Hence $P_{00}(2n) \leq \binom{2n}{n}\frac{1}{2^{2n}}\binom{n}{m\,m\,m}\frac{1}{3^n} \overset{\text{Stirling}}{\sim} \frac{c'}{n^{3/2}}$

So this is transient $\square$

$d>3 \rightsquigarrow$ see exercises.

For general graphs this kind of analysis is much harder.

Defn: A Markov chain with matrix P has invariant/stationary/equilibrium distribution π if when $X_0 \sim \pi$ then for any n, $X_n \sim \pi$

This means that π has to satisfy $\pi P = \pi$, i.e. π is a left eigenvector of the matrix P

Evita and Persi were talking about convergence to this invariant distribution in their examples.

Def: A MC with P, π is called reversible if for all N, when $X_0 \sim \pi$

$$(X_0, \dots, X_N) \underset{\substack{\uparrow \\ \text{same distribution}}}{\sim} (X_N, \dots, X_0)$$

Exercise: Reversibility is equivalent to: for all x, y $\pi(x)P(x,y) = \pi(y)P(y,x)$ [detailed balance equation]

~~If we want a solution to these equations which is a distribution~~

In a finite graph we can set $\pi(i) = \frac{\deg(i)}{\sum_j \deg j}$ gives an invariant, reversible distribution.

Next lectures: Electrical networks (finite graphs); extend to infinite graphs; spanning trees

Lecture 2 - Today we'll talk about general graphs

Let $G=(V,E)$ be a finite connected graph (unoriented). Endow it with non-negative weights $(c(e))_{e\in E}$ $c(e)>0$

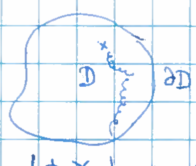
The edges are unoriented but we still write $c(x,y) = c(\{x,y\}) = c(y,x)$

A weighted RW on G has $P(x,y) = \frac{c(x,y)}{c(x)}$ where $c(x) = \sum_{z\sim x} c(x,z)$.

Check that $\pi(x) = \frac{c(x)}{\sum_{y\in V} c(y)}$ is invariant and the Weighted Random Walk is reversible.

Exercise: ~~If the chain is reversible~~ Every reversible Markov chain on finite Ω with transition matrix P is a weighted walk on some graph

Def: $h: \Omega \to \mathbb{R}$ is called harmonic at x if $h(x) = \sum_y P(x,y)h(y)$ (with respect to the MC (Ω, P))



f a function on the boundary. How can we extend it to the inside as an harmonic function.

Theorem Let X be irreducible on Ω with transition ~~Markov~~ matrix P. Let $B \subseteq \Omega$ and $f: B \to \mathbb{R}$. Then there exists a unique $h: \Omega \to \mathbb{R}$ such that $h(x) = f(x)$ on B and h is harmonic on $\Omega \setminus B$

h is given by $h(x) = \mathbb{E}_x\left[f(X_{\tau_B}) \right]$ where $\tau_B = \{ t \geq 0 : X_t \in B \}$

<u>Note</u>: there is a continuous version that produces continuous Harmonic functions using Brownian motion.

<u>Proof</u>: Obviously if $x \in B$ then $h(x) = f(x)$.

$$h(x) = \sum_y \mathbb{E}_x\left[f(X_{\tau_B}) \mid X_1 = y \right] P(x,y) \quad \text{(law of total probability)}$$

$$= \sum_y \mathbb{E}_y\left[f(X_{\tau_B}) \right] P(x,y) \quad \text{(since it is a Markov chain we can forget we started at } y\text{)}$$

$$= \sum_y P(x,y)\, h(y)$$

So h is harmonic.

<u>Uniqueness</u>: Let h' be another function with same properties and let $g = h' - h$. Then g is harmonic on $\Omega \setminus B$ and $g = 0$ on B.
~~Sketch~~ Need to show $g = 0$ on Ω. Let $A = \{ x : g(x) = \max_z g(z) \}$ need to show $\max(g) = 0$ (in same way we see $\min(g) = 0$)

Let $x \in A$. If $x \in B$ ✓.
If $x \notin B$ then let ~~g(x)~~ y be such that $P(x,y) > 0$ $\qquad g(x) = P(x,y) g(y) + \sum_{z \neq y} P(x,z) g(z)$
If $g(y) < g(x)$ then $g(x) < \max g$. So $g(y) = g(x)$. By irreducibility we will hit B $\qquad \square$

<u>Electrical networks</u>:

$G = (V, E)$ $\qquad$ $(c(e))$ conductances (weights)

resistence of e $\quad r(e) = \dfrac{1}{c(e)}$

Let a and b be 2 vertices of G

<u>Def</u>: A <u>flow</u> θ is a function on oriented edges $\vec{E}$ $\qquad \theta : \vec{E} \to \mathbb{R}$ which is anti-symmetric $\theta(x,y) = -\theta(y,x)$

divergence of θ $\qquad \boxed{\operatorname{div} \theta(x) = \sum_{y \sim x} \theta(x,y).}$

Note that $\sum \operatorname{div} \theta(x) = 0$.

<u>Def</u>: A <u>flow θ from $\overset{\text{source}}{a}$ $\overset{\text{sink}}{b}$</u> is a flow such that

- $\operatorname{div} \theta(x) = 0$ for all $x \notin \{a, b\}$ $\qquad$ Kirchoff's <u>node law</u> $\qquad$ (what comes in comes out)
- $\operatorname{div} \theta(a) \geq 0$

$\|\theta\| = \operatorname{div} \theta(a)$ is called the <u>strength</u> of θ / unit strength means $\|\theta\| = 1$.

Note that $\operatorname{div} \theta(b) = -\operatorname{div} \theta(a)$.

A <u>voltage</u> $W : V \to \mathbb{R}$ is a harmonic function on $V \setminus \{a, b\}$

By the Theorem above a voltage exists and is uniquely determined by $W(a)$ and $W(b)$

We define the current I associated to W by setting $I(x,y) = \dfrac{W(x) - W(y)}{r(x,y)} = c(x,y)\left(W(x) - W(y)\right)$

<u>Check</u>: I is a flow, and the unit current flow is unique (even though the voltage isn't).

$$W(x) - W(y) = r(x,y)\, I(x,y) \quad \text{is } \underline{\text{Ohm's law}}$$

If $\vec{e}_1, \dots, \vec{e}_n$ is a cycle then $\displaystyle\sum_{i=1}^{n} r(\vec{e}_i)\, I(\vec{e}_i) = 0$ $\qquad$ <u>cycle law</u>

<u>Lemma</u>: Let I be a current and θ a flow from a to b satisfying the cycle law for all cycles. If $\|\theta\| = \|I\|$ then $\theta = I$

<u>Proof</u>: Let $f = \theta - I$. Then f is a flow and $\operatorname{div} f(x) = 0$ for all x. Also f satisfies the cycle law for any cycle.
$f \neq 0$ then without loss $\exists\ \vec{e}_1$ so that $f(\vec{e}_1) > 0$ $\qquad \sum f(x,y) = 0$ for all $x \Rightarrow \exists\ \vec{e}_2$ $\xrightarrow{e_1}\xrightarrow{e_2}$ s.t. $f(\vec{e}_2) > 0$

But since the set is finite ~~....~~ the successive edges $\vec{e_i}$ will eventually close. But this contradicts the cycle law. □

Effective resistance :

replace by

what resistance should we put here so that with the given voltage on a, b the same current will flow?

Let W_0 be a voltage with $W_0(a)=1$ and $W_0(b)=0$.

Let W be another voltage. Then $\frac{W(x)-W(b)}{W(a)-W(b)} = W_0(x)$ by uniqueness.

Hence $W(x) = (W(a)-W(b))\,W_0(x) + W(b)$

Let I_0 be the current associated to W_0 and I the current associated to ~~W_0~~ W

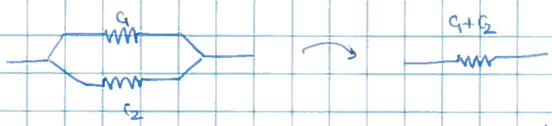
$$\|I\| = \sum_{x\sim a} I(a,x) = \sum_{y\sim a} c(a,x)\left(W(a)-W(x)\right) = \|I_0\|\left(W(a)-W(b)\right)$$

Thus $\frac{W(a)-W(b)}{\|I\|}$ is independent of I, W. This is what is called the effective resistance $R_{eff}(a,b)$

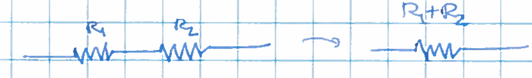
The effective conductance is $C_{eff}(a,b) = \frac{1}{R_{eff}(a,b)}$

Some laws for computing effective resistance:

Conductance in parallel

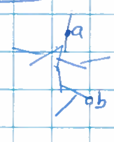


Resist in series



Gluing of points with same voltage

Exercise: If T is a finite connected tree with weights all equal. $R_{eff}(a,b) = d(a,b)$



Connection between random walks and electrical networks:

Theorem: Let X be a reversible MC on Ω (think of it as a WRW) and let $\tau_x = \min\{t \geq 0 : X_t = x\}$
$\tau_x^+ = \min\{t \geq 1 : X_t = x\}$

then $\mathbb{P}_a(\tau_b < \tau_a^+) = \frac{1}{c(a)\,R_{eff}(a,b)}$

This will be used to study transience and recurrence.

Exercise: Prove that effective resistance defines a metric on any graph