# Lisbon school July 2017: eversion of the sphere

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# Lecture's goal

We still want to prove that the two regular closed curves in the sphere
$f_1\colon [0,1] \to S^2, t \mapsto (cos 2\pi t, sin 2\pi t, 0)$ and
$f_2\colon [0,1] \to S^2, t \mapsto (cos 4\pi t, sin 4\pi t, 0)$
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But it is enough to prove that the two loops (for i=1,2)

$$\widehat{f_i}\colon [0,1] \longrightarrow TS^2\,,\; t \longmapsto \left( f(t)\,,\; \frac{f'(t)}{\|f'(t)\|} \right)$$

are not homotopic loops.
This is a problem of topology which can be solved with the tools of algebraic topology.

# The fundamental group of a space (Henri Poincaré $\simeq$ 1900 in "Analysis situs")

Let X be a space and $x_0 \in X$ a chosen point ("the base point").
A based loop is

$$\omega : [0, 1] \to X, t \mapsto \omega(t)$$

such that $\omega(0)=\omega(1)=x_0$.
A based homotopy is

$$\Omega : [0, 1] \times [0, 1] \to X, (t, u) \mapsto \Omega_u(t)$$

such that $\Omega_u(0)=\Omega_u(1)=x_0$ for all $u\in[0,1]$. We then write

$$[\Omega_0] = [\Omega_1].$$

We set

$$\pi_1(X, x_0) := \{[\omega] \,:\, \omega \text{ is a based loop in } X\}.$$

# The fundamental group is a group (Hatcher, 1.1)

Let $\alpha$,$\beta$:[0,1]$\to$ X be two based loops.
We can concatenate them in a new based loop

$$\alpha \cdot \beta \colon [0,1] \longrightarrow X \,,\ t \longmapsto \begin{cases} \alpha(2t) & \text{if } 0 \leq t \leq 1/2 \\ \beta(2t-1) & \text{if } 1/2 \leq t \leq 1 \end{cases}$$

Then we can easily check the following facts:
([$\alpha$]=[$\alpha$'] and [$\beta$]=[$\beta$'])$\Rightarrow$ [$\alpha \cdot \beta$]=[$\alpha' \cdot \beta'$]
therefore we have an induced operation $\cdot$ on $\pi_1$(X,$x_0$).

$(\alpha \cdot \beta) \cdot \gamma \neq \alpha \cdot (\beta \cdot \gamma)$ but $[(\alpha \cdot \beta) \cdot \gamma] = [\alpha \cdot (\beta \cdot \gamma)]$.

If $c_{x_0} \colon [0,1] \to X, t \mapsto x_0$ denotes the constant loop then
$[\alpha] \cdot [c_{x_0}] = [\alpha] = [c_{x_0}] \cdot [\alpha]$. Thus 1:=[$c_{x_0}$] plays the role of a unit.

If we set $\overline{\alpha} \colon [0,1] \to X, t \mapsto \alpha(1-t)$ then $[\alpha] \cdot [\overline{\alpha}] = 1 = [\overline{\alpha}] \cdot [\alpha]$.

Conclusion: $\pi_1$(X,$x_0$) is a group.
Challenge: find a space X such that the group $\pi_1$(X,$x_0$) is not abelian.

# First example of computation: $\pi_1(\mathrm{R}^n, 0) = \{1\}$

Proof that the fundamental group of $\mathbf{R}^n$ is trivial.

Consider the homotopy

$$H\colon \mathbf{R}^n \times [0,1] \to \mathbf{R}^n\,,\ (x,u) \mapsto H_u(x) := u \cdot x.$$

It is a homotopy between

- the constant map $H_0\colon x \mapsto 0$ and
- the identity map $H_1\colon x \mapsto x$.

Because of this homotopy (constant map)$\simeq$(identity map) we say that $\mathbf{R}^n$ is contractible.

IF $\omega$ is a based loop in $\mathbf{R}^n$ then $\Omega_\mathrm{u}(\mathrm{t}) := \mathrm{H}_\mathrm{u}(\omega(\mathrm{t}))$ gives a based homotopy between the constant loop $\mathrm{c}_0$ and the loop $\omega$.

QED

# Examples by pictures

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Challenge find a general formula for $pi_1(S)$ for a surface represented by a polygon where the edges are identified. Hint consider the case where all the vertices of the polygongo to a single point of the surface. Then $\pi_1(S)$ should be the "free" group generated by the edges modulo a relation whihc is the "word" describing the boundary of the polygon

# The fundamental group of the circle

To compute $\pi_1$(S$^1$,(1,0)) we use the "bundle" or "covering"

$$p\colon \mathbf{R} \to S^1, t \mapsto (\cos t, \sin t)$$

which has the path lifting property.
Given a based loop $\omega\colon$ [0,1]$\to$ S$^1$ based at 0 we can lift it to a path $\tilde{\omega} : [0,1] \to$.
This defines a map

$$deg : \omega \mapsto deg(\omega) := \tilde{\omega}(1).$$

It turns out that deg($\omega$) only depends on the homotopy class of the path. This is because we can not only lift paths but also homotopy of paths.
Moreover it gives an isomorphism (we will see a generalization of this latter)

$$deg\colon \pi_1(S^1) \cong \mathbf{Z}.$$

How to prove that p has the (homotopy of) path lifting property ?

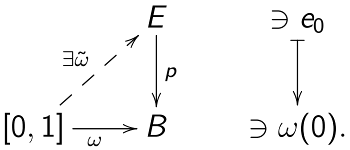
# The path lifting property

We say that a map p: E→ B has the path lifting property (or $\mathrm{HPL}_0$) if given

- a path $\omega\colon [0,1] \to B$ starting at some point $b_0 = \omega(0)$, and
- a point $e_0 \in E$ such that $p(e_0) = b_0$,

Then there exists a path $\tilde{\omega}\colon [0,1] \to E$ such that

1. $p\tilde{\omega} = \omega$, and
2. $\tilde{\omega}(0) = e_0$.



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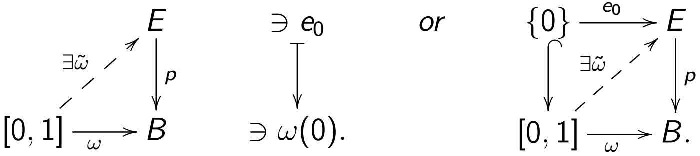
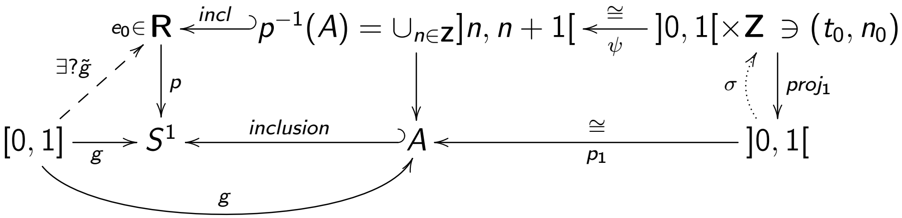


# Proof of the path lifting property of p: step 1.

Step 1: Prove the existence of a lifting under the extra assumption that the path $g\colon [0,1] \to S^1$ is missing the rightmost point (1,0) of the circle.

Hint: Set $A := S^1 \setminus \{1\}$ the circle minus that rightmost point. Thus g([0,1])⊂ A. We have a homeomorphism $p_1 : ]0,1[ \xrightarrow{\cong} A$.
Then $p^{-1}(A) = \mathbf{R} \setminus \mathbf{Z}$. Contemplate 10 minutes the following:




where $\psi$(t,n):=t+n, ($t_0$ ,$n_0$ ):=$\psi^{-1}$ ($e_0$) and $\sigma$(t):=(t,$n_0$).
Show that $\tilde{g}(t) := \psi(\sigma(p_1^{-1}(g(t))))$ is a lift of $g$ along p: $p\tilde{g} = g$.

# Proof of the path lifting property: remaining steps

Step 1bis: Prove the existence of a lifting under the extra assumption that the path $g\colon [0,1] \to S^1$ is missing the leftmost point (-1,0) of the circle. (Completely analoguous to step (1))

Step 2: show that we can decompose the interval $[0,1]$ in small subintervals $[t_i, t_{i+1}]$ with $0=t_0 < t_1 < \ldots < t_{N-1} < t_N = 1$ such that each restricted paths $g|[t_i, t_{i+1}]$ misses either (-1,0) or (1,0)

Step 3: Use the previous steps to construct inductively the lifting $\tilde{g}$ and finish the proof

Exercise: complete the details of the proof.

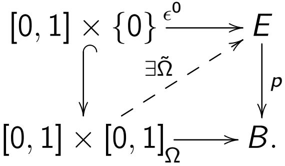
# The homotopy of paths lifting property

We say that a map p: E→ B has the homotopy of path lifting property (or $\mathrm{HPL}_1$) if given

- a homotopy $\Omega\colon [0,1] \times [0,1] \to B, (t,u) \mapsto \Omega(t,u)$, and
- a path $\epsilon^0 \in E$ such that $p(\epsilon^0(u)) = \Omega(0,u)$,

then there exists a homotopy $\tilde{\Omega}\colon [0,1] \times [0,1] \to E$ such that

1. $p\tilde{\Omega} = \Omega$, and
2. $\tilde{\omega}(0,u) = \epsilon^0(u)$ for all $u \in [0,1]$.




Exercise: prove the above homotopy lifting theorem. Hint Very similar the proof of the path lifting property. Look at Hacher "Algebraic topology", proof of property (c) on page 29 (freely available on the web). Dont be scared by the apparent complexity of Hatcher: have faith and struggle.

# Path-connected, simply-connected, and $\pi_0$

Let X be a space with a base point $x_0$.
We say that $X$ is path-connected if any two points in X can be connected by a path.
The path-component of a point x in X is the subset

$$\langle x\rangle := \{y \in X \ : \ x \text{ and } y \text{ are connected by a path in } X\}\,.$$

The space X is partitionned into its path components. The set of path components of X is denoted by

$$\pi_0(X) := \{\langle x\rangle \ : \ x \in X\}\,.$$

Thus a space is path-connected (and non-empty) if and only if $\pi_0$(X) is a singleton.
We say that $X$ is simply-connected if it is path-connected and if $\pi_1$(X,$x_0$) is the trivial group.
Example: the sphere $S^n := \{x \in \mathbf{R}^{n+1} \ : \ \|x\| = 1\}$ is simply-connected for $n \geq 2$.

# Bundle

A continuous map p: E$\rightarrow$ B is a bundle with fibre F if every point b$\in$ B admits a neighborhood N such that there is a homeomorphism $h_N \colon p^{-1}(N) \stackrel{\cong}{\rightarrow} N \times F$ such that the restriction of p to $p^{-1}(N)$ is the same as $proj_1$ $h_n$.

When the fibre F is a discrete space we say that the bundle is a covering.

Examples:

- p: R$\rightarrow$ $S^1$ ,t$\mapsto$ (cos(2$\pi$ t),sin(2$\pi$ t)) with fibre Z
- p: $R^{2\rightarrow}$ (torus T) with fibre $Z^2$
- p: $S^2$ $\rightarrow$ (projective plane P) with fibre Z/2
- $p \colon TS^2 \rightarrow S^2, (x,v) \mapsto x$ with fibre $S^1$
- p: $S^{3\rightarrow}$ SO(3) with fibre Z/2Z
- $p \colon B \times F \rightarrow B, (b,f) \mapsto b$ with fibre F (this is the "trivial bundle")

# Fundamental group of the base of a bundle

## Theorem

*Let $E \to B$ be a bundle with fibre $F$ and assume that $E$ is simply-connected. Then we have a bijection*

$$\partial \colon \pi_1(B, b_0) \stackrel{\cong}{\to} \pi_0(F).$$

Example: consider the bundle $\mathbf{R} \to S^1$ with fibre $\mathbf{Z}$. The total space $\mathbf{R}$ is simply-connected. Therefore we have a bijection $\partial \colon \pi_1(S^1, b_0) \stackrel{\cong}{\to} \pi_0(\mathbf{Z}) = \mathbf{Z}$.

Example: we have a bundle

$$p \colon \mathbf{R}^2 \to \mathit{torus}\,,\ (\theta, \phi) \mapsto (\cos(\theta)(R + r \sin\phi), \sin(\theta)(R + r \sin\phi), r \cos\phi)$$

whose fibre is $2\pi$ Z$\times 2\pi$ Z. Since R$^2$ is simply-connected we deduce that $\pi_1$(torus)$\cong 2\pi$ Z$\times 2\pi$ Z$\cong$ Z$\times$ Z

# Computation of $\pi_1$ of the projective plane P

Recall that the projective plane P is defined from the 2-disk D=$\{(u,v) \in R^2 : u^2+v^2 \leq 1\}$ after identification of the points (u,v) and (-u,-v) when (u,v) is on the boundary of the disk. We write $\overline{(u,v)}$ to denote that point in P.

Define a map p:$S^2 \to$ P as follows.

For (x,y,z)$\in S^2$ (thus $x^2+y^2+z^2=1$) we set

$$p(x,y,z) = \begin{cases} \overline{(x,y)} & \text{if } z \geq 0 \\ \overline{(-x,-y)} & \text{if } z \leq 0 \end{cases}$$

This map is well defined because if z=0 then (x,y) is on the boundary of the disk and thus $\overline{(x,y)} = \overline{(-x,-y)}$

One checks that it is a bundle with fibre F=Z/2.

Since $S^2$ is simply connected we deduce that

$\pi_1(P,p_0) \cong \pi_0(Z/2)$=Z/2.

# Computation of $\pi_1$ of SO(3)$\cong$ TS$^2$

There exists a well known bundle

$$p\colon S^3 \to SO(3).$$

with fibre Z/2.
One way to define this bundle is to look at S$^3$ as the space of quaternionic numbers of modulus 1 and use this to associate to z$\in$ S$^3$ the transformation h$\mapsto$ $z^{-1}$ $\cdot$ h$\cdot$ z of the pure quaternionic numbers (identified with R$^3$) which turns out to be an element of SO(3). See the exercises for more details.
Since S$^3$ is simpl-connected we deduce that
$\pi_1$(SO(3))$\cong$ $\pi_0$(Z/2)=Z/2.

# An explicit generator of the group $\pi_1(SO(3), I) \cong \mathbf{Z}/2$.

A non trivial loop is obtanined as follows. Consider the path $\tilde{\omega}\colon t \mapsto (cos(\pi t), \sin(\pi t), 0, 0)$ in $S^3$ connecting (1,0,0,0) to (-1,0,0,0). From the explicit definition of p one get that $\omega := p\tilde{\omega}$ is the based loop in SO(3) defined by

$$\omega(t) = \begin{pmatrix} \cos 2\pi t & -\sin 2\pi t & 0 \\ \sin 2\pi t & \cos 2\pi t & 0 \\ 0 & 0 & 1 \end{pmatrix} \in SO(3)$$

and this loop is not contractible since its lifts $\tilde{\omega}$ connects points in two different path components of the fibre.
Thus $[\omega]$ is the non trivial generator of $\pi_1$(SO(3)) and $[\omega \cdot \omega]$=1 because the square of any element in the group with two elements is the unit.

# The regular immersions $f_1$ and $f_2$ are not regularly homotopic

We come back to the two regular closed curves $f_i$ in $S^2$ (for i=1,2) and the associated loops $\hat{f}_1$ and $\hat{f}_2$.
We have $\hat{f}_1 = \omega$ and $\hat{f}_2 = \omega \cdot \omega$. Thus the base loops $\hat{f}_1$ and $\hat{f}_2$ are not based homotopic and therefore $f_1$ and $f_2$ are not regularly homotopic.
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To finish the proof you have to look at Hatcher exercise 6 page 38: the free homotopy classes of loops in X corresponds to the conjugacy classes in the group $\pi_1(X,x_0)$. Since $\pi_1(SO(3)) \cong Z/2$ is abelian the conjugacy classes are in bijection with the elements of the group.
Thus two based loops in SO(3) are unbased homotopic if and only if they are based homotopic.

# Fundamental group from the path lifting property

Let $p\colon E \to B$ be a map which has the path lifting property ($HLP_0$) and the homotopy of path lifting property ($HLP_1$).
Assume that E is simply-connected.
Then we have a bijection

$$\partial\colon \pi_1(B, b_0) \xrightarrow{\cong} \pi_0(p^{-1}(b_0)).$$

Construction of the map $\partial$:
Fix a point $e_{0\in}$ E such that $p(e_0)=b_0$. Set $F:=p^{-1}$ (F)
Let $\omega$ be a based loop in X and choose a lifting $\tilde\omega\colon [0,1] \to X$ starting at $e_0$. Set

$$\partial([\omega]) := \langle\tilde\omega(1)\rangle \in \pi_0(F).$$

Use the homotopy of path lifting property to prove that this map is well defined, injective and surjective.

# The path lifting property

We say that a map p: E→ B has the path lifting property if given

- a path $\omega\colon [0,1] \to B$ starting at some point $b_0 = \omega(0)$, and
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1. $p\tilde{\omega} = \omega$, and
2. $\tilde{\omega}(0) = e_0$.

We say that $\tilde{\omega}$ is a lifting of the path $\omega$ along p starting at $e_0$

There also a notion of homotopy of path lifting property

Proposition A bundle has the path lifting property and the homotopy of path lifting property