Lecture 5 : Recall how the Wilson algorithm works. ~~[illegible]~~
to find a spanning tree

Look at top card and introduce a direct edge from x to y

⇝ directed graph.

If no directed cycles then we get a spanning tree

If ∃ directed cycle we pop it (i.e. we discard the top card of the vertices that lie on the cycle and reveal the next cards)

~~If [illegible]~~ Continue like this until there is no cycle to be popped

(This will happen if the stacks are iid uniform).

Suppose we have an edge (x, S_x^i) . color it with color i. A colored cycle is a directed cycle (but not all edges need have the same color)

↑ i-th instruction under x

Lemma : Given any stacks under the stacks then every order of popping cycles will pop on no number of cycles or the same finite number of colored cycles thus leaving a colored spanning tree.

Proof : Let C be a colored cycle that can be popped in the order $C_1, C_2, .., C_n = C$.

Suppose $C' \neq C_1$ happens to be the first cycle we pop. Then we will show either $C' = C$ or C can still be popped after popping $C', C_1, .., C_{n-1}$

Suppose C' is disjoint from $C_1, .., C_n$. Then this is clearly true

If C' is not disjoint let k be the first index C_k which has a vertex in common with C'

let $x \in C' \cap C_k$, $x \notin C_1, .., C_{k-1}$

edge carry out of x in C' must have color 1 because it has not been seen before

The vertex that x leads to must be the same in C' and C_k. As x has not appeared before C_k, the edge carry out must also have color 1. Hence $y \in C' \cap C_k$. Hence the following edge will also have color 1. ~~[illegible]~~ Continuing in this way it follows $C' = C_k$

the end of the edge.

So either $C' = C = C_k$ or C can be popped ~~[illegible]~~ after $C', C_1, .., C_n$.

Proof that Wilson's algorithm provides a Unif. Span. Tree :

Erasing loops in order of cycle creation is one way of popping cycles. IID stacks ⟹ stop with a probability 1 at a spanning tree. Any order will yield the same distribution.

Let X be the set of all pairs (O, T) where O is a set of colored cycles and T is a spanning tree that can be obtained from O

If $(O,T) \in X$ then $(O, T') \in X$ for any other spanning tree T' (because we can have any other instructions on the cards under the vertices that lie on the cycles)

Let $X = X_1 \times X_2$ X_1 set of colored cycles (not all, they have to be compatible i.e. they can arise)
X_2 all spanning trees

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let $\psi(C) = \prod_{v \in C} \frac{1}{\deg(v)}$

$\uparrow$ cycle

O set of colored cycles $\quad \psi(O) = \prod_{C \in O} \psi(C)$

$$P\left((O,T) \text{ is the resulting pair}\right) = \left(\prod_{C \in O} \prod_{v \in C} \frac{1}{\deg(v)}\right)\left(\prod_{x \neq r} \frac{1}{\deg(x)}\right)$$

$\uparrow$ probability of spanning tree

independent of the tree T

not a prob. measure but may be normalized

$$= \psi(O) \cdot \left(\prod_{x \neq r} \frac{1}{\deg(x)}\right)$$

$\square$

$\Rightarrow$ O is independent of T $\Rightarrow$ T is a uniform spanning tree.