Exercises

September 5, 2019

1. **1**. Let $R(r)$ be the effective resistance between two given vertices of a finite network with edge-resistances $r = (r(e) : e \in E)$. Show that R is concave in that

$$\frac{1}{2} \cdot (R(r_1) + R(r_2)) \leq R\left(\frac{1}{2}(r_1 + r_2)\right).$$

2. Show that if, in a network with source a and sink z, vertices with different voltages are glued together, then the effective resistance from a to z will strictly decrease.

3. (i) Using the Nash Williams inequality give another proof that simple symmetric random walk on $\mathbb{Z}^2$ is recurrent.

(ii) Let $\rho \in (0, 2)$ and consider a binary tree with resistance between edges from level $n-1$ to level n equal to ρ^n. Calculate the effective resistance between 0 and ∞ and show that it is finite.

Show that for suitable values of ρ we can embed this tree into $\mathbb{Z}^3$.

Deduce that simple symmetric random walk on $\mathbb{Z}^3$ is transient.

4. (i) Let G be a finite connected graph on n vertices with conductances $(c(e))$ on the edges. Prove that

$$\sum_{e \in E} c(e) R_{\text{eff}}(e) = n - 1.$$

(ii) Let $G = (\mathbb{Z}_n^d, E(\mathbb{Z}_n^d))$ be the d-dimensional torus of side length n, i.e. $\mathbb{Z}_n^d = \{0, \ldots, n-1\}^d$ and $E(\mathbb{Z}_n^d) = \{(x, y) \in \mathbb{Z}_n^d \times \mathbb{Z}_n^d : \|x - y\| = 1\}$. Let $e \in E(\mathbb{Z}_n^d)$. Show that

$$R_{\text{eff}}(e; \mathbb{Z}_n^d) \to \frac{1}{d} \quad \text{as } n \to \infty.$$

5. Let $G = (V, E)$ be a connected subgraph of the finite connected graph G'. Let T and T' be uniform spanning trees of G and G' respectively. Show that for any edge e of G,

$$\mathbb{P}(e \in T) \geq \mathbb{P}\big(e \in T'\big).$$

More generally, let B be a subset of E, and show that $\mathbb{P}(B \subseteq T) \geq \mathbb{P}(B \subseteq T')$.