# Lisbon school July 2017: eversion of the sphere

Pascal Lambrechts

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# Today's lecture (and probably further)

Our goal today (probably continuing tomorrow) is to generalize Whitney-Graustein theorem about regular closed curves in the plane to regular closed curves in any surface (or even manifold).

We will need to introduce a very important tool in algebraic topology called the fundamental group (or Poincaré group) of a based space and denoted $\pi_1(X,x_0)$ which is the group of homotopy classes of based loops. Tomorrow we will introduce coverings to compute this group.

Some reference for today is Smale "Regular closed curves in Riemannian manifolds" and also the part of the chapter on the fundamnetal group in the book "Algebraic topology" of Hatcher (freely available on the web)

# Regular closed curves in the plane

A regular parametrized curve or immersion of the segment in the plane is a map

$$f : [0,1] \to \mathbf{R}^2 \, , \; t \mapsto f(t)$$

which is $\mathcal{C}^1$ and such that $f'(t) \neq (0,0)$ for each t∈[0,1].
We denote it by $f : [0,1] \looparrowright \mathbf{R}^2$.
It is closed or an immersion of the circle if moreover f(0)=f(1) and f'(0)=f'(1).

Each f'(t) is a tangent vector to the curve at the point f(t).
This tangent vector

1. never vanishes (because of the condition $f'(t) \neq (0,0)$)
2. varies continuously (because f is $\mathcal{C}^1$)

# Regular homotopies of regular closed curves

## Definition

*A regular homotopy between regular parametrized(closed curves is a map*

$$F\colon [0,1]\times[0,1] \longrightarrow \mathbf{R}^2\,,\ (t,u)\longmapsto F(t,u)=F_u(t)$$

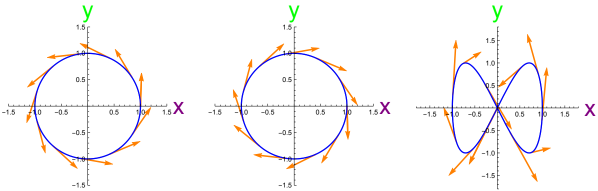
*such that*

1. *for each $u\in[0,1]$ $F_u$ is a regular parametrized closed curve*
2. *$F$ is continuous*
3. *$\frac{\partial F}{\partial t}\colon [0,1]\times[0,1]\to\mathbf{R}^2$ is continuous (continuity of the tangent vectors) (OUPS! Whitney forgot that...)*

*Two regular closed curves $f,g\colon[0,1]\looparrowright\mathbf{R}^2$ are regularly homotopic if there exists a regular homotopy such that $F_0=f$ and $F_1=g$.*

Remark this is not a pure topological notion !

# Three regular closed curves in the plane




$\gamma$(left)=+1 $\gamma$(middle)=-1 $\gamma$(right)=0

$\gamma(f) \in \mathbf{Z}$ is the rotation number of the closed regular curve defined as the degree of the map $\frac{f'}{\|f'\|} : S^1 \to S^1$ which counts the total number of full turns that the tangent vector makes along the curve.

Whitney-Graustein thm: f,g regularly homotopic $\Leftrightarrow$ $\gamma$(f)=$\gamma$(g).

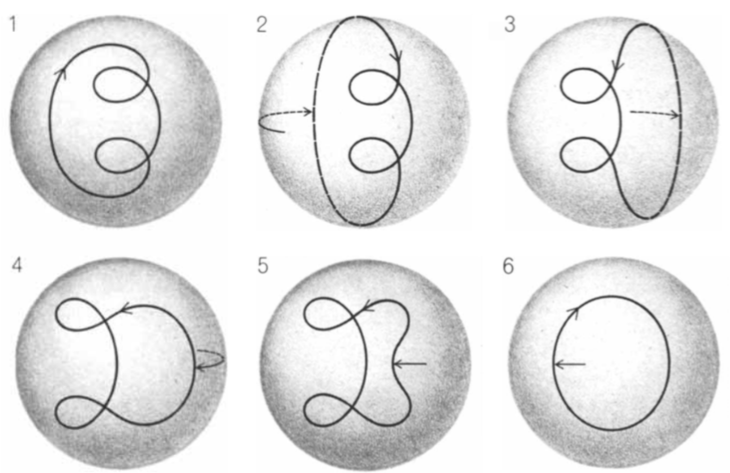
They are infinitely many different regular homotopy classes of closed curves in the plane.

# Regular closed curve in the sphere

Consider the two dimensional sphere $S^2 = \{(x,y,z) \in R^3 : x^2+y^2+z^2=1\}$.
A regular closed curve in the sphere is a function $f:[0,1] \to S^2$ of class $\mathcal{C}^1$ such that for each $t \in [0,1]$ $f'(t) \neq 0$
From the sheet of exercises:

1. The immersion $f_1 \colon t \mapsto (\cos(2\pi t), \sin(2\pi t), 0)$ traveling the equator in one direction is regularly homotopic to the one traveling the equator in the other direction $f_{-1} \colon t \mapsto (\cos(2\pi t), -\sin(2\pi t), 0)$.
   Easy solution: turn the curve of 180° along the y-axis
2. Show that $f_1$ is regularly homotopic to the immersion $f_3 \colon t \mapsto (\cos(6\pi t), \sin(6\pi t), 0)$ traveling the equator 3 times
   See next slide

# A regular homotopy of closed curve in $S^2$




**REGULAR HOMOTOPY ON THE SPHERE** is illustrated for two curves on sphere *A* of illustration at top. Broken segment of the curve has been shifted around the back of sphere.

From A. Phillips "Turning a surface inside out" Scientific American ,May 1966

# A regular closed curve in $S^2$ not regularly homotopic to traveling once the equator

We will prove that travelling twice the equator is not regularly homotopic to traveling once the equator !
Actually we will prove that they are exactly two regular homotopy closed curves in the sphere. When these curves have finitely many crossing points their regular homotopy classes are detected by counting the number of crossings mod 2.

# Loop in the unit tangent bundle of $R^2$

The "unit tangent bundle" of $R^2$ is the space of tangent vectors of length one of the plane:

$$\begin{aligned} T\mathbf{R}^2 &:= \{(x,v) : x \in \mathbf{R}^2, v \text{ tangent to } \mathbf{R}^2 \text{ at } x, \|v\| = 1\} \\ &\cong \mathbf{R}^2 \times S^1 \end{aligned}$$

A regular curve f: [0,1]$\looparrowright$ $\mathbf{R}^2$ induces a path

$$\hat{f}\colon [0,1] \to T\mathbf{R}^2\,,\; t \mapsto \left( f(t)\,,\, \frac{f'(t)}{\|f'(t)\|} \right).$$

If the curve f is closed then this path $\hat{f}$ is a loop.
KEY FACT: If two regular closed curves f and g are regularly homotopic then the associated loops $\hat{f}$ and $\hat{g}$ are homotopic as loops.

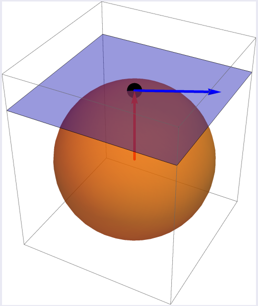
# Loop in the unit tangent bundle of $S^2$

Similarly, in order to classify regular closed curves f: $[0,1] \looparrowright S^2$ we should study the loops

$$\hat{f}\colon [0,1] \to TS^2\,,\ t \mapsto \hat{f}(t) = \left( f(t), \frac{f'(t)}{\|f'(t)\|} \right).$$

where $TS^2$ is the unit tangent bundle of the sphere.

## A point (x,v) in ...



## ... the unit tangent bundle $TS^2$

A point (x,v)$\in TS^2$ : x is the red vector which represents a point in the sphere and v is the blue vector which is tangent to the sphere at the point x and of unit length. Recall that v is tangent to $S^2$ at x if and only if v$\perp$ x.

# The unit tangent bundle of the sphere is SO(3)

$$
\begin{aligned}
TS^2 &:= \{(x,v) : x \in S^2, v \text{ tangent to } S^2 \text{ at } x, \|v\| = 1\} \\
&= \{(x,v) \in \mathbf{R}^3 \times \mathbf{R}^3 : \|x\| = 1, v \perp x, \|v\| = 1\} \\
&\cong \{(x, v, x \times v) : x, v \in \mathbf{R}^3, \|x\| = 1, v \perp x, \|v\| = 1\} \\
&\cong \{(x, v, w) \text{ orthonormal direct basis of } \mathbf{R}^3\} \\
&\cong \{A = (x, v, w) \in \mathbf{R}^{3\times 3} : A^t A = I, \det(A) = 1\} \\
&= SO(3) = \{\text{linear rotations about 0 of } \mathbf{R}^3\}.
\end{aligned}
$$

Here $x \times v$ is the cross product of vectors in $\mathrm{R}^3$.

# Def's: path, (based) loop and their homotopies

Let X be a space.
A path in X is a continuous map $\omega\colon [0,1]\to X, t\mapsto \omega(t)$.
This path connects the point $\omega(0)$ to the point $\omega(1)$.
A loop is a path such that $\omega(0)=\omega(1)$.
Two loops $\alpha$ and $\beta$ are homotopic loops if there exists a continuous map

$$\Omega\colon [0,1]\times[0,1] \longrightarrow X\,,\ (t,u)\longmapsto \Omega_u(t)$$

such that

1. $\forall$ $u\in[0,1]$: $\Omega_u(0)=\Omega_u(1)$ (i.e. $\Omega_u$ is a loop)
2. $\Omega_0=\alpha$
3. $\Omega_1=\beta$

Fix $x_0\in X$. A based loop at $x_0$ is a loop such that $\omega(0)=x_0$. A homotopy of based loops is a homotopy $\Omega$ of loops such that $\Omega_u(0)=\Omega_u(1)=x_0$ for each $u\in[0,1]$.

# The fundamental group

Let X be a space and $x_0 \in X$ a chosen point (called the base point).
Let $\omega$:[0,1]$\rightarrow$ X with $\omega(0)=\omega(1)=x_0$ be a based loop.
We denote by $[\omega]$ the equivalence class w.r.t based homotopy of the based loop $\omega$:[0,1]$\rightarrow$ X with $\omega(0)=\omega(1)=x_0$.
In other words $[\omega] = [\psi]$ if and only if $\omega$ and $\psi$ are homotopic as based loops.
We set $\pi_1(X,x_0)=\{[\omega]:\omega$ is based loop of $X\}$.
We can concatenate loops and this define a "multiplication" on $\pi_1(X,x_0)$ which is

1. associative
2. has a unit represented by the constant loop at $x_0$
3. each class of loop as an inverse for that multiplication.

Thus $\pi_1(X,x_0)$ is a group called the fundamental group of X.
The proof of (1)-(3) is not difficult but not immediate.

# Examples of fundamental groups

- $\pi_1(R^2,(0,0))\cong\{0\}$
- $\pi_1(S^1,(1,0)) \cong Z$
- $\pi_1(\text{torus } T,x_0)\cong Z\times Z$
- $\pi_1(S^2,(1,0,0)) \cong\{0\}$
- $\pi_1(\text{projective plane } P,x_0) \cong Z/2Z$
- $\pi_1(TR^2 ,x_0)\simeq \pi_1(R^2 \times S^1, x_0) \cong Z$
- $\pi_1(TS^2 ,x_0)\cong \pi_1(SO(3),x_0) \cong Z/2Z$

All of this is intuitive (except the last) but need proofs. The notion of covering space that we will see in the next lecture is a usefull tool to compute $\pi_1$.