Lisbon school July 2017: eversion of the sphere

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What have we seen so far ?

(Half-)converting a geometric question to a topological question

GEOMETRY: Consider regular closed curves $f_i\colon [0,1] \looparrowright M$ where M=S^2 or M=R^2, for i=0,1.
Question: Is f_0 regularly homotopic to f_1 as closed regular curves ?

We translated this question into:
TOPOLOGY: Consider loops $\hat{f}_i\colon [0,1] \to TM$.
Question: Is $\hat{f}_0$ homotopic to $\hat{f}_1$ as a loops ?

The conversion (up to now) is in one way: if the answer to the topological question is NO then the answer to the geometric question is NO.
It is often that way: topology give "obstructions" to solution of geometric problem.

Sometimes it works in both ways. This is the case this problem on immersions thanks to Smale theory.

Some algebraic topology invariants of spaces

π_0(F)= the set of path components of F
π_1(B)= the group of based homotopy classes of based loops in B

We have seen some tools to compute these invariants:

- Baby Van Kampen: tool to prove that a space is simply connected by viewing it as the union of two simply-connected subspaces with connected intersection.
- Contractibility: if the identity map of X is homotopic to a constant map then π_i={0}
- Connecting morphism: next slide

The connecting morphism $\partial\colon \pi_1(B) \longrightarrow \pi_0(F)$

If p:E→ P has the $\mathrm{HLP_k}$ property for k=0,1 and if we set $F := p^{-1}(b_0)$ then we have a well defined morphism

$$\partial\colon \pi_1(B) \longrightarrow \pi_0(F).$$

If moreover E is simply-connected then ∂ is a bijection.

Important examples of maps satisfying $\mathrm{HLP_k}$ are given by bundles.

A fundamental example of this is the degree of self-maps of the circle

$$deg = \partial\colon \pi_1(S^1) \stackrel{\cong}{\to} \pi_0(\mathbf{Z}) = \mathbf{Z}.$$

We use that to compute $\pi_1(\mathrm{TS}^2 \cong \mathrm{SO}(3))$=Z/2Z.

A complete (and not half) classification of immersions.

The space of immersions

Fix a manifold M=R^2 or M=S^2 (or M any riemannian manifold).
We define the space of immersions of the segment

$$\mathcal{I} := Imm([0,1], M) := \{f\colon [0,1] \looparrowright M \;:\; f \text{ is an immersion}\}$$

equipped with the distance

$$d(f,g) := \sup_{t\in[0,1]} dist(f(t), g(t)) + dist(f'(t), g'(t)).$$

This is called the $\mathcal{C}^1$ topology on that set of immersions.

A regular curve f in M (not closed) is a point $f \in \mathcal{I}$.
A regular homotopy betwen f and g is ???? a path in $\mathcal{I}$ connecting f to g.

The space of immersions with prescribed origin and/or endpoint

Denote by T_0M the space of non zero tangent vectors in M:

$$T_0M := \{(x,v) \colon x \in M, v \text{ tangent to M at x }, v \neq 0\}.$$

Fix $b_0=(x_0, v_0) \in T_0M$ and $b_1=(x_1, v_1) \in T_0M$.
Define the subspaces of immersions

$$\begin{aligned}
\mathcal{I}_{b_0} &:= \left\{f \in \mathcal{I} \,:\, (f(0), f'(0)) = b_0\right\} \\
\mathcal{I}_{b_0,b_1} &:= \left\{f \in \mathcal{I} \,:\, (f(0), f'(0)) = b_0\,,\, (f(1), f'(1)) = b_1\right\}
\end{aligned}$$

Example: $\mathcal{I}_{b_0,b_0}$ is the space of all regular closed curves based at b_0

Contractibility of $\mathcal{I}_{b_0}$

Recall that a space X is contractible if there exists a homotopy between the identity map and some constant map. Visually it means that there is a continuous global way to shrink evey point in X to some prefered point c. Caution: every point in the sphere S^2 can be moved continuously to the north pole but S^2 is not contractible: there is no way to define this shrinking continuously over all the sphere.

Theorem

$\mathcal{I}_{b_0}$ is contractible.

Idea of proof. Fix the geodesic c with starting vector $\mathbf{b_0}$. This is a point $c \in \mathcal{I}_{b_\mathbf{0}}$. We can define a homotopy $H\colon [0,1] \times \mathcal{I}_{b_\mathbf{0}} \to \mathcal{I}_{b_\mathbf{0}}$ with $\mathbf{H_0}$ the constant map on c and $\mathbf{H_1}$=identity:

$$\begin{cases} H_t(f) \text{ shortens f near to } x_\mathbf{0} \text{ so that it is almost linear} & 2/3 \leq t \leq 1 \\ H_t(f) \text{ projects the shortened f to the start of the geodesic c} & 1/3 \leq t \leq 2/3 \\ H_t(f) \text{ rescale this start of geodesic to all c} & 0 \leq t \leq 1/3 \end{cases}$$

The projection of $\mathcal{I}_{b_0}$ on $\mathrm{T}_{0\mathrm{M}}$

Define the map

$$p\colon \mathcal{I}_{b_0} \longrightarrow T_0 M\,,\ f \longmapsto (f(1), f'(1)).$$

This map is continuous because of the $\mathcal{C}^1$ topology.

Theorem

$p\colon \mathcal{I}_{b_0} \to T_0 M$ has the HLP_k property for all $k \geq 0$

For k=0 this amount to say that

- given a continuous path ω in $T_0 M$, $\omega\colon [0,1] \to T_0 M, u \mapsto \omega_u$
- given an immersion $\varepsilon\colon [0,1] \looparrowright M$ such that $(\varepsilon(0), \varepsilon'(0)) = b_0$ and $p(\varepsilon) = (\varepsilon(1), \varepsilon'(1)) = \omega_0$,

then there exists a path $\tilde{\omega}\colon [0,1] \to \mathcal{I}_{b_0}, u \mapsto \tilde{\omega}_u$ such that

- $\tilde{\omega}_0 = \varepsilon$ and
- $p(\tilde{\omega}_u) = (\tilde{\omega}_u(1), \tilde{\omega}_u(1)) = \omega_u$ for all $0 \leq$ u≤ 1.

How to prove that p has the HLP property ?

Smale ("Regular curves on Riemannian manifolds") does this by defining explicitely the lift $\tilde{\omega}$ through some explicit equation deforming the original immersion ε.
This is not too complicated (formulas on top of p.499 and bottom of p.501.)
The only subtelty is to make sure that each $\tilde{\omega}_v$ is an immersion, that is $\tilde{\omega}_v(t) \neq 0$ for all t. For this the deformation of ϵ makes use of some non-zero vector u orthogonal to $\varepsilon'(1)$. Thereofore we need that the immersion is in codimension at least 1.

The path components of the regular closed curves based at b_0

Fix $b_{0\in}$ T_0M. The fibre of p at b_0 is:

$$F := p^{-1}(b_0) = \mathcal{I}_{b_0,b_0} = \{\text{regular closed curves based at } b_0\}$$

Theorem

We have a bijection $\partial\colon \pi_1(TM_0, b_0) \xrightarrow{\cong} \pi_0(\mathcal{I}_{b_0,b_0})$.

$$\pi_0(Imm_{b_0,b_0}([0,1],\mathbf{R}^2)) \cong \pi_1(T_0\mathbf{R}^2) \cong \mathbf{Z} \quad (1)$$

$$\pi_0(Imm_{b_0,b_0}([0,1],S^2)) \cong \pi_1(T_0S^2) \cong \mathbf{Z}/2\mathbf{Z} \quad (2)$$

(1) $\Rightarrow$ γ is a <u>complete</u> invariant of regular homotopy classes of closed curves in the plane.
(2) $\Rightarrow$ the only regular closed curves in the spheres are "O" and "8"

Finally the eversion of the sphere (1)

Consider V=S^2 and M=R^3.
We want to prove that Imm(S^2 ,R^3) is path connected.
Let H be the upper hemisphere slightly extented below the equator.
Let A be a collar about the boundary of H so that A is a thickening of the equator.
Thus we have an inclusion $i\colon A \hookrightarrow H$ and any immersion of H restricts to an immersion of A.
Fix a 2-frame b_0 in R^3 and some 2-frame a_0 in the tangent bundle of A.
Consider immersions f of A or H into R^3 that sends a_0 to b_0:

$$\begin{aligned} Imm_0(H, R^3) &:= \left\{ f\colon H \looparrowright \mathbf{R}^3 : df(a_0) = b_0 \right\} \\ Imm_0(A, R^3) &:= \left\{ f\colon H \looparrowright \mathbf{R}^3 : df(a_0) = b_0 \right\} \end{aligned}$$

etc. . .

Perspectives

Immersions of general manifolds

Smale-Hirsh: they convert the study of $Imm(V, M) := \{f\colon V \looparrowright M\}$ into a pure topology problem.
Example of basic question: when is Imm(V,M) non-empty ?
Examples:

- Every surface can be immersed in R^3 (an example of immersion of the projective plane is the "Boy surface"; you have seen an immersion of the Klein bottle in R^3).
- What about manifolds of higer dimensions ? Whitney proved (1945) that every n-manifold can be immersed in R^{2n}-1.
- On the other hand it is impossible to immerse the projective space $\mathbf{R}P^4$ in $\mathbf{R}^6$, and more generaly $\mathbf{R}P^{2^k}$ immerses in $\mathbf{R}^{2^{k+1}-1}$ but not in $\mathbf{R}^{2^{k+1}-2}$.
- Obstructions to immerses V in $\mathbf{R}^N$ are given by some cohomology classes $w_i \in H^i(V;Z/2)$ called the Stiefel-Whitney classes.

Embeddings

Instead of looking at immersions we can look at embeddings:
$f : V \hookrightarrow M$ is an embedding of manifolds if it is an immersion and is injective.
From S. Smale "The classicication of spheres in Euclidean spaces", Annals of Mathematics (1959):

> *Question: develop an analogous theory for imbeddings. Presumably this will be quite hard. However, even partial results in this direction would be interesting.*

This was answered by T. Goodwillie - J. Klein - M. Weiss in the 2000's, inventing "embedding calculus" to analyse the homotopy type of Emb(V,M).
The study of embedding spaces is hard: $\pi_0(\mathrm{Emb}(S^1, R^3))$ is the set of isotopy classes of knot. Classifying knots is a hard and rich problem. At least 3 Fields medalists (Witten, Jones, Kontsevich) worked on that...

h-principle

Smale strategy has been generalized by A. Phillips, Y. Elisachberg, and M. Gromov who came up with the very general "h-principle".
Roughly studying immersions is studying differentiable maps f:V$\rightarrow$ M such that some inequality is satisfied, i.e. for all t, $f'(t) \neq 0$. Smale reduce that problem to a topology or homotopy problem.
Gromov's h-principle generalize this to more general differential in-equations.

A few suggestions of readings (1)

Here are a fwe suggestions of readings (from the 3 lecturers). They differ in difficulty and style; find the one that fits your taste and knowledge.

- A. Hatcher "Algebraic topology" freely available on the web. Plenty of fun and beautiful exercises to work.
- J. P. May: "A Concise Course in Algebraic Topology". https://www.math.uchicago.edu/~may/CONCISE/ConciseRevised.pdf (beyond Hatcher with a quite different style)
- J. Milnor "Topology from a differential viewpoint". Very short book and starts gently. Give a very nice viewpoint on the degree, how it can be used to prove the fundamental theorem of algebra and goes up to "frame cobordism" which are basic tool for classification of manifolds.
- http://analysis-situs.math.cnrs.fr/ a web site with a lot of goodies in algebraic topology

A few suggestions of readings (2)

More advanced readings

- J. Milnor - J. Stasheff "Characteristic classes: if you do a PhD in algebraic topology you should read that book (if not now, latter). He explains there for example that $\mathbf{R}P^{2^k}$ does not immerse in $\mathbf{R}^{2^k-2}$.
- J. Dieudonne "A history of algebraic topology -1960" A historical perspective on the beginning of salgebraic topology. Thick book, covering algebraic topology until 1960.
- H. Geiges "h-Principles and Flexibility in Geometry" (memoirs AMS); the last chapter is a very accessible proof of Smale's eversion giving a clue of a deep generalization of Smales' ideas due to Gromov and called the h-principle.

A few suggestions of readings (3)

Enjoyable lighter readings:

- Jeff Weeks: "The shape of space"
- E. Abbott: "Flatland" http://www.geom.uiuc.edu/~banchoff/Flatland/
- Two histories of (Algebraic) Topology
  - http://www-groups.dcs.st-and.ac.uk/~history/HistTopics/Topology_in_mathematics.html
  - http://www.mathnet.or.kr/real/2009/3/McCleary_col.pdf