Lecture 1 suggested problems

Evita Nestoridi

August 28, 2019

Exercise on S_n: Choose a permutation $\sigma \in S_n$ uniformly at random.

(a) Prove that $\mathbb{P}\,(1$ and 2 are in the same cycle of $\sigma) = 1/2$.

(b) Prove that $\mathbb{P}\,(1, 2, \ldots, n$ are in the same cycle of $\sigma) = 1/n$.

(c) Let $2 \leq j \leq n$. Make a conjecture on what the value of $\mathbb{P}\,(1, 2, \ldots, j$ are in the same cycle of $\sigma)$ should be and try to prove it.

Random to top: Let X_t be the configuration of a deck of n cards after having performed the random to top shuffle performed t times. Let T be the first time that $n-1$ different cards have been selected. For the random to top card shuffle of a deck of n cards,

(a) if $n = 3$ prove that
$$\mathbb{P}\,(X_2 = \sigma | T = 2) = 1/6,$$
for any $\sigma \in S_3$.

(b) if $n = 4$ prove that
$$\mathbb{P}\,(X_3 = \sigma | T = 2) = 1/24,$$
for any $\sigma \in S_4$.

(c) For general n, prove that T is a strong stationary time for X_t.

Random to random: The random to random card shuffle suggests to pick a card uniformly at random, remove it from the deck and insert it to a uniformly random position. Let T be the first time that all cards have been picked. Prove that T is NOT a strong stationary time for the random to random card shuffle.

Lazy random walk on the hypercube : Let $\Omega = \{0, 1\}^n$. Let (X_t) be the following process on Ω: at time t we pick a uniformly random coordinate of X_{t-1}. We flip a fair coin. If tails, then we update the chosen coordinate to be equal to one. If tails we update it to be zero. Prove that

(a) X_t has the uniform measure as stationary measure.

(b) for the separation distance we have that if $t_{n,c} = n \log n + cn$ then
$$s(t_{n,c}) \leq e^{-c}.$$

Chernoff Bound: Let Ω be the path with n nodes. Let X_t be the random walk on Ω that either moves to the right with probability 2/3 or to the left with probability 1/3. If $X_t = n$ then either we stay fixed with probability 2/3 or we move to the left with probability 1/3. Similarly if $X_t = 1$ then either we stay fixed with probability 1/3 or we move to the right with probability 2/3. If $X_0 = n$, let $T = \min\{t \geq 1 | X_t = n\}$. Prove that

$$\mathbb{P}\left(T > x\right) \leq \left(\frac{2\sqrt{2}}{3}\right)^x$$