TÉCNICO LISBOA

Perla Sousi - lecture 1

Will also be talking about Markov chains but in infinite countable state spaces

Will discuss different questions such as how long a ~~[illegible]~~ random walk takes to come back to the starting space.

(transience and recurrence)

<u>Definition</u>: A sequence of random variables $(X_n)_{n\geq 0}$ taking values in a space E is called a <u>Markov chain</u> if $\forall\, x_1, \dots, x_n \in E$ such that $P(X_1 = x_1, \dots, X_n = x_n) > 0$ we have

$$\mathbb{P}(X_{n+1} = x_{n+1} \mid X_n = x_n, \dots, X_0 = x_0) = P(X_{n+1} = x_{n+1} \mid X_n = x_n) = P(X_1 = x_{n+1} \mid X_0 = x_n)$$

"Past is independent of the future given the present" — past $\perp\!\!\!\perp$ future | present — time homogeneous

We'll assume E is <u>countable</u>

The Markov chain is characterized by the transition matrix P $\quad P(i,j) = P(X_1 = j \mid X_0 = i)$

<u>Notation</u>: Let A be an event $\quad \mathbb{P}_i(A) \overset{\text{def}}{=} \mathbb{P}(A \mid X_0 = i)$

$$\mathbb{E}_i[Y] = \mathbb{E}[Y \mid X_0 = i]$$

<u>Check</u>: $\mathbb{P}(X_n = j \mid X_0 = i) = P^n(i,j)$ — also written $P_{ij}(n)$

n-th matrix power

<u>Def</u>: A Markov chain is called <u>irreducible</u> if $\forall_{x,y\in E}\ \exists_{n\geq 0}$ such that $P^n(x,y) > 0$

<u>Example</u>: A random walk on a graph can be irreducible only if the graph is connected.

<u>Def</u>: For every i define $T_i = \inf\{n \geq 1 : X_n = i\}$

From now on assume that all Markov chains are irreducible

<u>Def</u>: A state i is called <u>recurrent</u> if $\mathbb{P}_i(T_i < \infty) = 1$. Otherwise a state is called is transient.

<u>Exercise</u>: If $\exists\, i$ such that $\mathbb{P}_i(T_i < \infty) = 1$ then for all j, $\mathbb{P}_j(T_j < \infty) = 1$

So a Markov chain (MC) can be either ~~transient~~ <u>recurrent</u> or <u>transient</u>

<u>Lemma</u>: Let $X_0 = i$ and $V_i = \sum_{n=0}^{\infty} 1(X_n = i)$ the total number of visits to i. Then V_i has the geometric distribution: $\quad P_i(V_i > r) = (P_i(T_i < \infty))^r$

<u>Proof</u>: $\mathbb{P}_i(V_i > 1) = \mathbb{P}_i(T_i < \infty)$

$\mathbb{P}_i(V_i > 2) = \mathbb{P}_i(T_i < \infty)^2$ (because of the Markov property) and so on for all r $\square$

So $\quad \mathbb{E}_i[V_i] = \dfrac{1}{1 - \mathbb{P}_i(T_i < \infty)}$

<u>Criterion for recurrence</u>: A MC is recurrent iff $\sum_{n\geq 0}^{\infty} P_{ii}(n) = \infty$

<u>Proof</u>: $\Rightarrow$ If $P(T_i < \infty) = 1$ then for all r $\mathbb{P}_i(V_i > r) = 1$ and so $\mathbb{P}_i(V_i = \infty) = 1 \Rightarrow \mathbb{E}[V_i] = \infty$

But $\mathbb{E}[V_i] = \sum_{n\geq 0}^{\infty} \mathbb{P}_i(X_n = i) = \sum_{n\geq 0}^{\infty} P_{ii}(n)$

$\Leftarrow$ If $\sum P_{ii}(n) = \infty$ then $\mathbb{E}_i[V_i] = \infty \Rightarrow \mathbb{P}_i(T_i < \infty) = 1$ $\square$

__Note__: Recurrence and transitivity is only interesting for infinite graphs.

__Def__: A simple random walk on a (undirected, locally finite) graph G (always assumed connected)

$$G = (V, E)$$

vertices, edges

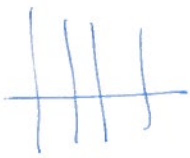
$$P(i,j) = \begin{cases} \frac{1}{\deg(i)} & \text{if } (i,j) \text{ is an edge} \\ 0 & \text{otherwise.} \end{cases}$$

($\deg(i) = \#$ neighbors of i)

(i,j) means the unordered pair; edges can connect a vertex to itself and there can be multiple edges.

__Theorem__ (Polya 1920) A simple random walk on $\mathbb{Z}^d$ is recurrent iff $d=1$ or $d=2$.

(this theorem was motivated by the fact that he kept running into his flatmate in the park.
On $\mathbb{Z}^d$ 2 walks meeting is the same as recurrence because you can subtract but this is not always the case. For instance



$\subset \mathbb{Z}^2$ is recurrent as we will see later but two random walks do not always meet)

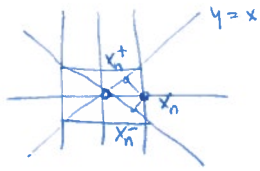
__Proof__: __$d=1$__:

$$P_{00}(2n) = \binom{2n}{n}\left(\frac{1}{2}\right)^{2n}$$ return probability after $2n$ steps

Stirling's formula: $n! \sim n^n \sqrt{2\pi n}\, e^{-n}$

Hence $P_{00}(2n) \sim \frac{c}{\sqrt{n}}$ for some $c > 0$. By the previous criterion we see the walk is recurrent.

__$d=2$__. This uses a very nice trick



Let X_n be the walk and
X_n^+ its projection to $y = x$
X_n^- " " to $y = -x$

You can easily check that X_n^+, X_n^- are independent simple random walks on $\mathbb{Z}/\sqrt{2}$ so we have decomposed our walk in 2 dimensions into 2 random walks in 1-dimension.

$$P_{00}(2n) = P_0(X_{2n}^+ = 0, X_{2n}^- = 0) = \mathbb{P}_0(X_{2n}^+ = 0)\,\mathbb{P}(X_{2n}^- = 0) = \left(\binom{2n}{n}\left(\frac{1}{2}\right)^{2n}\right)^2 \sim \frac{c^2}{n} \xRightarrow{\text{criterion}} d=2 \text{ is recurrent.}$$

__$d=3$__: Here we can't project and get 3 independent walks

$$\mathbb{P}(X_{2n} = 0) = \sum_{\substack{i,j,k \geq 0 \\ i+j+k=n}} \frac{(2n)!}{(i!\,j!\,k!)^2}\left(\frac{1}{6}\right)^{n} = \binom{2n}{n}\frac{1}{2^{2n}} \sum_{\substack{i,j,k\geq 0 \\ i+j+k=n}} \binom{n}{i\ j\ k}^2 \left(\frac{1}{3}\right)^{2n}$$

$$\sum_{\substack{i,j,k\geq 0 \\ i+j+k=n}} \binom{n}{i\ j\ k} = 3^n \qquad \text{if } \begin{matrix} n = 3m \\ i+j+k=n \end{matrix} \text{ then } \binom{n}{i\ j\ k} \leq \binom{n}{m\ m\ m} \quad \text{[check!]}$$

Hence $P_{00}(2n) \le \binom{2n}{n} \frac{1}{2^{2n}} \binom{n}{m\ m\ m} \frac{1}{3^n} \overset{\text{Stirling}}{\sim} \frac{c'}{n^{3/2}}$

So this is transient $\square$

$d>3 \rightsquigarrow$ see exercises.

For general graphs this kind of analysis is much harder.

Defn: A Markov chain with matrix P has invariant/stationary/equilibrium distribution π if when $X_0 \sim \pi$ then for any n, $X_n \sim \pi$

This means that π has to satisfy $\pi P = \pi$, i.e. π is a left eigenvector of the matrix P

Evita and Persi were talking about convergence to this invariant distribution in their examples.

Def: A MC with P, π is called reversible if for all N, when $X_0 \sim \pi$

$$(X_0, \dots, X_N) \underset{\substack{\uparrow \\ \text{same distribution}}}{\sim} (X_N, \dots, X_0)$$

Exercise: Reversibility is equivalent to: for all x,y $\pi(x)P(x,y) = \pi(y)P(y,x)$ [detailed balance equation]

~~[illegible]~~

In a finite graph we can set $\pi(i) = \frac{\deg(i)}{\sum_j \deg j}$ gives an invariant, reversible distribution.

Next lectures: Electrical networks (finite graphs) ; extend to infinite graphs ; spanning trees