Shapes of Spaces III

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Homology – of $SO(3)$ and in general
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The orthogonal groups

Let $O(n)$ be the space of all real orthogonal $n \times n$-matrices. I.e., the real $n \times n$-matrices A satisfying

$$A^t A = I$$

Includes reflections: ignoring these we have the *special orthogonal group*

$$SO(n) = \{A \in O(n) \mid \det A = 1\}$$

consisting exactly of the rotations.

$O(n) = \{A \in M_n(\mathbb{R}) \mid A^t A = I\}$
subspace of $M_n(\mathbb{R}) = \mathbb{R}^{n^2}$
A^t: the transpose of A
$A^t A = I \Leftrightarrow$ the columns in A are orthonormal

The space $SO(3)$ of rotations in $\mathbb{R}^3$.

Understanding the space $SO(3)$ of rotations in $\mathbb{R}^3$ is all important for

- robotics/prosthetics
- computer visualisation/games
- navigation

It is a 3D-subspace of $M_3\mathbb{R} = \mathbb{R}^9$, but curves and folds up on itself in a manner that makes the flat 9D coordinates useless. At this point I hope you all did the hands-on exercise about rotations on the first exercise sheet.

The space $SO(3)$ of rotations in $\mathbb{R}^3$.

A nontrivial rotation in $\mathbb{R}^3$ has a unique axis: the eigenspace (w/eigenvalue 1) of the rotation matrix.
Choosing an eigen vector corresponds to choosing an orientation of the rotation.

$$\begin{aligned} D^3 &\to SO(3) \\ p &\mapsto \text{ the rotation around } p \text{ by } |p|\pi \text{ radians} \end{aligned}$$

A rotation around p by $|p|\pi$ radians is the same as a rotation around $-p$ by $-|p|\pi$ radians, which isn't a problem until $|p| = 1$.

$SO(3) \cong D^3/(|p| = 1 \Rightarrow p \sim -p) \cong \mathbb{RP}^3$

$SO(3)$ may be identified with the quotient of D^3 by the equivalence relation that a point on the boundary is identified with its antipodal point, which is a model for the real projective space $\mathbb{RP}^3$.

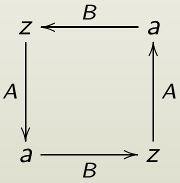
$D^3 = \{p \in \mathbb{R}^3 \mid |p| \leq 1\}$

The shape of $SO(3)$

Hence, to understand the shape of the space of rotations, it suffices to understand the projective space $\mathbb{R}\mathrm{P}^3$ obtained by identifying antipodal points on the boundary of D^3.

More generally, the (real) projective n-space $\mathbb{R}\mathrm{R}^n$ is obtained by identifying antipodal points on the boundary of D^n.

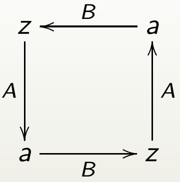
E.g. $\mathbb{R}\mathrm{P}^2$ is obtained from the (filled) square, with points on the boundary identified accordingly: [1]



As a toy example towards understanding the shape of $SO(3)$, we'll calculate $H_*\mathbb{R}\mathrm{P}^2$.

[1] Shout if I forget to talk about electrons

The shape of $\mathbb{RP}^2$



This is not a simplicial complex. We *could* model $\mathbb{RP}^2$ by a simplicial complex, but it would be much bigger e.g. requiring six vertices (as opposed to two), but H_* can be calculated directly for "CW-complexes" of which the above is an example:[2] it has

two "0-cells": z, a,

two "1-cells": A, B and

one "2-cell": the (filled) square S itself.

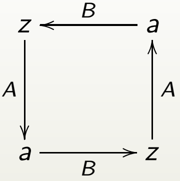
The chain complex

$$C_0 \xleftarrow{\partial_1} C_1 \xleftarrow{\partial_2} C_2 \xleftarrow{\partial_3}$$

Homology has an axiomatic description on CW-complexes

[2] Oh, no! he is repeating Dan's lecture

The shape of $\mathbb{RP}^2$



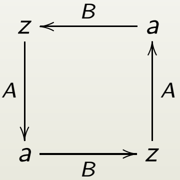
two "0-cells": z, a,
two "1-cells": A, B and
one "2-cell": the (filled) square S itself.

$$C_0 \xleftarrow{\partial_1} C_1 \xleftarrow{\partial_2} C_2 \xleftarrow{\partial_3} \dots$$

$$\langle z, a\rangle \xleftarrow[z-a \mapsfrom B]{a-z \mapsfrom A} \langle A, B\rangle \xleftarrow{2A+2B \mapsfrom S} \langle S\rangle$$

$$\begin{aligned} Z_0 &= \langle z, a\rangle, & B_0 &= \langle z - a\rangle, & H_0 &= \mathbb{Z} \\ Z_1 &= \langle A + B\rangle, & B_1 &= \langle 2(A + B)\rangle, & H_1 &= \mathbb{Z}/2\mathbb{Z} \\ Z_2 &= 0, & B_2 &= 0, & H_2 &= 0. \end{aligned}$$

The shape of $\mathbb{RP}^2$

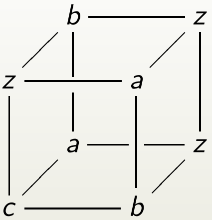


$H_0 \mathbb{RP}^2 = 0$ - path connected
$H_1 \mathbb{RP}^2 = \mathbb{Z}/2\mathbb{Z}$ - has a hole, but "going around it twice, it vanishes".[3]

[3]This can also be viewed as a manifestation of the fact that S^2 is the "universal covering space" of $\mathbb{RP}^2$ from another lecture.

$H_* SO(3)$

$SO(3) = \mathbb{RP}^3$:



Zero cells: z, a, b, c, 1-cells: za, zb, zc, ab, ac, bc, 2-cells: F, S, T ("front, side, top"), 3-cell: C. Cell complex

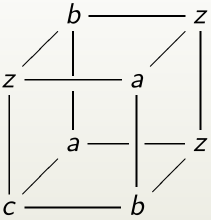
$$\left\langle \begin{smallmatrix} z & a \\ b & c \end{smallmatrix} \right\rangle \xleftarrow{\partial_1} \left\langle \begin{smallmatrix} za & zb & zc \\ ab & ac & bc \end{smallmatrix} \right\rangle \xleftarrow{\partial_2} \langle F, S, T \rangle \xleftarrow{\partial_3} \langle C \rangle$$

$$[\partial_1] = \begin{bmatrix} -1 & -1 & -1 & . & . & . \\ 1 & . & . & -1 & -1 & . \\ . & 1 & . & 1 & . & -1 \\ . & . & 1 & . & 1 & 1 \end{bmatrix}, [\partial_2] = \begin{bmatrix} 1 & . & -1 \\ . & -1 & 1 \\ -1 & 1 & . \\ 1 & 1 & . \\ . & -1 & -1 \\ 1 & . & 1 \end{bmatrix}, [\partial_3] = 0$$

$H_*(SO(3))$ is found by finding the null space and column spaces of $[\partial_*]$.

The shape of the space of rotations

$SO(3) = \mathbb{RP}^3$:



Zero cells: z, a, b, c, 1-cells: za, zb, zc, ab, ac, bc, 2-cells: F, S, T ("front, side, top"), 3-cell: C.

$$
\begin{aligned}
Z_0 &= \langle \begin{smallmatrix} z & a \\ b & c \end{smallmatrix} \rangle, & B_0 &= \langle \begin{smallmatrix} z-a & z-b \\ z-c & \end{smallmatrix} \rangle, & H_0 &= \mathbb{Z} \\
Z_1 &= \langle \begin{smallmatrix} za+ab+bc-zc \\ zc-ac+ab-zb \\ ab+bc-ac \end{smallmatrix} \rangle, & B_1 &= \langle \begin{smallmatrix} za+ab+bc-zc \\ zc-ac+ab-zb \\ 2(ab+bc-ac) \end{smallmatrix} \rangle, & H_1 &= \mathbb{Z}/2\mathbb{Z} \\
Z_2 &= 0, & B_2 &= 0, & H_2 &= 0 \\
Z_3 &= \langle C \rangle & B_3 &= 0 & H_3 &= \mathbb{Z}.
\end{aligned}
$$

The shape of the space of rotations

$$H_i SO(3) = \begin{cases} \mathbb{Z} & \text{if } i = 0, 3 \\ \mathbb{Z}/2\mathbb{Z} & \text{if } i = 1 \\ 0 & \text{otherwise.} \end{cases}$$

- H_0 - path connected
- H_1 - reflects the technical complications with giving good coordinates for computer graphics, the irritating and surprising behavior of the cables of a computer, robotics, aviation...[4]
- $H_2 = 0$ - no problem, so nobody talks about it.
- H_3 - reflects that $SO(3)$ is a "closed 3D manifold" - a space that "locally looks like $\mathbb{R}^3$, compact (like S^3), and no loops take you to your mirror image".

[4] Again, S^3 – the space of "unit quaternions" – is the universal covering space of $SO(3) = \mathbb{RP}^3$

Homology

1. As we have seen, given a good description of your space it is possible to calculate homology, giving us a powerful tool in understanding the shape and properties of the space.
2. However, we have still not exhausted the good algebraic properties of homology.
3. Homology is characterized by a list of properties, and in practice, this list is *exactly* what you want to use for calculations - the construction is immaterial.

Kernel, image...[5]

Exact sequence = a sequence

$$\dots \xrightarrow{\partial_{n+2}} M_{n+1} \xrightarrow{\partial_{n+1}} M_n \xrightarrow{\partial_n} \dots, \quad s.t. \quad \ker \partial_n = \mathrm{im}\partial_{n+1}$$

(exact sequence = "complex whose homology vanishes")
Abelian groups

[5] ya,ya - Dan did all of this, didn't he?
Except, that he said he didn't want to teach you how to do linear algebra over the integers...
and I won't either

Cell attachments

A *cell attachment* $A \subseteq X$ is the inclusion of spaces you get from a continuous $\phi\colon S^n \to A$, setting [6]

$$X = A \coprod_{\phi} D^{n+1} = A \coprod D^{n+1}/\phi(p) \sim p$$

A *CW-complex* is a space obtained by repeatedly attaching cells (of increasing dimension, starting with the empty set).

Disjoint union:
$A_1 \coprod A_2 = \{(i, a) \mid i = 1, 2, a \in A_i\}$
$S^{-1} = \emptyset$, $D^0 = \{0\}$.

[6] for $p \in S^n$. A subset of X is open if "it's preimage in A and D^{n+1} are".

CW-complexes

CW-complex:

a space obtained by repeatedly attaching cells (of increasing dimension, starting with the empty set).
A *CW-pair* (X, A) is a CW complex X and a closed subspace $A \subseteq X$ consisting of some of the cells of X.
A *pointed CW-complex* is a CW-pair $(X, \{\text{point}\})$. Maps between pointed CW-complexes preserve the "base point".

Example: if (X, A) is a CW-pair, the quotient space

$$X/A = X/a \simeq a' \text{ for } a, a' \in A$$

is a pointed CW-space ($A/A \subseteq X/A$ is the "base point")

Axioms for homology

A (reduced) *homology theory* is a sequence $\{\tilde{h}_n\}_n$ of homotopy functors such that [7] for each CW-pair (X, A) there is a *natural* [8] exact sequence

$$\dots \xrightarrow{\partial} \tilde{h}_n A \xrightarrow{i_n} \tilde{h}_n X \xrightarrow{q_n} \tilde{h}_n(X/A) \xrightarrow{\partial} \tilde{h}_{n-1} A \xrightarrow{i_{n-1}} \dots$$

where $i\colon A \subseteq X$ is the inclusion and $q\colon X \to X/A$ is the quotient map.

Given an abelian group M. On pointed finite CW-complexes there is a *"unique"* homology theory s.t.

$$\tilde{h}_n(S^0) = \begin{cases} M & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases}$$

[7] the "wedge axiom" which is not relevant to us is omitted

[8] "natural": crucial word: discussed in the exercises.

The uniqueness is "up to natural isomorphism"

Many calculations follow directly from the axioms

$D^0 = S^0/S^0$, so we have a short exact sequence

$$\dots \to H_j S^0 = H_j S^0 \xrightarrow{0} H_j D^0 \xrightarrow{0} H_{j-1} S^0 = H_{j-1} S^0 \to \dots$$

implying that $\tilde{h}_* D^0 = 0$.

For all n, the identity map on D^n is homotopic to the constant map (D^n is "contractible"), hence $\tilde{h}_* D^n \cong \tilde{h}_* D^0 = 0$.

$S^n = D^n/S^{n-1}$, so we have a long exact sequence

$$\dots \to \tilde{h}_j D^n \xrightarrow{0} \tilde{h}_j S^n \to \tilde{h}_{j-1} S^{n-1} \xrightarrow{0} \tilde{h}_{j-1} D^n \to \dots$$

so $\tilde{h}_* S^n = \tilde{h}_{*-1} S^{n-1}$. In particular, if

$$\tilde{h}_j(S^0) = \begin{cases} M & \text{if } j = 0 \\ 0 & \text{otherwise,} \end{cases}$$

then

$$\tilde{h}_j(S^n) = \begin{cases} M & \text{if } j = n \\ 0 & \text{otherwise.} \end{cases}$$