Lisbon school July 2017: eversion of the sphere

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Teaser: the eversion of the sphere

In 1958 Stephen Smale (Fields medal 1966) proved that the sphere can be turned inside-out in the 3-dimensional space. =======> VIDEO
The sphere (or 2-sphere) is

$$S^2 := \{(x, y, z) \in \mathbf{R}^3 : x^2 + y^2 + z^2 = 1\}$$

We have the standard embedding

$$f_0 : S^2 \longrightarrow \mathbf{R}^3, (x, y, z) \longmapsto (x, y, z)$$

and the reversed standard embedding

$$f_1 : S^2 \longrightarrow \mathbf{R}^3, (x, y, z) \longmapsto (x, y, -z)$$

The Smale eversion is a continuous deformation

$$f_t : S^2 \longrightarrow \mathbf{R}^3 \qquad 0 \leq t \leq 1$$

such that each $\mathrm{f_t}$ and the deformation are "nice".
(Technically: it is a regular homotopy of immersions)

Five-days and one-day plans

Goal of my series of five lectures: give (the ingredients) of Smale's proof of the existence of the eversion. The proof is not by explicitely describing the deformtation f_t.

Goal of today lecture: study the analog problem for immersions of curves in the plane.
In other words we replace the 2-sphere by the circle $S^1 = \left\{(x, y) \in \mathbf{R}^2 : x^2 + y^2 = 1\right\}$ and the 3-dimensional space by the plane $\mathbf{R}^2$.

Reference for today: paper of Whitney "Closed regular curves in the plane" 1936 (cf. web site of the school)
If my lecture goes too slowly: read carefully Whitney's paper instead (Challenge: spot the mistake in theorem 1 of that paper.)

Regular parametrized curve in the plane

A parametrized regular curve or immersion of the segment in the plane is determined by a function

$$f: [0,1] \to \mathbf{R}^2\,,\; t \mapsto f(t)$$

with the following properties:

1. f is differentiable and $f' = \frac{df}{dt}$ is continuous (i.e. f is of class $\mathcal{C}^1$);
2. $f'(t) \neq 0, \forall t \in \mathbf{R}$.

Condition (1) means that there is a tangent vector at any point of the curve and that these tangent vectors vary continuously along the curve. Condition (2) means that this tangent vector never vanishes.

A regular curve may have many self-intersections.
=======> MATHEMATICA WhitneyGrausteinV4

Regular parametrized closed curve in the plane

A parametrized regular closed curve or immersion of the circle in the plane is a parametrized regular curve $f\colon [0,1]\to R^2$ such that moreover

- $f(0)=f(1)$ and $f'(0)=f'(1)$

Its image in a closed curve in the plane with a continuous never vanishing tangent vector field. =======> MATHEMATICA WhitneyGrausteinV

Regular homotopies of regular closed curves

Question: Can one "continuously deform" any regular closed curve into any other one through regular closed curves ?

Definition

A *regular homotopy* *between regular parametrized closed curves is a map*

$$f : [0,1] \times [0,1] \longrightarrow \mathbf{R}^2 \,,\; (u,t) \longmapsto f_u(t)$$

such that

1. *for each* $u \in [0,1]$ f_u *is a regular parametrized closed curve*
2. f *is continuous*
3. $\frac{\partial f}{\partial t} : [0,1] \times [0,1] \to \mathbf{R}^2$ *is continuous (continuity of the tangent vectors)*

When such a map f exists we say that the immersions f_0 and f_1 are regularly homotopic

Pool:Are those closed curves regularly homotopic?

Is the "standard embedding of the circle" immersion regularly homotopic to the "reversed embedding of the circle" immersion ?

1. $f_0(t)=(\cos(2\pi\ t),\sin(2\pi\ t))$
2. $f_1(t)=(\cos(2\pi\ t),-\sin(2\pi\ t))$

In other words can we turn the circle inside out in the plane through immersions ? =======> MATHEMATICA fkeversion

Is the "standard embedding of the circle" immersion regularly homotopic to the "figure eight" immersion ?

1. $f_0(t)=(\cos(2\pi\ t),\sin(2\pi\ t))$
2. $f_1(t)=(\sin(-2\pi\ t),\sin(-4\pi\ t))$

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To answer those questions look at the tangent vector along the immersion.

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The normalized tangent vector

Let f:[0,1] be a closed immersion.
Because of the regularity the derivative $f' = df/dt$ never vanishes so we can consider the map

$$\hat{f} \colon [0,1] \longrightarrow [0,1]\,,\; t \mapsto \frac{f'(t)}{\|f'(t)\|}.$$

For any t we have $\|\hat{f}(t)\| = 1$ and $\hat{f}(0) = \hat{f}(1)$. Also $\hat{f}$ is continuous because f is $\mathcal{C}^1$.
$\hat{f}$ is the normalized tangent vector field of the curve.

Therefore we can look at $\hat{f}$ as a continuous self-map of the circle S^1:

$$\hat{f} \colon S^1 \to S^1.$$

The degree of a self map of the circle

Consider the circle

$$S^1 := \left\{(x,y) \in \mathbf{R}^2 \,:\, x^2+y^2=1\right\}.$$

A continuous self-map of the circle $\mathrm{g}\colon\ \mathrm{S}^1 \to \mathrm{S}^1$ will "wraps" the source circle onto the target circle a number of times.
This (signed) integer is called the degree of the self map.

Notation: Denote by $[\theta]$ the point $(\cos(\theta),\sin(\theta))\in \mathrm{S}^1$.
Example: Fix n$\in$ Z. The map

$$g_n\colon S^1 \to S^1, [\theta] \mapsto [n\theta]$$

wraps the circle around itself n times and so $\deg(\mathrm{g_n})$=n. When n is negative the circle is wrapped in the reversed direction.

When f: [0,1]$\to$ R^2 is a regular closed curve then $\hat{f} = f'/\|f'\|$ is a selfmap of the circle which has a degree $deg(\hat{f}) \in \mathbf{Z}$

Whitney-Graustein theorem (thm 1 of W. article)

Let f: [0,1]$\to$R^2 be a regular closed curve (equivalently f can be seen as an immersion $S^1 \looparrowright \mathbf{R}^2$.)

The rotation number of the immersion f is

$$\gamma(f) := deg\left(\frac{f'}{\|f'\|}\right) = deg(\hat{f}) \in \mathbf{Z}.$$

- γ("standard embedding of the circle")=1
- γ("reversed embedding of the circle")=-1
- γ("figure eight")=0
- γ("circle traveled twice")=2

Theorem

Two regular closed curves $f_0, f_1 : [0,1] \to \mathbf{R}^2$ are regularly homotopic if and only if $\gamma(f_0) = \gamma(f_1)$

Precise definition of the degree

Let g: $S^1 \to S^1$. How to define deg(g)$\in$Z?
Consider the "universal covering" of the circle S^1 by the line R:

$$p\colon \mathbf{R} \longrightarrow S^1\,,\ t \longmapsto (\cos(2\pi t), \sin(2\pi t)).$$

The restriction p_1:=p|[0,1]: [0,1]$\to S^1$ identifies the circle S^1 to the interval [0,1] with their two endpoints 0 and 1 glued together.
Claim there exist a continuous map $\tilde{g}\colon [0,1] \to \mathbf{R}$ such that the following diagram commutes

$$\begin{tikzcd}
 & & \mathbf{R} \arrow[d, "p"] \\
{[0,1]} \arrow[urr, dashed, "\exists\tilde{g}"] \arrow[r, "p_1"'] & S^1 \arrow[r, "g"'] & S^1
\end{tikzcd}$$

in other words $p\tilde{g} = gp_1$.
We say that $\tilde{g}$ is a lift of the path gp_1 along p.
The degree of g is defined by $deg(g) := \tilde{g}(1) - \tilde{g}(0)$.

Exemple of lifting and formula for the degree

Consider g:$S^1 \to S^1$ defined by g($[\theta]$)=[-3θ].
In other words

$$g(\cos(\theta),\sin(\theta)) = (\cos(-3\theta),\sin(-3\theta)).$$

Then a lift $\tilde{g}$ of gp_1 along p is given by the map

$$\tilde{g}\colon [0,1] \to \mathbf{R}\,,\ t \mapsto -3t.$$

Indeed $\tilde{g}$ is continuous and for all t$\in$[0,1] we have

$$\begin{cases} p(\tilde{g}(t)) = p(-3t) = (\cos(2\pi(-3t)),\sin(2\pi(-3t))) \\ g(p_1(t)) = g((\cos(2\pi t),\sin(2\pi t)) = (\cos(-3(2\pi t),\sin(-3(2\pi t)), \end{cases}$$

thus $p\tilde{g} = gp_1$.

Therefore $deg(g) = \tilde{g}(1) - \tilde{g}(0) = -3\cdot 1 - -3\cdot 0 = -3$.

Note that $\breve{g}(t) := 5 - 3t$ is a different lift of gp_1 along p but we also have $\breve{g}(1) - \breve{g}(0) = (5 - 3\cdot 1) - (5 - 3\cdot 0) = -3$

Homotopic maps have the same degree

Let f,g: $S^1 \to S^1$ be continuous self-maps of the circle.
Let H: $S^1 \times [0,1] \to S^1$,$([\theta],u) \mapsto H_u([\theta])$ be a continuous map such that H_0=f and H_1=g.
We say that f and g are homotopic and write f$\simeq$ g.
<u>Claim</u> $f \simeq g \implies deg(f) = deg(g)$

$$
\begin{tikzcd}
 & & \mathbf{R} \arrow[d, "p"] \\
{[0,1]} \arrow[r, "p_1"'] \arrow[urr, dashed, "\exists \tilde{f}", "\exists \tilde{g}"'] & S^1 \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & S^1
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & \mathbf{R} \arrow[d, "p"] \\
{[0,1] \times [0,1]} \arrow[r, "p_1 \times \mathrm{id}"'] \arrow[urr, "\exists \tilde{H}"] & S^1 \times [0,1] \arrow[r, "H"'] & S^1
\end{tikzcd}
$$

$deg(f) = \tilde{f}(1) - \tilde{f}(0)$=deg($H_0$) $\qquad$ $deg(g) = \tilde{g}(1) - \tilde{g}(0)$=deg($H_1$)
$deg(H_u) = \widetilde{H_u}(1) - \widetilde{H_u}(1) \in \mathbf{Z}$ $\qquad$ $0 \le u \le 1$
deg(H_u) continuous in u$\in$[0,1] but $\in$Z $\implies deg(H_u)$ is constant!

Proof of one direction of Whitney-Graustein

Let f_0 and f_1 be regular closed curves in the plane.
Let $\{f_u\}_{u\in[0,1]}$ be a regular homotopy between them.
By condition(3) of the definition of a regular homotopy, $\partial f_u/\partial t$ is continuous in both variables and so

$$\hat{f}_u(t) := \frac{f_u'(t)}{\|f_u'(t)\|}$$

is also continuous where $f_u'(t) = \frac{\partial f_u}{\partial t}(t)$ is the derivative in the t-direction.
Therefore $\left\{\hat{f}_u\right\}_{u\in[0,1]}$ is a homotopy between $\hat{f}_0$ and $\hat{f}_1$ and

$$\gamma(f_0) = deg(\hat{f}_0) = deg(\hat{f}_1) = \gamma(f_1).$$

QED

The path lifting property of $p\colon \mathbf{R} \longrightarrow S^1$

Theorem

Let

- $g\colon [0,1] \to S^1$ *be a continuous map, and*
- $e_0 \in$ *such that* $p(e_0) = g(0)$,

Then there exists a continuous map $\tilde{g}\colon [0,1] \to \mathbf{R}$ *such that*

1. $p\tilde{g} = g$, *and*
2. $\tilde{g}(0) = e_0$.

We say that *$\tilde{g}$ is a lifting of the path g along p starting at e_0*

$$
\begin{tikzcd}
 & \mathbf{R} \arrow[d, "p"] & \ni e_0 \arrow[d, mapsto] \\
{[0,1]} \arrow[ur, dashed, "\exists \tilde{g}"] \arrow[r, "g"'] & S^1 & \ni g(0).
\end{tikzcd}
$$

Proof of the path lifting property of p: step 1.

Step 1: Prove the existence of a lifting under the extra assumption that the path $g\colon [0,1] \to S^1$ is missing the rightmost point (1,0) of the circle.
<u>Hint</u>: Set $A := S^1 \setminus \{1\}$ the circle minus that rightmost point. Thus g([0,1])$\subset$ A. We have a homeomorphism $p_1 :]0,1[\overset{\cong}{\to} A$.
Then $p^{-1}(A) = \mathbf{R} \setminus \mathbf{Z}$. Contemplate 10 minutes the following:

$$
\begin{tikzcd}
 & e_0 \in \mathbf{R} \arrow[d, "p"] & p^{-1}(A) = \cup_{n\in\mathbf{Z}}]n, n+1[\arrow[l, hook', "incl"'] \arrow[d] &]0,1[\times\mathbf{Z} \ni (t_0, n_0) \arrow[l, "\cong", "\psi"'] \arrow[d, "proj_1"] \\
[0,1] \arrow[r, "g"'] \arrow[ur, dashed, "\exists?\tilde{g}"] \arrow[rr, bend right, "g"'] & S^1 & A \arrow[l, hook', "inclusion"'] &]0,1[\arrow[l, "\cong", "p_1"'] \arrow[u, dotted, bend left, "\sigma"]
\end{tikzcd}
$$

where ψ(t,n):=t+n, (t_0 ,n_0):=ψ^{-1} (e_0) and σ(t):=(t,n_0).
Show that $\tilde{g}(t) := \psi(\sigma(p_1^{-1}(g(t))))$ is a lift of g along p: $p\tilde{g} = g$.

Proof of the path lifting property: remaining steps

Step 1bis: Prove the existence of a lifting under the extra assumption that the path $g\colon [0,1] \to S^1$ is missing the leftmost point (-1,0) of the circle. (Completely analoguous to step (1))

Step 2: show that we can decompose the interval $[0,1]$ in small subintervals $[t_i, t_{i+1}]$ with $0=t_0 < t_1 < \ldots < t_{N-1} < t_N = 1$ such that each restricted paths $g|[t_i, t_{i+1}]$ misses either (-1,0) or (1,0)

Step 3: Use the previous steps to construct inductively the lifting $\tilde{g}$ and finish the proof

Exercise: complete the details of the proof.

The homotopy lifting property

Theorem

Let Y be a space.

- $g\colon [0,1] \times Y \to S^1$ *be a continuous map, and*
- $e_0\colon \{0\} \times Y \to$ *be a map such that* $pe_0 = g|\{0\} \times Y$,

Then there exists a continuous map $\tilde{g}\colon [0,1] \to \mathbf{R}$ *such that*

1. $p\tilde{g} = g$, *and*
2. $\tilde{g}(0) = e_0$.

We say that $\tilde{g}$ is a lifting of the homotopy g along p starting at e_0

When Y is the one-point space, this is the path lifting property.

When Y=[0,1] is the unit interval, this is the lifting of homotopy of paths that we needed before.

Exercise: prove the above homotopy lifting theorem. Hint Very similar the proof of the path lifting property. Look at Hacher "Algebraic topology", proof of property (c) on page 29 (freely available on the web). Dont be scared by the apparent complexity of Hatcher: have faith and struggle.

Whitney's magic formula for the turning number

Let Γ be a regular closed curve parametrized by $f\colon [0,1] \to \mathbf{R}^2$.
Assume that Γ has only finitely many self-intersections and that they are crossing points with exactly 2 branches with linearly independent tangent vectors at that point (i.e. Γ is "transverse" to itself.)
Assume also that Γ has been translated and reparametrized so that its lowest point is $f(0)$ and is on the x-axis.
For each crossing point $P = f(t_1) = f(t_2)$ of the curve with $0 \leq t_1 < t_2 \leq 1$ note that by assumption $f'(t_1)$ and $f'(t_2)$ are linearly independent thus the 2×2 matrix $(f'(t_1)\ f'(t_2))$ has a non-zero determinant whose sign is the orientation of the basis $(f'(t_1), f'(t_2))$.
Moreover the tangent vector at $t = 0$ is parallel to the x-axis and thus $f'(0)$ has a non zero x-component denoted by $x'(0) \in \mathbf{R}$.
Then (see Whitney'paper, theorem 2):

$$\gamma(f) = \mathrm{sign}(x'(0)) + \sum_{0\leq t_1<t_2\leq 1\,,\, f(t_1)=f(t_2)} \mathrm{sign}\left(\det\left(f'(t_1)\ f'(t_2)\right)\right).$$

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EXERCISES: Examples and non examples of immersions in the plane

For each of the following functions $f : \mathbf{R} \to \mathbf{R}^2$ determine whether it is a $\mathcal{C}^1$ -immersion. Draw the corrseponding curve in the plane.

- $f(t)=(t,t^2)$
- $f(t)=(\cos(2\pi\ t),\sin(2\pi\ t))$
- $f(t)=(t^3,\ t^2)$
- $f(t) = (t, \sqrt[3]{t^2})$
- $f(t)=(\sin(2\pi\ t),\sin(4\pi\ t))$
- $f(t)=\cos(6\pi\ t)\cdot(\ \cos(2\pi\ t),\sin(2\pi\ t)\ )$
- $f(t)=(t^3\ ,t^3)$

Prove that there exists no immersion of a closed curve in the real line.

EXERCISES: on the degree

- Prove that if a self map of the circle, g:$S^1 \to S^1$, is not surjective then it is homotopic to a constant map.
- Prove that the degree of a self map of S^1 is always an integer.
- Find and prove a formula for the degree of the composition of two self-maps of the circle in terms of the degrees of each of the maps.
- Prove that the degree of a map is well defined and does not depend on the choice of the lifting.
- Complete the details of the proof of the existence of a lifting along p of path in the circle. After that do the same for the general homotopy lifting theorem for p: $R \to S^1$.
- Prove that if two self maps of the circle have the same degree then they are homotopic. Hint: Assume first that the two maps have the same value at (0,1). Show that their liftings can be taken with same origins and same endpoint. Show that there is a homotopy between the liftings fixing these extremities and project this homotopy on the circle.

Challenges

- Spot a mistake in Whitney's paper.
  Hint: with his definition of deformation on p. 278 (not same as ours), the proof of the first half of his theorem 1 is wrong.
  Find an explicit counterexample to his theorem 1 (with his definition).
  Fix his definition to make theorem 1 correct.
- Consider closed regular curves f: [0,1]$\to$ S^2 in the 2-sphere $S^2 = \{(x,y,z) \in \mathbf{R}^2 \,:\, x^2+y^2+z^2=1\}$ instead of the plane.
  1. Show that the immersion $f_1\colon t \mapsto (\cos(2\pi t), \sin(2\pi t), 0)$ traveling the equator in one direction is regularly homotopic to the one traveling the equator in the other direction, $f_{-1}\colon t \mapsto (\cos(2\pi t), -\sin(2\pi t), 0)$
  2. Show that f_1 is regularly homotopic to the immersion $f_3\colon t \mapsto (\cos(6\pi t), \sin(6\pi t), 0)$ traveling the equator 3 times
  3. Is f_1 regularly homotopic to the immersion $f_2\colon t \mapsto (\cos(4\pi t), \sin(4\pi t), 0)$ traveling the equator 2 times ?

Further challenges

- Prove the magic formula of Whitney in the special case when they are no self-intersections. This is the "Umlaufsatz of Hopf".
  Hint define $\psi(t_1, t_2) = (f(t_2) - f(t_1))/dist(t_1, t_2)$ on the half-open solid triangle $0 \leq t_1 < t_2 \leq 1$ and extend continuously on the boundary $t_1 = t_2$ by the derivative $f'(t_1)$. Look at `http://www.mathematik.com/Hopf/` for a pictorial proof.