Shapes of Spaces I

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The idea of Algebraic Topology through examples
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Example: the Hairy Ball theorem

"Somewhere on the earth the wind is still – always."

HOW *can one possibly prove such a thing???*

1. Mountains and valleys – it makes no difference.
2. However, arches (one or two or...)...

Algebraic topology

the idea: translate the problem to a problem in algebra which is

easy/easier/hopefully possible

to solve.

An easier example illustrating the method

The intermediate value theorem

Given a continuous function

$$g\colon [-1,1] \to [-1,1]$$

with $g(-1) = -1$ and $g(1) = 1$ there is a $t_0 \in [-1,1]$ s.t. $g(t_0) = 0$

IF there were *no* such t_0, the expression $f(t) = \frac{g(t)}{|g(t)|}$ would give a well defined continuous function

$$f\colon [-1,1] \to \{-1,1\}$$

with $f(-1) = -1$ and $f(1) = 1$.

There are many proofs of IVT; we give a sketch of an approach that works in much wider context. If one is not careful, the presentation below is circular, but still illustrates the point.

We are reduced to showing that

IVT2

There is *no* continuous function

$$f\colon [-1,1] \to \{-1,1\}$$

with $f(-1) = -1$ and $f(1) = 1$.

Assume you somehow had access to a device, which

1. to a space X gives a *vector space* $\tilde{H}_0(X)$ s.t.

$$\tilde{H}_0([-1,1]) = 0 \text{ and } \tilde{H}_0(\{-1,1\}) = \mathbb{R}$$

   and

2. to a continuous function $f\colon X \to Y$ gives a *linear map* $\tilde{H}_0 f\colon \tilde{H}_0 X \to \tilde{H}_0 Y$ s.t.
   1. $\tilde{H}_0(f \circ g) = \tilde{H}_0 f \circ H_0 g$ and
   2. $\tilde{H}_0(\text{identity map on } X) = \text{identity map on } \tilde{H}_0(X)$.

what's a *space*?
vector space?
continuous?
linear?
$f \circ g$?
identity?

SUPPOSE there were a continuous function

$$f\colon [-1,1] \to \{-1,1\}, \qquad f(-1) = -1 \text{ and } f(1) = 1.$$

IF so,

1. $\tilde{H}_0 f$ is the linear map $0 \to \mathbb{R}$, and
2. $\tilde{H}_0$ of $\{-1,1\} \subseteq [-1,1]$ is the linear map $\mathbb{R} \to 0$.
3. The composite

$$\{-1,1\} \subseteq [-1,1] \to \{-1,1\}$$

is the identity,
so $\tilde{H}_0$ of it is the identity $\mathbb{R} = \mathbb{R}$

BUT, ☺, the identity $\mathbb{R} = \mathbb{R}$
is not the zero map $\mathbb{R} \to 0 \to \mathbb{R}$!!!

a space X gives a *vector space* $\tilde{H}_0 X$ s.t.
– $\tilde{H}_0([-1,1]) = 0$
– $\tilde{H}_0(\{-1,1\}) = \mathbb{R}$
a continuous function $f\colon X \to Y$ gives a *linear map* $\tilde{H}_0 f\colon \tilde{H}_0 X \to \tilde{H}_0 Y$ s.t.
– $\tilde{H}_0(f \circ g) = \tilde{H}_0 f \circ \tilde{H}_0 g$ and
– $\tilde{H}_0(\text{id map on } X) = \text{id map on } \tilde{H}_0(X)$.

Summary IVT proof

We showed the intermediate value theorem by showing that [1]

there is **no** *continuous function* $D^1 \to S^0$ *such that the composite* $S^0 \subseteq D^1 \to S^0$ *is the identity,*

which is true simply because

$\tilde{H}_0 S^0 = \mathbb{R}$, $\tilde{H}_0 D^1 = 0$ *and*
the identity map $\mathbb{R} = \mathbb{R}$ *is* **not** *zero.*

a space X gives a *vector space* $\tilde{H}_0 X$ s.t.
- $\tilde{H}_0 D^1 = 0$
- $\tilde{H}_0 S^0 = \mathbb{R}$

a continuous function $f : X \to Y$ gives a *linear map* $\tilde{H}_0 f : \tilde{H}_0 X \to \tilde{H}_0 Y$ s.t.
- $\tilde{H}_0(f \circ g) = \tilde{H}_0 f \circ \tilde{H}_0 g$ and
- $\tilde{H}_0(\text{id map on } X) = \text{id map on } \tilde{H}_0(X)$.

[1] Notation: $D^1 = [-1, 1]$, $S^0 = \{-1, 1\}$

An idea that extends

there is no continuous function $D^1 \to S^0$ such that the composite $S^0 \subseteq D^1 \to S^0$ is the identity,

True because the identity map $\mathbb{R} = \mathbb{R}$ is **not** zero.
Notation:
$D^1 = [-1, 1]$,
$S^0 = \{-1, 1\}$

"$\tilde{H}_0$ is a **functor**" s.t.
$-\tilde{H}_0 D^1 = 0$
$-\tilde{H}_0 S^0 = \mathbb{R}$

a space X gives a *vector space* $\tilde{H}_0 X$ s.t.
$-\tilde{H}_0 D^1 = 0$
$-\tilde{H}_0 S^0 = \mathbb{R}$
a continuous function $f\colon X \to Y$ gives a *linear map* $\tilde{H}_0 f\colon \tilde{H}_0 X \to \tilde{H}_0 Y$ s.t.
$-\tilde{H}_0(f \circ g) = \tilde{H}_0 f \circ \tilde{H}_0 g$ and
$-\tilde{H}_0(\text{id map on } X) = \text{id map on } \tilde{H}_0(X)$.

An idea that extends

there is no continuous function $D^{n+1} \to S^n$ such that the composite $S^n \subseteq D^{n+1} \to S^n$ is the identity

True because the identity map $\mathbb{R} = \mathbb{R}$ is **not** zero.

Notation:

$D^{n+1} = \{p \in \mathbb{R}^{n+1} \,|\, |p| \leq 1\}$,

$S^n = \{p \in \mathbb{R}^{n+1} \,|\, |p| = 1\}$

"$\tilde{H}_n$ is a **functor**" s.t.

$-\tilde{H}_n D^{n+1} = 0$

$-\tilde{H}_n S^n = \mathbb{R}$

a space X gives a *vector space* $\tilde{H}_n X$ s.t.

$-\tilde{H}_n D^{n+1} = 0$

$-\tilde{H}_n S^n = \mathbb{R}$

a continuous function $f : X \to Y$ gives a *linear map* $\tilde{H}_n f : \tilde{H}_n X \to \tilde{H}_n Y$ s.t.

$-\tilde{H}_n(f \circ g) = \tilde{H}_n f \circ \tilde{H}_n g$ and

$-\tilde{H}_n(\text{id map on } X) = \text{id map on } \tilde{H}_n(X)$.

An application: Games without cooperations

In 1951 John F. Nash published a 10 page paper in *Ann. Math.* showing that

Any finite game has an *equilibrium point*.

It is a fairly simple application of

there is no continuous function $D^{n+1} \to S^n$ such that the composite $S^n \subseteq D^{n+1} \to S^n$ is the identity,

bagging him the "Nobel Prize" in Economics in 1994.[2]

Equilibrium point: a set of strategies for each of the players, such that for every player the current strategy is optimal provided that all the other players stick to their strategies

[2] but is not the justification for his 2016 Abel Prize

Brouwer's fixed point theorem

The "equilibrium point" of Nash is an instance of a *fixed point*, and Nash' teorem follows from

Brouwer's fixed point theorem

Any continuous function $f\colon D^{n+1} \to D^{n+1}$ has a fixed point, *i.e.* a point $p \in D^{n+1}$ s.t. $f(p) = p$.

Brouwer's fixed point theorem has a wide variety of applications inside and outside mathematics.[3]

[3] I believe BFT is the main reason why there regularly are economy majors in my classes

Brouwer's fixed point theorem

Brouwer's fixed point theorem

Any continuous function $f\colon D^{n+1} \to D^{n+1}$ has a fixed point, *i.e.* a point $p \in D^{n+1}$ s.t. $f(p) = p$.

Proof: Assume there were no fixed points:

$$\textit{for all } p \in D^{n+1},\ f(p) \neq p.$$

Then the *unique line* through p and $f(p)$ intersects S^n in two points. Let $F(p)$ be the intersection point closer to p. Then F is a continuous function

$$D^{n+1} \to S^n$$

and the composite

$$S^n \subseteq D^{n+1} \to S^n$$

is the identity. No such F exists. ☺

a space X gives a *vector space* $\tilde{H}_n X$ s.t.
$-\tilde{H}_n D^{n+1} = 0$
$-\tilde{H}_n S^n = \mathbb{R}$
a continuous function $f\colon X \to Y$ gives a *linear map* $\tilde{H}_n f\colon \tilde{H}_n X \to \tilde{H}_n Y$ s.t.
$-\tilde{H}_n(f \circ g) = \tilde{H}_n f \circ \tilde{H}_n g$ and
$-\tilde{H}_n(\text{id map on } X) = \text{id map on } \tilde{H}_n(X)$.

Populist summary

Knowing that the identity map $\mathbb{R} = \mathbb{R}$ is not zero can lead to the "Nobel prize" in economics.

or somewhat more seriously:

Simple facts of algebra have profound consequences for the behavior of spaces and continuous functions.

This is what algebraic topology is all about.

Somewhere the wind is still

Returning to the claim that somewhere on (a spherical) earth the wind is still, there is one more aspect of the vector spaces and linear maps coming from $\tilde{H}_n$ we need to know about:

Homotopy invariance.

When talking about wind, we observed that there being windless places was not dependent on deformations,[4]

[4] e.g. as long as we do not make holes, it doesn't really matter if there are mountains

Definition

Two continuous functions $f, g\colon X \to Y$ are said to be *homotopic*, written $f \simeq g$, if there is a continuous function

$$H\colon X \times [0,1] \to Y,$$

so that for $p \in X$ we have that
$H(p,0) = f(p)$ and
$H(p,1) = g(p)$

We see that for each $p \in X$ we get a path from $f(p)$ to $g(p)$ by sending $t \in [0,1]$ to $H(p,t)$.
The important thing is that $H(p,t)$ is continuous in $(p,t) \in X \times [0,1]$.[5]
We say that

H is a *homotopy* from f to g.

[5]you may think of H as a "continuous family of paths" or a "deformation of f into g"

Recall: $\tilde{H}_n$ is a functor:
cont. $f\colon X \to Y$ gives *linear map* $\tilde{H}_n f\colon \tilde{H}_n X \to \tilde{H}_n Y$ s.t.
$-\tilde{H}_n(f \circ g) = \tilde{H}_n f \circ \tilde{H}_n g$ and
$-\tilde{H}_n(\text{id map on } X) = \text{id map on } \tilde{H}_n(X)$.

$\tilde{H}_n$ is *homotopy invariant.*

If $f, g\colon X \to Y$ are homotopic, then $\tilde{H}_n f = \tilde{H}_n g$ as linear maps $\tilde{H}_n X \to \tilde{H}_n Y$.

We'll see that it sometimes is easy to calculate $\tilde{H}_n f$.
For instance, for the *antipodal map*

$$\mathrm{ant}\colon S^2 \to S^2, \quad \mathrm{ant}(p) = -p,$$

we have that

$$\tilde{H}_2(\mathrm{ant})\colon \tilde{H}_2 S^2 \to \tilde{H}_2 S^2$$

is

$$\mathbb{R} \to \mathbb{R}, \quad t \mapsto -t,$$

and so, ant *is not* homotopic to

$$\mathrm{id}_{S^2}\colon S^2 \to S^2, \quad p \mapsto p.$$

$f, g\colon X \to Y$ are *homotopic* - $f \simeq g$ - if there is a continuous $H\colon X \times [0,1] \to Y$, s.t. $H(p,0) = f(p)$ and $H(p,1) = g(p)$

$\tilde{H}_n$ is a *homotopy functor:*

cont. $f\colon X \to Y$ gives *linear map* $\tilde{H}_n f\colon \tilde{H}_n X \to \tilde{H}_n Y$ s.t.
$-\tilde{H}_n(f \circ g) = \tilde{H}_n f \circ \tilde{H}_n g$,
$-\tilde{H}_n(\text{id map on } X) = \text{id map on } \tilde{H}_n(X)$,
$-$if $f \simeq g$ then $\tilde{H}_n f = \tilde{H}_n g$.

Summary: $\tilde{H}_n$ is a homotopy functor

cont. $f\colon X \to Y$ gives linear map $\tilde{H}_n f\colon \tilde{H}_n X \to \tilde{H}_n Y$ s.t.
$-\tilde{H}_n(f \circ g) = \tilde{H}_n f \circ \tilde{H}_n g$,
$-\tilde{H}_n(\text{id map on } X) = \text{id map on } \tilde{H}_n(X)$,
$-$if $f \simeq g$ then $\tilde{H}_n f = \tilde{H}_n g$.

Application

The antipodal map

$$\mathrm{ant}\colon S^2 \to S^2, \quad p \mapsto -p,$$

is **not** homotopic to the identity map

$$\mathrm{id}_{S^2}\colon S^2 \to S^2, \quad p \mapsto p$$

because $\tilde{H}_2(\mathrm{ant})\colon \tilde{H}_2 S^2 \to \tilde{H}_2 S^2$ is $\mathbb{R} \to \mathbb{R}, \quad t \mapsto -t$.

$f \simeq g$ iff there is a continuous $H\colon X \times [0,1] \to Y$, s.t. $H(p,0) = f(p)$ and $H(p,1) = g(p)$

Somewhere the wind is still

Given a point $p = (p_1, p_2, p_3)$ on the unit sphere $S^2 = \{p \in \mathbb{R}^3 \,|\, |p| = 1\}$, there are infinitely many great circles through p

A nonzero tangent vector v at p gives a *unique* great circle.

Travelling from p to $-p$ along this great circle, starting in the direction of v, gives a continuous path $f\colon [0,1] \to S^2$ from p to $-p$ on S^2.

$S^2 \to S^2, \quad p \mapsto -p,$
is not homotopic to
$S^2 \to S^2, \quad p \mapsto p.$

Somewhere the wind is still

For each $p \in S^2$, the wind gives a tangent vector v_p, and so - if v_p is nonzero - a preferred great circle arc from p to $-p$. **Moreover**, v_p varies *continuously* with p.[6]
In conclusion - *IF the wind is never still* - the family of arcs gather to form a homotopy from the identity on S^2 to the antipodal map,
BUT no such thing exists!!! ☺

$S^2 \to S^2, \quad p \mapsto -p,$
is not homotopic to
$S^2 \to S^2, \quad p \mapsto p.$

[6]anyone worried about quantum effects or other real-world complications is asked to keep quiet about it.