Exercises

September 3, 2019

1. Prove that every reversible Markov chain is a weighted random walk on some graph.

2. Let e_1 and e_2 be two edges sharing the same endvertices. Show that if we replace them by a single edge of conductance the sum of the conductances, then all currents and potentials in $G \setminus \{e_1, e_2\}$ remain unchanged and the current $I(\overrightarrow{e}) = I(\overrightarrow{e_1}) + I(\overrightarrow{e_2})$.

Let $v \in V \setminus \{a, z\}$ be a node of degree 2 with neighbours v_1 and v_2. Show that if we replace the edges (v, v_1) and (v, v_2) by a single edge of resistance the sum of the resistances, then all currents and potentials in $G \setminus \{v\}$ remain unchanged and the current that flows from v_1 to v_2 equals $I(\overrightarrow{v_1 v}) = I(\overrightarrow{v v_2})$.

3. Let a and b be two vertices on a finite connected tree T. Prove that the effective resistance $R_{\text{eff}}(a, b)$ is equal to the distance on the tree between a and b.

4. (i) Let X be a reversible Markov chain on a finite state space Ω. Let $\tau_z = \min\{t \geq 0 : X_t = z\}$. Prove that

$$\mathbb{E}_a\left[\sum_{t=0}^{\tau_z} \mathbf{1}(X_t = a)\right] = c(a) R_{\text{eff}}(a, z).$$

(ii) Suppose that X is a simple symmetric random walk on $\mathbb{Z}^2$ started from $x \in (-100, 100)^2$. Let τ be the first exit time from the square $(-100, 100)^2$, i.e. $\tau = \min\{t \geq 0 : X_t \notin (-100, 100)^2\}$. Show that for all $y \in [-100, 100]^2 \setminus (-100, 100)^2$

$$\mathbb{E}_x\left[\sum_{t=0}^{\tau} \mathbf{1}(X_t = x)\right] = \mathbb{E}_x\left[\sum_{t=0}^{\tau} \mathbf{1}(X_t = x) \,\middle|\, X_\tau = y\right].$$

5. Let X be a simple symmetric random walk on $\{0, \ldots, n\}$ with $P(0, 0) = 1/2 = P(n, n)$. Using the network reduction rules show that for all $x \in \{0, \ldots, n\}$

$$\mathbb{P}_x(\tau_n < \tau_0) = \frac{x}{n}.$$

6. Show that effective resistances form a metric on any network with conductances $(c(e))$.