

Definition:

A Kähler manifold is a tuple (M, ω, g, J) such that:

- M is a complex manifold;
- ω is a symplectic form;
- g is a Riemannian metric;
- J is a complex structure;
- The triple (g, ω, J) is compatible.

Flows in Imaginary Time

Suppose we have

- (M, ω, g, J) compact Kähler manifold where all the structures are real-analytic.
- $h : M \rightarrow \mathbb{R}$ real-analytic function.

Let X_h be the Hamiltonian vector field induced by h .

→ The flow of X_h , ψ_t , is also real-analytic.

Flows in Imaginary Time

Given $p \in M$, there exists a neighborhood U of p and holomorphic coordinates $\{z^i\}_{i=1,\dots,n}$ defined in U .

We act with the flow of X_h on $\{z^i\}_{i=1,\dots,n}$:

$$z_t^i = z^i \circ \psi_t$$

M and ψ_t real analytic $\Rightarrow z_t^i = \sum_{k=0}^{\infty} a_i t^k$

Flows in Imaginary Time

$$\sum_{k=0}^{\infty} a_i t^k \rightsquigarrow \sum_{k=0}^{\infty} a_i \tau^k, \quad \tau \in \mathbb{C}$$

There exists $T > 0$ such that:

$$\tau \in \{\tau \in \mathbb{C} : |\tau| < T\} \implies \{z_{\tau}^i\}_{i=1,\dots,n} \text{ induce } J_{\tau}$$

In a neighborhood $V \subset U$.

Flows in Imaginary Time

M compact $\Rightarrow J_\tau$ can be made global.

And J_τ is compatible with ω and defines for
 $p \in M$ and $u, v \in T_p M$,

$$g_{\tau p}(u, v) := \omega_p(u, J_{\tau p} v)$$

Which is a Riemannian metric.

Flows in Imaginary Time

This means that

$$(M, \omega, g_\tau, J_\tau)$$

Is a Kähler manifold!

And it will not be (in general) isomorphic to the original manifold (M, ω, g, J) .

So the compact case is understood, let's look at an example of a non compact manifold.

$$H = \{(x, y) \in \mathbb{R}^2 : y > 0\}$$

- Global chart $(\iota_{H \rightarrow \mathbb{C}}, H)$,
- J standard complex structure,
- $g = \frac{1}{y^2}(dx \otimes dx + dy \otimes dy)$,
- $\omega = \frac{1}{y^2}dx \wedge dy$.

Select $h : H \rightarrow \mathbb{R}$.

$$h(x, y) = f(y)$$

$$X_h = y^2 f'(y) \frac{\partial}{\partial x}$$

We can calculate z_τ by

$$z_\tau = \sum_{k=0}^{\infty} \frac{x_h^k(z)}{k!} \tau^k$$

Since

$$X_h(z) = X_h(x + iy) = y^2 f'(y)$$

$$X_h^2(z) = X_h(y^2 f'(y)) = 0$$

We have

$$z_\tau = (x + iy) + y^2 f'(y) \tau$$

Writing $\tau = r + is$, we have

$$z_\tau = \underbrace{(x + ry^2 f'(y))}_{x_\tau} + i \underbrace{(y + sy^2 f'(y))}_{y_\tau}$$

$$z_\tau \longrightarrow J_\tau$$

And using the compatibility relation between ω and J_τ :

$$g_\tau(\cdot, \cdot) = \omega(\cdot, J_\tau(\cdot))$$

We obtain

$$g_\tau = \frac{1}{y^2(1 + s\psi(y))}(dx_\tau \otimes dx_\tau + dy_\tau \otimes dy_\tau)$$

Which is a Riemannian metric if we have, for example, $s > 0$ and $\psi(y) > 0$ for all $y > 0$, where

$$\psi(y) = (y^2 f'(y))' = 2yf'(y) + y^2 f''(y)$$

So in those conditions $(H, \omega, g_\tau, J_\tau)$ will be a Kähler manifold.

For each appropriate $\tau \rightarrow (H, \omega, g_\tau, J_\tau)$ is a Kähler manifold.

For each appropriate $\tau \rightarrow (H, \omega, g_\tau, J_\tau)$ is a Kähler manifold.

Consider the space of Kähler metrics for H .

For each appropriate $\tau \rightarrow (H, \omega, g_\tau, J_\tau)$ is a Kähler manifold.

Consider the space of Kähler metrics for H .

If we have $\tau = is, s \in \mathbb{R}$,

It can be shown that there exists a connection for which the curve

$$s \mapsto g_\tau$$

Is actually a geodesic of this space.

Let's compute some properties of the new manifolds. For example, their scalar curvature will be

$$S = \frac{s\psi''(y)y^2 - 2(1 + s\psi(y))^2 - 2sy\psi'(y)(1 + s\psi(y) + sy\psi'(y))}{(1 + s\psi(y))^3}$$

Let's compute some properties of the new manifolds. For example, their scalar curvature will be

$$S = \frac{s\psi''(y)y^2 - 2(1 + s\psi(y))^2 - 2sy\psi'(y)(1 + s\psi(y) + sy\psi'(y))}{(1 + s\psi(y))^3}$$

$$\lim_{s \rightarrow \infty} S = 0$$

$$g_\tau\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}\right) = \frac{1}{y^2(1+s\psi(y))} \rightarrow_{s \rightarrow \infty} 0$$

$$g_\tau\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) = \frac{r\psi(y)}{y^2(1+s\psi(y))} \rightarrow_{s \rightarrow \infty} 0$$

$$g_\tau\left(\frac{\partial}{\partial y}, \frac{\partial}{\partial y}\right) = \frac{r^2\psi(y)^2 + (1+s\psi(y))^2}{y^2(1+s\psi(y))} \rightarrow_{s \rightarrow \infty} \infty$$

$$\frac{\partial}{\partial z_\tau} = \frac{1}{2} \left(\frac{\partial}{\partial x_\tau} - i \frac{\partial}{\partial y_\tau} \right)$$

$$\begin{aligned} \lim_{s \rightarrow \infty} \left\langle \frac{\partial}{\partial z_\tau} \right\rangle &= \lim_{s \rightarrow \infty} \left\langle \frac{1}{2} \left(\frac{\partial}{\partial x_\tau} - i \frac{\partial}{\partial y_\tau} \right) \right\rangle = \\ &= \lim_{s \rightarrow \infty} \left\langle \left(1 + \frac{ir\psi(y)}{1+s\psi(y)} \right) \frac{\partial}{\partial x} - \frac{i}{1+s\psi(y)} \frac{\partial}{\partial y} \right\rangle = \left\langle \frac{\partial}{\partial x} \right\rangle \end{aligned}$$

It can also be interesting to apply the flow of X_h on x and repeat the process:

$$w_\tau = \sum_{k=0}^{\infty} \frac{X_h^k(x)}{k!} \tau^k = x + y^2 f'(y) \tau$$

$$\tau = r + is$$

$$w_\tau = \underbrace{x + ry^2 f'(y)}_{x_\tau} + i \underbrace{sy^2 f'(y)}_{y_\tau}$$

Checking if w_τ is a good complex coordinate:

$$dw_\tau \wedge d\overline{w}_\tau = -2is\psi(y)dx \wedge dy \neq 0$$

If, for instance, $s > 0$ and $\psi(y) > 0$ for all $y > 0$.

Checking if w_τ is a good complex coordinate:

$$dw_\tau \wedge d\overline{w}_\tau = -2is\psi(y)dx \wedge dy \neq 0$$

If, for instance, $s > 0$ and $\psi(y) > 0$ for all $y > 0$.

Repeating the computations we did before, we obtain:

$$g_\tau = \frac{1}{y^2 s \psi(y)} (dx_\tau \otimes dx_\tau + dy_\tau \otimes dy_\tau)$$

And once again, if we let $s > 0$ and $\psi(y) > 0$ for all $y > 0$, g_τ is a Riemannian metric.

$$S = \frac{-2(\psi(y))^2 - 2y^2(\psi'(y))^2 + y\psi(y)(-2\psi'(y) + y\psi''(y))}{s(\psi(y))^3} \rightarrow_{s \rightarrow \infty} 0$$

$$g_\tau \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial x} \right) = \frac{1}{y^2 s \psi(y)} \rightarrow_{s \rightarrow \infty} 0$$

$$g_\tau \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right) = \frac{r\psi(y)}{y^2 s \psi(y)} \rightarrow_{s \rightarrow \infty} 0$$

$$g_\tau \left(\frac{\partial}{\partial y}, \frac{\partial}{\partial y} \right) = \frac{(s^2 + r^2)(\psi(y))^2}{y^2 s \psi(y)} \rightarrow_{s \rightarrow \infty} \infty$$

$$\lim_{s \rightarrow \infty} \left\langle \frac{\partial}{\partial w_\tau} \right\rangle = \left\langle \frac{\partial}{\partial x} \right\rangle$$

What does $\psi(y) > 0$ mean in terms of $f(y)$?

$$\psi(y) = (y^2 f'(y))' = 2y f'(y) + y^2 f''(y)$$

What does $\psi(y) > 0$ mean in terms of $f(y)$?

$$\psi(y) = (y^2 f'(y))' = 2y f'(y) + y^2 f''(y)$$

Consider the function

$$u(y) = -\frac{1}{y}$$

What does $\psi(y) > 0$ mean in terms of $f(y)$?

$$\psi(y) = (y^2 f'(y))' = 2yf'(y) + y^2 f''(y)$$

Consider the function

$$u(y) = -\frac{1}{y}$$

If we change coordinates

$$(x, y) \rightarrow (x, u)$$

And write $\psi(y)$ in terms of u , we obtain

$$\psi(y(u)) = u^2 \frac{d^2 f(y(u))}{du^2}$$

Which means that

$$\psi(y) > 0 \Leftrightarrow f \text{ is convex in the coordinate } u$$

Some formulas we obtained can be written in a simpler way using these new coordinates.

Some formulas we obtained can be written in a simpler way using these new coordinates.

For starters,

$$\omega = \frac{1}{y^2} dx \wedge dy = dx \wedge du$$

Some formulas we obtained can be written in a simpler way using these new coordinates.

For starters,

$$\omega = \frac{1}{y^2} dx \wedge dy = dx \wedge du$$

The curvature in the case w_τ :

$$S = -\frac{1}{s} \frac{d^2}{du^2} \left(\frac{1}{\frac{d^2 f}{du^2}} \right)$$

In conclusion,

- From our initial Kähler manifold (H, ω, g, J) , we can obtain new Kähler manifolds $(H, \omega, g_\tau, J_\tau)$,
- These new Kähler structures define a geodesic in the space of Kähler metrics,
- Despite H being non-compact, these geodesics exist for infinite time,
- It's interesting to note the similarities between our manifold and other examples of this construction such as toric manifolds.