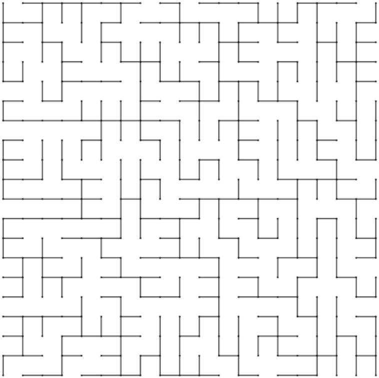
USF and LERW

Perla Sousi [1]

[1] University of Cambridge

UST on $\mathbb{Z}^2$



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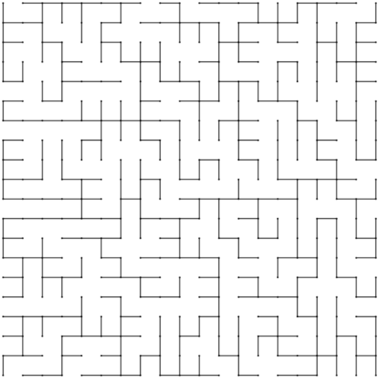

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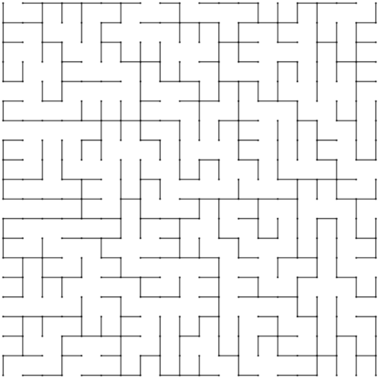

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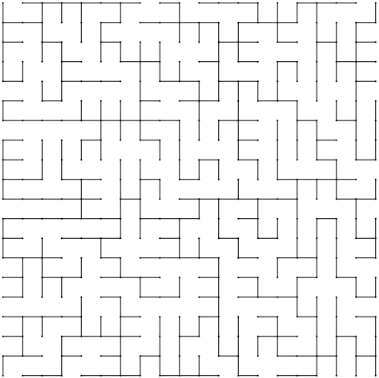

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This is the uniform spanning tree of $\mathbb{Z}^2$

Higher dimensions

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What happens in $\mathbb{Z}^3$*? What is the canonical way to define a UST?*

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Is the USF connected?

USF – connectivity

Theorem (Pemantle (1991))

The USF on $\mathbb{Z}^d$ has one tree with probability 1 for $d \leq 4$ and infinitely many trees for $d \geq 5$.

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Proof idea

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The USF on $\mathbb{Z}^d$ has one tree with probability 1 for $d \leq 4$ and infinitely many trees for $d \geq 5$.

Proof idea

By Wilson's algorithm, it reduces to the question of intersections of two independent SRW's on $\mathbb{Z}^d$.

USF – more properties

More refined connectivity properties of the USF change with dimension.

Theorem (Benjamini, Kesten, Peres and Schramm (2004))

For $5 \leq d \leq 8$ every tree in the USF is adjacent to every other tree in the USF. For $d \geq 9$ there are trees that are adjacent to none.

Lots of other properties of USF's change with dimension.

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Proof uses electrical network theory.

Recurrence of USF

Conjecture

Consider the tree containing 0 *in the USF in* $\mathbb{Z}^d$ *for* $d \geq 5$. *Take the induced subgraph of* $\mathbb{Z}^d$ *that this defines (connect all vertices of the tree that are adjacent in* $\mathbb{Z}^d$*). Is this graph recurrent?*

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Yes, when $d \geq 8$*.*

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Open what happens for $d = 5, 6, 7$.

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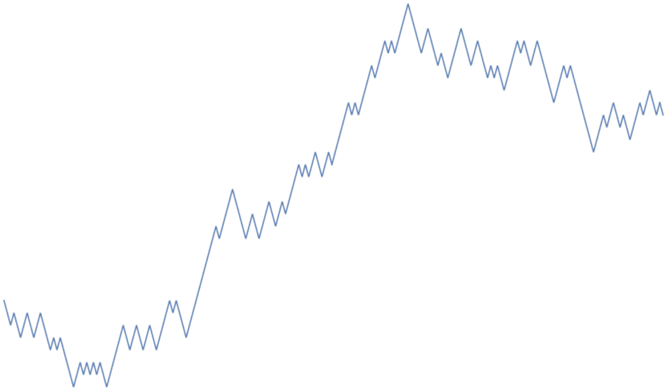

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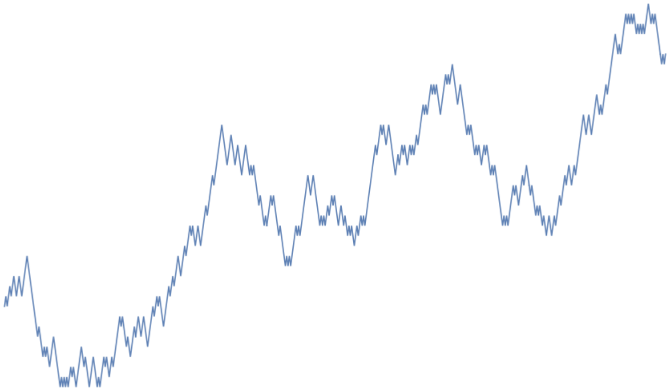

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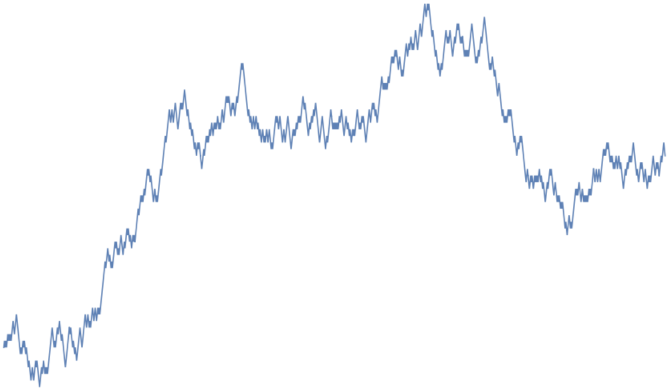

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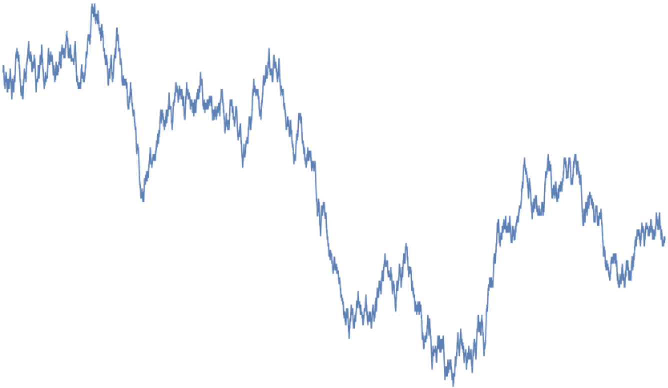

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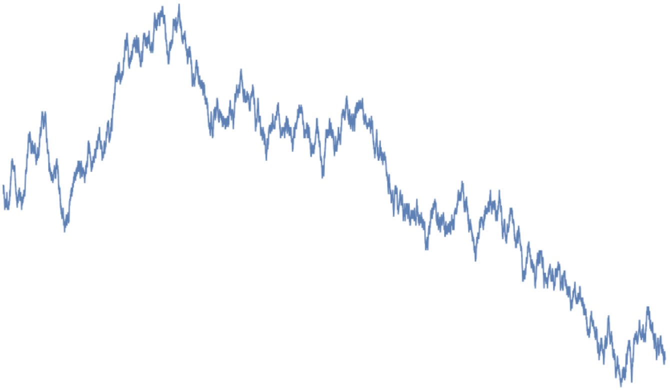

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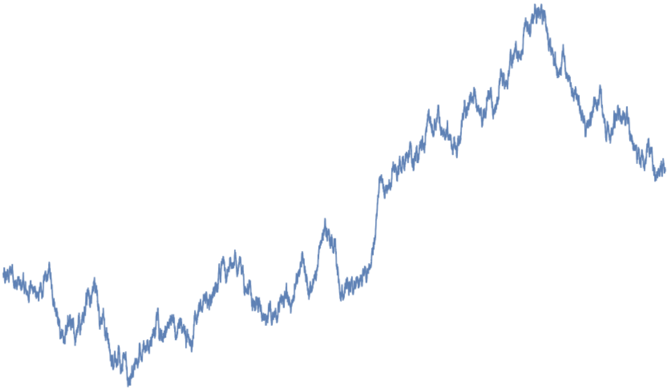

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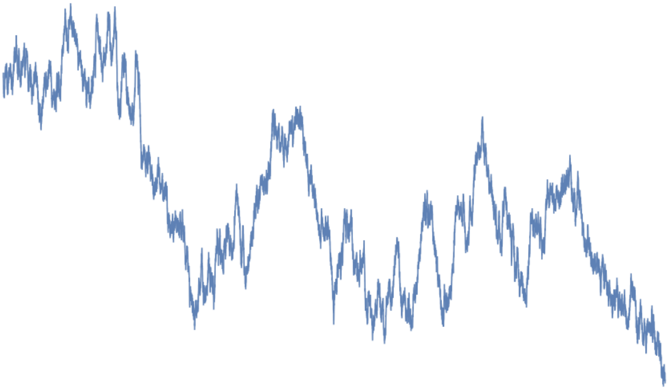

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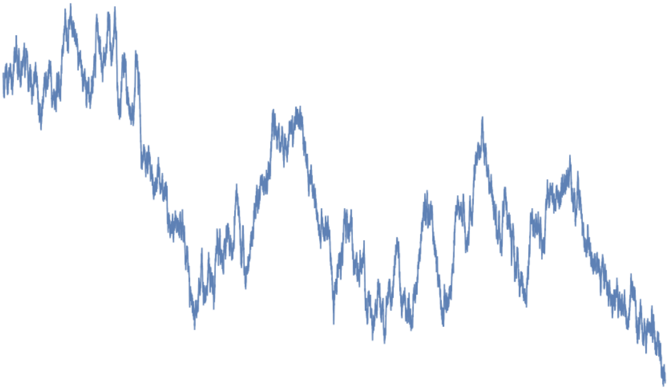

Brownian motion is a fractal curve – central object in *modern probability*.

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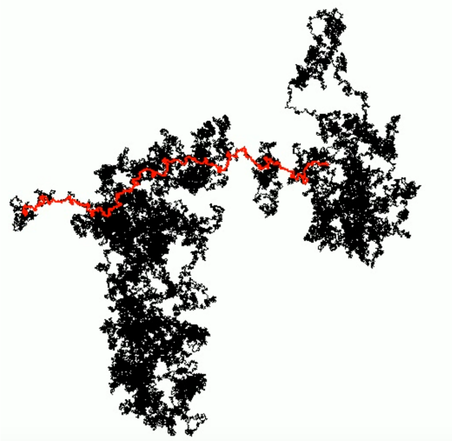

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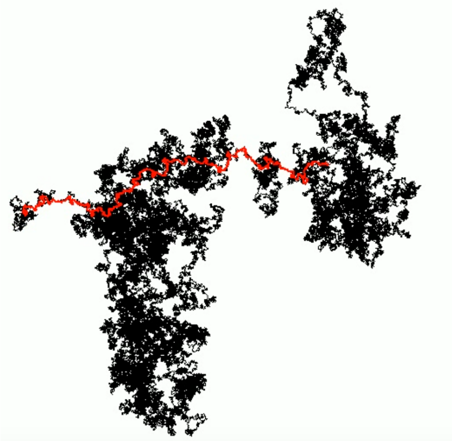

SLE(κ) for different values of κ are the proven or conjectured scaling limits of macroscopic interfaces in several models from statistical physics.

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Question

$\boxed{d = 3}$ *What is the scaling factor? What is the expected displacement?*

Thanks!