# Shapes of Spaces II

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Homology: a practical approach
Summer School, Lisbon, July 2017[1]

[1]Nothing of what follows is due to me. See the list of references

Last lecture we saw that something called *homology* efficiently translated questions about shapes of spaces that otherwise seemed hard to answer into easy linear algebra.[2]

**Today we look at it from a practical angle:**

can homology tell us something about real data sets, and how can we calculate it?

[2] Shout if I forget to digress on fusion reactors and Dan's talk

# Topological Data Analysis

How can one extract information about the *shape* underlying data?

**Carlsson's list** [3]

1. Coordinates are not natural (enter geometry)
2. Metrics are not theoretically justified (enter topology)
3. Qualitative information needed (enter alg. topology)
4. Summaries are more valuable than individual parameter choices (enter functoriality)

[3] This list could equally well advocate what statisticians call "cluster analysis" - which in some ways is the zero dimensional version of the below.

# **Example:** natural images (pictures!)



Digital pictures must be compressed.
Low contrast areas are uninteresting.
What do the high contrast $3 \times 3$ patches of natural images look like?
Simplify to look at gray scale images – so that we're in 9D [4]
The distribution is not uniform: there's a region of vastly higher density than the rest. We need to understand this region. [5]
How can we capture the *shape* of this region?

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[4] really discrete...

[5] compare w. linear regression. However, here we're not approximating with lines, but some other spaces – like ($\simeq$ to) circles, spheres...– capable of expressing the data better.
See refs at Natural Klein

# Building spaces from point clouds



**Example: the Čech complex** of a point cloud

$$V = \{v_0, \ldots, v_N\} \subseteq \mathbb{R}^d$$

Given $r \geq 0$.
A subset $\sigma = \{v_{i_0}, \ldots, v_{i_k}\} \subseteq V$ is a ($k$-dimensional) *face/simplex* in the *Čech complex* $\check{C}(r)$ iff $\sigma$ is contained in a closed ball of radius $r$
The Čech complex $\check{C}(r)$ is something called a[6] *simplicial complex* for which computing the homology is just linear algebra.
An important feature is that $r$ can be varied, displaying the *persistent* shapes of the cloud.

[6]insert "n abstract" if you want to

# Building spaces from point clouds



**Example: the Vietoris-Rips complex** of a point cloud
Given $r \geq 0$. Draw a

1. line between each pair of points (in the cloud) at most $2r$ apart,
2. triangle (filled) between each triple of points at most $2r$ apart
3. tetrahedron (filled) for each quadruple of points at most $2r$ apart...

The result is the *Vietoris-Rips complex*[7] which is something called a *simplicial complex* for which computing the homology is just linear algebra.
An important feature is that $r$ can be varied, displaying the *persistent* shapes of the cloud.

[7] $VR(r)$ or $VR(2r)$ depending on source

# Simplicial complexes





A (finite, ordered) *simplicial complex* is a (finite, ordered) set

$$V = \{v_0, \dots, v_N\}$$

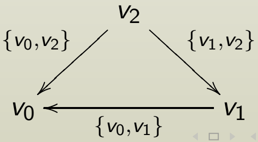
together with a set $F$ of nonempty subsets (called faces or simplices) of $V$, having the properties that

- if $\sigma \in F$ and $\tau$ is a subset of $\sigma$, then $\tau \in F$.
- $\{v_i\} \in F$ for all $i = 1, \dots, N$

The *dimension* of a face $\sigma \in F$ is $\dim \sigma = \#\sigma - 1$ (so that $\dim\{v_3\} = 0$)

**Example 1**: The **standard two-simplex** $\Delta^2 = (V, F)$ with $V = \{v_0, v_1, v_2\}$, $F$ *all* nonempty subsets.

**Example 2**: Its boundary, $\Sigma^1 = (V, F)$ with $V = \{v_0, v_1, v_2\}$, $F$ all *proper* subsets ($V \notin F$)



# Realization of $(V, F)$, $V = \{v_0, \ldots, v_N\}$





The *realization* of a face $\sigma = \{v_{i_0}, \ldots, v_{i_k}\} \in F$ is the convex hull of the unit vectors $\{e_{i_0}, \ldots, e_{i_k}\}$ in $\mathbb{R}^{N+1}$.
The *realization* of the simplicial complex $(V, F)$ in $\mathbb{R}^N$ is the union of the realization of its faces.

**Examples**

**1**: The realization of $\Delta^2$ is a 2-simplex: a filled triangle (and so $\simeq$ a 2-disc).
**2** The realization of $\Sigma^1$ is a triangle (and so $\simeq$ a circle).

*convex hull* of $v_0, \ldots, v_n$: the set of all linear combinations $a_0 v_0 + \ldots a_n v_n$ with the $a_i$ non-negative reals and $\sum a_i = 1$
– it is a "$k$-simplex".

Note: we index the coordinates of $\mathbb{R}^{N+1}$ starting with zero.
*Standard unit vectors:*
$e_0 = (1, 0, \ldots, 0), \ldots e_N = (0, \ldots, 0, 1)$

# The chain complex, $(C_*K, \partial)$, for $K = (V, F)$



Let $C_nK$ be the vector space of all linear combinations

$$a_0\sigma_0 + \cdots + a_k\sigma_k \text{ with } a_i \in \mathbb{F} \text{ and } \sigma_i \in F \text{ of dimension } n$$

Given $\sigma = \{v_{i_0}, \ldots, v_{i_n}\}$ (with $i_0 < i_1 < \cdots < i_n$) let its *boundary* be

$$\partial_n\sigma = \{\widehat{v_{i_0}}, v_{i_1}, \ldots, v_{i_n}\} - \{v_{i_0}, \widehat{v_{i_1}}, \ldots, v_{i_n}\} + \cdots + (-1)^n\{v_{i_0}, \ldots, v_{i_{n-1}}, \widehat{v_{i_n}}\}$$

Let $\partial_n \colon C_nK \to C_{n-1}K$ be the resulting linear map.

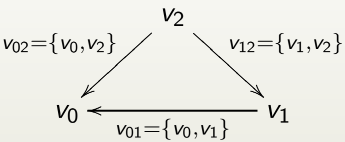
$\mathbb{F}$ is a "field": think of $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{F}_2 = \{0, 1\}$.

Notation: $(v_0, \widehat{v_1}, v_2) = (v_0, v_2)$, so that $\partial_2(v_0, v_1, v_2) = (v_1, v_2) - (v_0, v_2) + (v_0, v_1)$.

# Example, $\Sigma^1$

$\Sigma^1 = (V, F)$ with $V = \{v_0, v_1, v_2\}$, $F$ all *proper* subsets.

$\Sigma^1 =$ boundary of triangle:




dim $C_0K = 3$, dim $C_1K = 3$, dim $C_2K = 0 \ldots$.
– $C_0K$: basis $(v_0, v_1, v_2)$,
– $C_1K$: basis $(v_{12}, v_{02}, v_{01})$,
– $\partial_1 v_{ij} = v_j - v_i$.
In these bases, the matrix for $\partial_1$ is

$$[\partial_1] = \begin{bmatrix} 0 & -1 & -1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Note that $\partial_1(v_{12} - v_{02} + v_{01}) = 0$; we have a "cycle".

Simplify: write $v_1$ for $\{v_1\}$ and $v_{12}$ for $\{v_1, v_2\}$

# Homology: that which counts



- $Z_nK = \ker \partial_n$ is the subspace of *cycles*
  **only cycles should count**: that $\partial_n v = 0$ measures that $v$ "wraps up" possibly capturing a "hole".
- $B_nK = \operatorname{im} \partial_{n+1}$ is the subspace of *boundaries*
  **boundaries should not count**, that $v = \partial_{n+1} z$ means that the hole is *filled* by $z$ (there *is* no hole!).

We define the $n$th *homology* as **that which counts:**

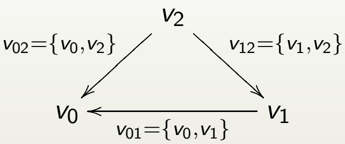
$$H_nK = Z_nK/B_nK$$
$$= Z_nK/ \underset{z_1 - z_2 \in B_nK}{\overset{z_1 \sim z_2}{\Updownarrow}}$$

$C_nK$: linear combinations of $n$-simplices

$Z_nK = \ker \partial_n$ "cycles" (null space)
$B_nK = \operatorname{im} \partial_{n+1}$ "boundaries" (column space)
Fact/exercise: $B_nK \subseteq Z_nK$

# $\Sigma^1$: $V = \{v_0, v_1, v_2\}$, $F$ all *proper* subsets

$\Sigma^1 =$ boundary of triangle:




$$0 \xleftarrow{\partial_0} C_0\Sigma^1 \xleftarrow{\quad \partial_1 \quad} C_1\Sigma^1 \xleftarrow{\partial_2} C_2\Sigma^1 \dots$$

$$0 \longleftarrow \langle v_0, v_1, v_2\rangle \xleftarrow{\begin{bmatrix} 0 & -1 & -1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}} \langle v_{12}, v_{02}, v_{01}\rangle \longleftarrow 0 \dots .$$

$Z_0\Sigma^1 = C_0\Sigma^1 = \mathbb{F}\langle v_0, v_1, v_2\rangle$,
$B_0\Sigma^1 = \langle v_1 - v_2, v_0 - v_2, v_0 - v_1\rangle = \langle v_1 - v_2, v_0 - v_2\rangle$,
$Z_1\Sigma^1 = \langle v_{01} - v_{02} + v_{12}\rangle$, $B_1 K = 0$.

$$\dim H_k\Sigma^1 = \begin{cases} 1 & \text{if } k = 0, 1 \\ 0 & \text{otherwise.} \end{cases}$$

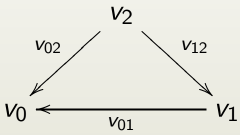
$Z_n K$: the *null space* of $[\partial_n]$
$B_n K$: the *column space* of $[\partial_{n+1}]$
$H_n K = Z_n K / \underset{z_1 - z_2 \in B_n K}{\overset{z_1 \sim z_2}{\Updownarrow}}$

# Example $\Sigma^1$ (cont.)



$\Sigma^1 = (V, F)$ with $V = \{v_0, v_1, v_2\}$, $F$ all *proper* subsets.

bdry of $\Delta^2$




$$\dim H_k \Sigma^1 = \begin{cases} 1 & \text{if } k = 0, 1 \\ 0 & \text{otherwise.} \end{cases}$$

$\dim H_1 \Sigma^1 = 1$ (with generator $v_{12} - v_{02} + v_{01}$) corresponds to $\dim \tilde{H}_1 S^1 = 1$ from Monday.
$\tilde{H}_1 D^2 = 0$ corresponds to $H_1 \Delta^2 = 0$: there

$$\partial_2 v_{012} = v_{12} - v_{02} + v_{01}$$

$\dim \tilde{H}_n S^n = 1$,
$\dim \tilde{H}_n D^{n+1} = 0 \Rightarrow$
"Nobel Prize"

# Example $\Sigma^2$

$\Sigma^2 = (V, F)$ with $V = \{v_0, v_1, v_2, v_3\}$, $F$ all *proper* subsets.
$\Sigma^2 =$ boundary of $\Delta^3$ (with *all* nonempty subsets of $V$).
$\Sigma^2$ realizes to $\simeq S^2$.

$$C_0\Sigma^2 \xleftarrow[\partial_1 v_{ij} = v_j - v_i]{\partial_1} C_1\Sigma^2 \xleftarrow[\partial_2 v_{ijk} = v_{jk} - v_{ik} + v_{ij}]{\partial_2} C_2\Sigma^2$$

$$\mathbb{F}\left\langle \begin{smallmatrix} v_0 & v_1 \\ v_2 & v_3 \end{smallmatrix} \right\rangle \xleftarrow{\begin{bmatrix} -1 & -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}} \mathbb{F}\left\langle \begin{smallmatrix} v_{01} & v_{02} \\ v_{03} & v_{12} \\ v_{13} & v_{23} \end{smallmatrix} \right\rangle \xleftarrow{\begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}} \mathbb{F}\left\langle \begin{smallmatrix} v_{012} & v_{013} \\ v_{023} & v_{123} \end{smallmatrix} \right\rangle$$

$H_2\Sigma^2 = Z_2\Sigma^2 = \langle v_{123} - v_{023} + v_{013} - v_{012} \rangle$.
Corresponds to $\dim \tilde{H}_2 S^2 = 1$ from Monday.
$\tilde{H}_2 D^3 = 0$ corresponds to $H_2\Delta^3 = 0$: there

$$\partial_3 v_{0123} = v_{123} - v_{023} + v_{013} - v_{012}$$

$\dim \tilde{H}_n S^n = 1$,
$\dim \tilde{H}_n D^{n+1} = 0 \Rightarrow$
"Nobel Prize"

# Ex: $\Sigma^n$ (all proper)/$\Delta^{n+1}$ (all nonempty subsets)



Finishing off the linear algebra [8] gives for $n > 0$

$$\dim H_k\Sigma^n = \begin{cases} 1 & \text{if } k = 0, n \\ 0 & \text{otherwise} \end{cases}$$

$$\dim H_k\Delta^{n+1} = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases}$$

essentially telling us that

– $\Sigma^n$ and $\Delta^{n+1}$ are connected ($\dim H_0 = 1$)
– $\Delta^{n+1}$ has no "holes" ($H_k\Delta^{n+1} = 0$, $k > 0$)
– $\Sigma^n$ has one "hole" in dim $n$.

Disclaimer: that this homology of simplicial complexes can serve the purpose of the invariants discussed on Monday is nontrivial (but *somewhere* I had to cheat!).

[8] or, if you're lazy, letting a computer do it

# The Čech complex



Given a finite set $V \subset \mathbb{R}^d$ and $r \geq 0$

**Definition** *Čech complex*

$\check{C}(r) = \{\sigma \subseteq V \mid \sigma \text{ is contained in a closed ball of radius } r\}$

$\check{C}(r)$ grows with $r$.

**Extremes uninteresting:**

For small $r$: $\check{C}(r) \leftrightarrow V$ (discrete).

For big $r$: $\check{C}(r) \leftrightarrow \Delta^{\#V-1}$.

**Idea:**

Important features of the point cloud shows up somewhere in the middle, and is more *persistent* than the noise.

# Example - the sixth roots of unity in $\mathbb{C} = \mathbb{R}^2$



$$V = \{e^{\frac{2\pi i}{6}k} \mid k = 0, \ldots, 5\} \subseteq \mathbb{C}.$$

$$\check{C}(r) = \begin{cases} V & \text{if } 0 \leq r < 1/2 \\ \text{hexagon } [\simeq S^1] & \text{if } 1/2 \leq r < \sqrt{3}/2 \\ \text{cylinder } [\simeq S^1] & \text{if } \sqrt{3}/2 \leq r < 1 \\ \Delta^5 \, [\simeq D^5] & \text{if } 1 \leq r \end{cases}$$

$$\dim H_0 \check{C}(r) = \begin{cases} 6 & \text{if } 0 \leq r < \frac{1}{2} \\ 1 & \text{if } \frac{1}{2} \leq r < \infty \end{cases}$$

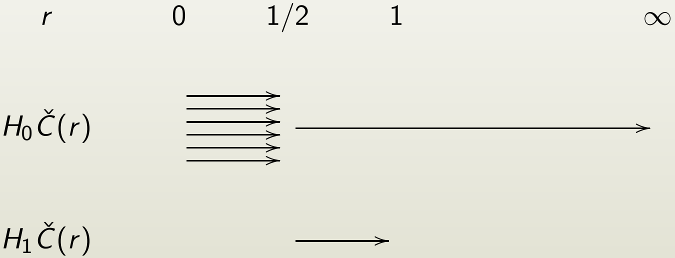
$$\dim H_1 \check{C}(r) = \begin{cases} 0 & \text{if } 0 \leq r < \frac{1}{2}, 1 \leq r < \infty \\ 1 & \text{if } \frac{1}{2} \leq r < 1 \end{cases}$$

$$\dim H_j \check{C}(r) = 0 \qquad \text{for } j \neq 0, 1$$

$\check{C}(r) = \{\sigma \subseteq V \mid \sigma \text{ in a closed } r\text{-ball}\}$

Bar codes, persistent homology

Typically, the homology classes are represented by *bar codes*.




In the example, the homology of a circle is fairly persistent.

# The Vietoris-Rips complex



Given a finite set $V \subset \mathbb{R}^d$ and $r \geq 0$

**Definition** *Vietoris-Rips complex*

$VR(r) = \{\sigma \subseteq V \mid \operatorname{diam} \sigma \leq 2r\}$

$VR(r)$ grows with $r$.
**Extremes uninteresting:**
For small $r$: $VR(r) \leftrightarrow V$ (discrete).
For big $r$: $VR(r) \leftrightarrow \Delta^{\# V - 1}$.

**Idea:**

Important features of the point cloud shows up somewhere in the middle, and is more *persistent* than the noise.

$\operatorname{diam} \sigma = \max\{|v - w| \mid v, w \in \sigma\}$
$s \leq t \Rightarrow VR(s) \subseteq VR(t)$

# Example - the sixth roots of unity in $\mathbb{C} = \mathbb{R}^2$





$$V = \{e^{\frac{2\pi i}{6}k} \mid k = 0, \dots, 5\} \subseteq \mathbb{C}.$$

$$VR(r) = \begin{cases} V & \text{if } 0 \le r < 1/2 \\ \text{hexagon } [\simeq S^1] & \text{if } 1/2 \le r < \sqrt{3}/2 \\ \text{octahedron } [\simeq S^2] & \text{if } \sqrt{3}/2 \le r < 1 \\ \Delta^5 [\simeq D^5] & \text{if } 1 \le r \end{cases}$$

$$\dim H_0 VR(r) = \begin{cases} 6 & \text{if } 0 \le r < \frac{1}{2} \\ 1 & \text{if } \frac{1}{2} \le r < \infty \end{cases}$$

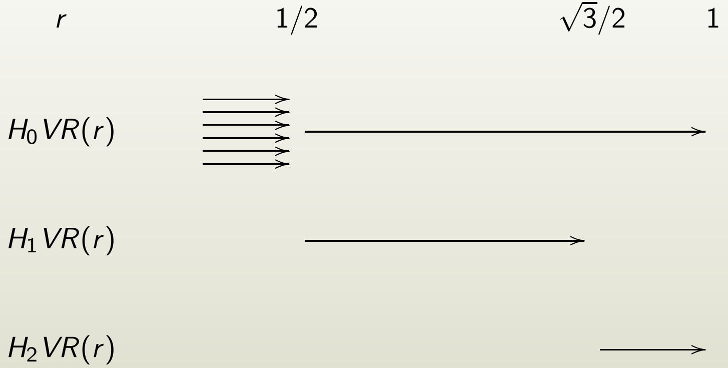
$$\dim H_1 VR(r) = \begin{cases} 0 & \text{if } 0 \le r < \frac{1}{2}, \frac{\sqrt{3}}{2} \le r < \infty \\ 1 & \text{if } \frac{1}{2} \le r < \frac{\sqrt{3}}{2} \end{cases}$$

$$\dim H_2 VR(r) = \begin{cases} 0 & \text{if } 0 \le r < \frac{\sqrt{3}}{2}, 1 \le r < \infty \\ 1 & \text{if } \frac{\sqrt{3}}{2} \le r < 1 \end{cases}$$

$$VR(r) = \{\sigma \subseteq V \mid \operatorname{diam} \sigma \le 2r\}$$

## Bar codes, persistent homology

Typically, the homology classes are represented by *bar codes*.




In the example, the most persistent feature is the homology of a circle, but it also shows a deficit of the Vietoris-Rips complex: fake high-D features can appear.[9]

[9] a shape inside the plane can't possibly have nontrivial $H_2$s

# Persistent homology

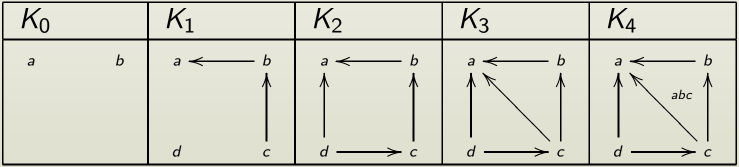
Can't compute the homology for every $r \geq 0$[10]

**Idea** Introduce "time"=radius *into* the algorithm. The (filtration) *degree* of a simplex is the time of its birth.

This changes the "scalars" for the homology from $\mathbb{F}$ to $\mathbb{F}[t]$, the *polynomials* in $t$.

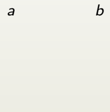
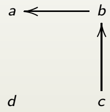
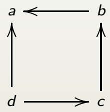
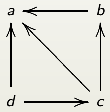
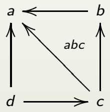
Multiplication by $t$ represent a time step.

**Example** (stolen)




[10] the homology of the Čech or Vietoris-Rips complex changes only finitely many times, and so is a finite problem. However, for large $V$ it is a BIG finite problem.

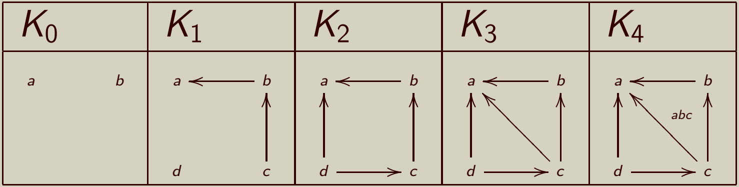
# Persistent homology **Example**

| $K_0$ | $K_1$ | $K_2$ | $K_3$ | $K_4$ |
|---|---|---|---|---|
|  |  |  |  |  |
| $H_0 =$ $\mathbb{F}\langle a, b\rangle$ | $\mathbb{F}\langle a, d\rangle$ | $\mathbb{F}\langle a\rangle$ | $\mathbb{F}\langle a\rangle$ | $\mathbb{F}\langle a\rangle$ |
| $H_1 =$ $0$ | $0$ | $\mathbb{F}\langle z\rangle$ | $\mathbb{F}\langle z, w\rangle$ | $\mathbb{F}\langle z\rangle$ |

$z = ab + bc + cd - ad$,

$w = ac + cd - ad$

Draw *bar codes* for $a, b, d, z, w$!

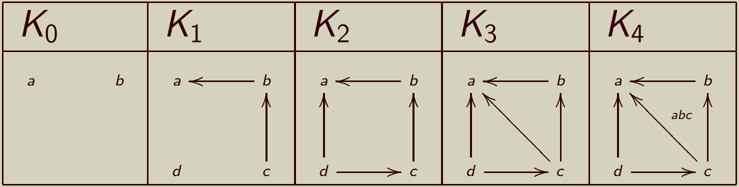



Filtration degrees of simplices:

| a | b | c | d | ab | bc | ad | cd | ac | abc |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | 3 | 4 |

Keep the calculations as before, but insert a "$t$" for each degree shift. Examples:

$$\partial(bc) = c - tb$$
$$\partial(abc) = t^3 bc - tac + t^3 ab$$




| a | b | c | d | ab | bc | ad | cd | ac | abc |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | 3 | 4 |

$$\mathbb{F}[t]\langle \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \rangle \xleftarrow{\partial_1} \mathbb{F}[t]\langle \begin{smallmatrix} ab & bc & ad \\ cd & ac & \end{smallmatrix} \rangle \xleftarrow{\partial_2} \mathbb{F}[t]\langle abc \rangle$$

Keep the calculations as before, but insert a "$t$" for each degree shift.

$$[\partial_1] = \begin{bmatrix} -t & . & -t^2 & . & -t^3 \\ t & -t & . & . & . \\ . & 1 & . & -t & t^2 \\ . & . & t & t & . \end{bmatrix}$$

$$[\partial_2] = \begin{bmatrix} t^3 \\ t^3 \\ . \\ . \\ -t \end{bmatrix}$$

| $H_0$, | a | b | c | d | ab | bc | ad | cd | ac | abc |
|---|---|---|---|---|---|---|---|---|---|---|
| | 0 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | 3 | 4 |



$$Z_0 = \mathbb{F}[t]\langle \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \rangle \xleftarrow{\partial_1} \mathbb{F}[t]\langle \begin{smallmatrix} ab & bc & ad \\ cd & ac & \end{smallmatrix} \rangle \xleftarrow{\partial_2} \mathbb{F}[t]\langle abc \rangle$$

$$[\partial_1] = \begin{bmatrix} -t & \cdot & -t^2 & \cdot & -t^3 \\ t & -t & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & -t & t^2 \\ \cdot & \cdot & t & t & \cdot \end{bmatrix}, \quad \mathrm{Col}[\partial_1] = \langle \begin{bmatrix} -t \\ t \\ \cdot \\ \cdot \end{bmatrix}, \begin{bmatrix} \cdot \\ -t \\ 1 \\ \cdot \end{bmatrix}, \begin{bmatrix} -t^2 \\ \cdot \\ \cdot \\ t \end{bmatrix} \rangle$$

$B_0 = \langle -ta + tb, -tb + c, -t^2a + td \rangle \leftrightarrow \mathrm{Col}[\partial_1]$, so that $H_0$ is spanned by $a, b, c, d$ over $\mathbf{F}[t]$ subject to the relations $ta \sim tb$, $c \sim tb$ and $td \sim t^2a$.

$$H_0 = \mathbb{F}[t]\langle a \rangle \oplus \frac{\mathbb{F}[t]\langle b - a \rangle}{t(b-a)} \oplus \frac{\mathbb{F}[t]\langle d - ta \rangle}{t(d - ta)}$$

The class $a$ is born at 0 and never dies,
the class $b - a$ is born at 0 and dies at 1
the class $d - ta$ is born at 1 and dies at 2.

$\langle \dots \rangle$ is still "Span", but now over $\mathbb{F}[t]$

| $H_1$, | a | b | c | d | ab | bc | ad | cd | ac | abc |
|---|---|---|---|---|---|---|---|---|---|---|
| | 0 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | 3 | 4 |

$$\mathbb{F}[t]\langle \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \rangle \xleftarrow{\partial_1} \mathbb{F}[t]\langle \begin{smallmatrix} ab & bc & ad \\ cd & ac & \end{smallmatrix} \rangle \xleftarrow{\partial_2} \mathbb{F}[t]\langle abc \rangle$$

$$[\partial_1] = \begin{bmatrix} -t & . & -t^2 & . & -t^3 \\ t & -t & . & . & . \\ . & 1 & . & -t & t^2 \\ . & . & t & t & . \end{bmatrix}, \qquad [\partial_2] = \begin{bmatrix} t^3 \\ t^3 \\ . \\ . \\ -t \end{bmatrix}$$

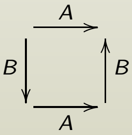
Let $z = tab + tbc + cd - ad$, $w = t^2bc - ac + t^2ab$. Then

$$Z_1 = \ker \partial_1 = \langle z, w \rangle$$
$$B_1 = \mathrm{im} \partial_2 = \langle tw \rangle$$
$$H_1 = Z_1/B_1 = \mathbb{F}[t]\langle z \rangle \oplus \frac{\mathbb{F}[t]\langle w \rangle}{tw}$$

The class $z$ is born at 2 and never dies
The class $w$ is born at 3 and dies at 4.

# Moral: Persistent homology is computable

– it is a combinatorial/linear algebra question safely left to computers.
With this tool one can discover the shape of point clouds.
For instance, one finds that very many natural images lie in a region with the homology of a circle – corresponding to horizontal edges.
Another one corresponds to vertical edges.
Slightly less persistent, a third circle appears that intersect the other two – and analyzing the persistent homology one discovers a high density surface (see picture) containing these three circles.




Here TDA was a tool useful for *discovering* a phenomenon.