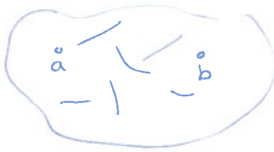
But since the set is finite ~~xxxxx~~ the successive edges $\vec{e_i}$ will eventually close. But this contradicts the cycle law. $\square$

Effective resistance:



replace by

what resistance should we put here so that with the given voltage on a, b the same current will flow?

Let W_0 be a voltage with $W_0(a)=1$ and $W_0(b)=0$.

Let W be another voltage. Then $\dfrac{W(x)-W(b)}{W(a)-W(b)} = W_0(x)$ by uniqueness.

Hence $W(x) = (W(a)-W(b))\,W_0(x) + W(b)$

Let I_0 be the current associated to W_0 and I the current associated to W

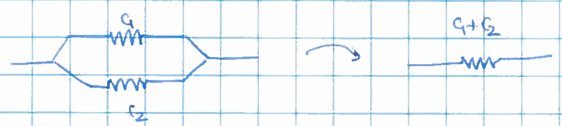
$$\|I\| = \sum_{x\sim a} I(a,x) = \sum_{x\sim a} c(a,x)\,(W(a)-W(x)) = \|I_0\|\,(W(a)-W(b))$$

Thus $\dfrac{W(a)-W(b)}{\|I\|}$ is independent of I, W. This is what is called the effective resistance $R_{eff}(a,b)$

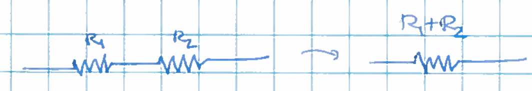
The effective conductance is $C_{eff}(a,b) = \dfrac{1}{R_{eff}(a,b)}$

Some laws for computing effective resistance:

Conductance in parallel

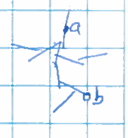


Resist in series



Gluing of points with same voltage

Exercise: If T is a finite connected tree with weights all equal. $R_{eff}(a,b) = d(a,b)$



Connection between random walks and electrical networks:

Theorem: Let X be a reversible MC on Ω (think of it as a WRW) and let $\tau_x = \min\{t\geq 0 : X_t = x\}$

$\tau_x^+ = \min\{t\geq 1 : X_t = x\}$

then $\mathbb{P}_a(\tau_b < \tau_a^+) = \dfrac{1}{c(a)\,R_{eff}(a,b)}$

This will be used to study transience and recurrence.

Exercise: Prove that effective resistance defines a metric on any graph

Lecture 3 Recall from last time

Theorem: X rev Ω finite

$$\mathbb{P}_a(\tau_b < \tau_a^+) = \frac{1}{c(a)\,R_{eff}(a,b)}$$

Proof: Let $f(x) = \mathbb{P}_x(\tau_b < \tau_a)$. Clearly $f(a)=0$, $f(b)=1$

f is harmonic on $\Omega\setminus\{a,b\}$. If W is a voltage on $\Omega\setminus\{a,b\}$. By uniqueness $f(x) = \dfrac{W(x)-W(a)}{W(b)-W(a)}$

$$\mathbb{P}_a(\tau_b<\tau_a^+) = \sum_{x\sim a} P(a,x)\,f(x) = \sum_x \frac{c(a,x)}{c(a)}\,\frac{W(x)-W(a)}{W(b)-W(a)} = \frac{1}{c(a)(W(b)-W(a))}\sum_{x\sim a} I(a,x)$$

$$= \frac{1}{c(a)}\,\frac{\|I\|}{W(b)-W(a)} = \frac{1}{c(a)\,R_{eff}(a,b)}$$

$\square$

Theorem (Thompson's principle) G finite connected graph, $(c(e))_e$. Then

$$R_{eff}(a,b) = \inf \{ \mathcal{E}(\theta) : \theta \text{ unit flow from } a \text{ to } b \}$$

energy $= \mathcal{E}(\theta) \stackrel{def}{=} \sum_e \theta(e)^2 r(e)$

The unique minimizer is the unit current flow $a \to b$.

Proof: Let i be the unit current flow with voltage φ

1) $R_{eff}(a,b) = \mathcal{E}(i)$:

$$\mathcal{E}(i) = \frac{1}{2} \sum_{\substack{u,v \\ u\sim v}} i(u,v)^2 r(u,v) = \frac{1}{2} \sum_{\substack{u,v \\ u\sim v}} i(u,v)\,(\varphi(u)-\varphi(v)) = \frac{\varphi(a)-\varphi(b)}{\underbrace{\|i\|}_{=1}} = R_{eff}(a,b).$$

2) Let j be another unit flow from a to b

let $k = j-i$ $\qquad \mathcal{E}(j) = \mathcal{E}(k+i) = \mathcal{E}(k) + \mathcal{E}(i) + 2\sum_e k(e)\, i(e)\, r(e)$

Now $\displaystyle\sum_e k(e)\, i(e)\, r(e) = \frac{1}{2} \sum_{\substack{u,v \\ u\sim v}} k(u,v) \underbrace{i(u,v)\, r(u,v)}_{\varphi(u)-\varphi(v)} = 0$ because k has 0 strength and satisfies the cycle law.

So $\mathcal{E}(j) \geq \mathcal{E}(i)$ with equality iff $\mathcal{E}(k) = 0 \Leftrightarrow k = 0$. $\square$

Theorem (Rayleigh's monotonicity principle) Let G be a finite connected network and $(r(e))_e$ and $(r'(e))_e$ be two assignments of resistances to the edges with $r(e) \leq r'(e)$ for all e. Then

$$R_{eff}(a,b,r) \leq R_{eff}(a,b,r')$$

Proof: Let i, i' be the 2 unit current flows.

$$R_{eff}(a,b,r) \stackrel{\text{Thompson}}{=} \mathcal{E}(i) = \sum_e (i(e))^2 r(e) \stackrel{\text{because } i' \text{ is a unit flow}}{\leq} \sum_e (i'(e))^2 r(e) \leq \sum_e (i'(e))^2 r'(e) \stackrel{\text{Thompson}}{=} R_{eff}(a,b,r') \quad \square$$

~~Removing an edge / adding an edge of 0 resistance ... effect~~

Corollary: Gluing vertices together decreases effective resistance

Corollary: Suppose we add an edge not adjacent to a. Then $P_a(\tau_b < \tau_a^+)$ increases.

Proof: $\displaystyle P_a(\tau_b < \tau_a^+) = \frac{1}{c(a) R_{eff}(a,b)}$ $\square$

goes down (not having edge is like having one with ∞ resistance)

It's not clear how to prove this with random walks but it becomes easy with effective resistance.

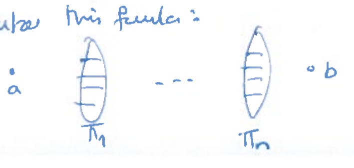
Def: A set of edges Π is called an edge cutset separating a from b if every path from a to b uses an edge of Π.

Nash-Williams inequality $\quad \Pi_k$ edge disjoint cutsets separating a from b. Then

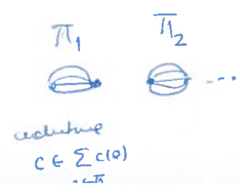
$$R_{eff}(a,b) \geq \sum_k \Big(\sum_{e \in \Pi_k} c(e) \Big)^{-1}$$

Way to remember this formula:

Suppose



glue endpoints of edges in $\Pi_1, \ldots$ $\rightsquigarrow$



now these are in series so the resistances add.

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<u>Proof</u> It is enough to show that for any unit flow θ from a to b

$$\sum_e (\theta(e))^2 r(e) \geq \sum_k \left(\sum_{e\in\Pi_k} c(e)\right)^{-1}$$

$$\sum_e (\theta(e))^2 r(e) \geq \sum_k \sum_{e\in\Pi_k} (\theta(e))^2 r(e)$$

$$\left(\sum_{e\in\Pi_k} \theta(e)^2 r(e)\right)\left(\sum_{e\in\Pi_k} c(e)\right) \overset{\text{Cauchy-Schwarz}}{\geq} \sum_{e\in\Pi_k} \left(|\theta(e)| \sqrt{r(e)}\sqrt{c(e)}\right)^2$$

$$= \left(\sum_{e\in\Pi_k} |\theta(e)|\right)^2 \geq \|\theta\|^2 = 1 \qquad \square$$

Exercise: For any cutset separating a from b show that $\sum_{e\in\pi} |\theta(e)| \geq \|\theta\|$.

In some cases this inequality is best possible up to constants.

<u>Commute-time identity</u> (time to go from a to b and then back to a)

For all a, b $\mathbb{E}_a[\tau_b] + \mathbb{E}_b[\tau_a] = c(G)\, R_{eff}(a,b)$

$$c(G) = \sum_{x\in V} c(x)$$

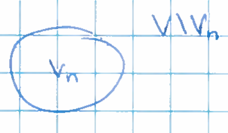
Will not prove it: ~~[illegible]~~ but it is a very useful property to know.

<u>Transience ~~and~~ vs recurrence</u>

Let G be a countable graph. $G=(V,E)$ and 0 a distinguished vertex of G.

Let $G_n=(V_n,E_n)$ be an exhaustion of G by finite graphs, i.e. $0\in V_n$, $V_n\subset V_{n+1}$, $\bigcup_n V_n = V$ and

E_n = set of edges of E connecting points of V_n.



Glue $V\setminus V_n$ into a single point z_n. Take $c(x,z_n) = \sum_{z\in V\setminus V_n} c(x,z)$

Call this new graph G_n^*

By Rayleigh monotonicity we have (decreasing sequence) $\lim_{n\to\infty} R_{eff}(a,z_n,G_n^*) \underset{\text{def}}{=:} R_{eff}(a,\infty)$

$$\mathbb{P}_0(\tau_0^+=\infty) = \mathbb{P}_0\left(\bigcap_n \{\tau_{z_n} < \tau_0^+\}\right) = \lim_{n\to\infty} \mathbb{P}_0(\tau_{z_n} < \tau_0^+) = \lim_{n\to\infty} \frac{1}{c(0)R_{eff}(0,z_n,G_n^*)} = \frac{1}{c(0)R_{eff}(0,\infty)}$$

<u>Criterion</u>: The graph G is recurrent iff $R_{eff}(0,\infty)=\infty$

There is another theorem that says a graph is transient if there is a flow with finite energy but this requires more advanced techniques

<u>Corollary</u>: Let $G \subset G'$ be connected. If G' is recurrent, so is G. If G is transient then so is G'.

Lecture 4

<u>Uniform spanning trees</u>: Will consider the case of unit conductances ~~[illegible]~~ (there is no additional difficulty in handling the general case).

Let G be a finite-connected graph. A <u>spanning tree</u> is a connected subgraph of G that contains all vertices of G and has no cycles. Let $\mathcal{T}$ be the collection of all spanning trees.

Consider the uniform distribution on $\mathcal{T}$, and pick a tree at random. This is a UST.

(If there were conductances with weigh each tree with product of conductances)

This is ~~[illegible]~~ a very interesting object mathematically and a very important in computer science (e.g. travelling salesman problem) for randomized algorithms.

Kirchoff's ~~[illegible]~~ in ~~[illegible]~~ 1847 : $R_{eff}(e) = \dfrac{\#\text{ spanning trees containing } e}{\#\text{ spanning trees}} = \mathbb{P}(e \in T)$

<u>Theorem 1</u> $\mathbb{P}(e \in T) = R_{eff}(e)$

<u>Theorem 2</u> f, g distinct edges of G. $\mathbb{P}(f \in T \mid g \in T) \leq \mathbb{P}(f \in T)$ (negative association of the uniform spanning tree)

<u>Detour</u>: $\mathcal{F}$ = collection of all forest of G (forest means no cycles but may not be connected)

$F \in \mathcal{F}$ uniformly at random $\qquad \mathbb{P}(f \in F \mid g \in F)$ should be $\leq \mathbb{P}(f \in F)$ but this is an open question!

Proof of the Theorems. They depend on defining a certain unit flow

<u>Def</u>: Let s, t be the source and the sink respectively. We define $\mathcal{N}(s,a,b,t) = \left\{ \begin{array}{l} \text{spanning trees such that the unique} \\ \text{path from } s \text{ to } t \text{ crosses } \{a,b\} \text{ from} \\ a \text{ to } b \end{array} \right\}$

$a, b \in E$

$N(s,a,b,t) = |\mathcal{N}(s,a,b,t)|$

<u>Lemma</u>: Define $i(a,b) = \dfrac{N(s,a,b,t) - N(s,b,a,t)}{N}$ where N is the total number of spanning trees. Then i is a unit flow from s to t satisfying Kirchoff's node and cycle laws.

<u>Proof of Theorem 1</u>: Let $e = (s,t)$

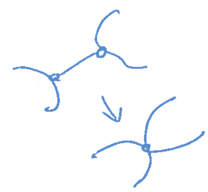
$$i(s,t) = \frac{N(s,s,t,t) - \overbrace{N(s,t,s,t)}^{=0}}{N} = \frac{\#\text{ spanning trees containing } e=(s,t)}{N} = \mathbb{P}(e \in T).$$

$i(s,t) = \varphi(s) - \varphi(t) = R_{eff}(s,t)$ $\qquad$ So $\mathbb{P}(e \in T) = R_{eff}(e)$ $\qquad \square$

voltage associated to i

<u>Proof of Theorem 2</u>: Let G_g be the graph obtained when we glue both endpoints of g into a single vertex

spanning trees of G containing g are in 1-1 correspondence with spanning trees of G_g



$\mathbb{P}(f \in T \mid g \in T) = \dfrac{\#\text{ spanning trees of } G_g \text{ containing } f}{\#\text{ spanning trees of } G_g}$

$\parallel$

$\mathbb{P}(\text{UST of } G_g \text{ contains } f) = R_{eff}(f, G_g) \leq R_{eff}(G) = \mathbb{P}(f \in T)$ $\qquad \square$

by Rayleigh

<u>Proof of Lemma</u>: i is antisymmetric ✓

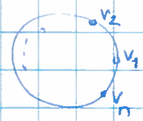
Kirchoff's node law: Let $a \notin (s,t)$ need to show that $\sum_{x \sim a} i(a,x) = 0$

In order to compute $\sum_{x \sim a} i(a,x)$ we calculate the contribution of each spanning tree to this sum. Let T be a spanning tree

- If $a \notin$ path from s,t then contribution is zero
- If a on the path there are two edges e_1, e_2 in path contributing $\frac{1}{N}$ and $-\frac{1}{N}$ respectively. So the contribution is still 0.

Hence $\operatorname{div} i = 0$ outside of $\{s,t\}$.

Cycle law: Let $v_1, v_2, \ldots, v_{n+1} = v_1$ be a cycle



Need to show $\displaystyle\sum_{i=1}^{n} i(v_i, v_{i+1}) = 0$

B is called an s/t bush if it is a forest consisting of exactly 2 trees T_s, T_t with $s \in T_s$, $t \in T_t$

$\mathcal{B}(s,a,b,t) = \{$ s/t bushes s.t $a \in T_s$ and $b \in T_t \}$. Clearly

$$\mathcal{B}(s,a,b,t) \overset{1-1}{\longleftrightarrow} N(s,a,b,t)$$

If B is an s/t bush then the contribution to $\displaystyle\sum_{j=1}^{n} i(v_j, v_{j+1})$ is $\dfrac{|F_+| - |F_-|}{N}$ where

$F_+ = \{(v_j, v_{j+1}) : B \in \mathcal{B}(s, v_j, v_{j+1}, t)\}$ and analogously for F_-

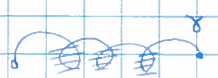
Since we have a cycle, $|F_+| = |F_-|$ by the acyclic property and hence the contribution of B is 0. Hence the sum is 0.

It remains to show that it is a unit flow

$$i(s,x) = \frac{N(s,s,x,t) - \cancelto{0}{N(s,x,s,t)}}{N} = \frac{N(s,s,x,t)}{N}$$

But clearly $\displaystyle\sum_{x \sim s} N(s,s,x,t) = N$. $\square$

Wilson's algorithm: Let $\gamma = (x_0, \ldots, x_k)$. The loop erasure of γ is the path obtained by erasing loops in the order they are created.



Set $y_0 = x_0$
If $x_k = x_0$ then set $m = 0$ and stop, otherwise let $i = \max\{j \geq 0 \mid x_j = x_0\}$ and set $y_1 = x_{i+1}$
If $y_1 = x_k$ stop, otherwise continue in same way.

$G = (V,E)$ finite graph. Distinguish a vertex r of G that we will call the root and give an arbitrary ordering to $V \setminus \{r\}$

Set $T_0 = \{r\}$. Inductively suppose we have defined T_i. If T_i spans G stop, otherwise take the first (in the ordering) vertex not in T_i, say x

Now start a SRW from x and run it until it hits T_i for the first time. Set $T_{i+1} = T_i \cup$ (loop erasure of path from x). Continue in the same way until T_i spans the whole graph.

Theorem (Wilson) This algorithm gives a uniform spanning tree.

Remark: The choice of root r and the ordering have no effect on the distribution.