

On regular closed curves in the plane ¹⁾

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We consider in this note closed curves with continuously turning tangent, with any singularities. To each such curve may be assigned a „rotation number” γ , the total angle through which the tangent turns while traversing the curve. (For a simple closed curve, $\gamma = \pm 2\pi$.) Our object is two-fold; to show that two curves with the same rotation number may be deformed into each other, ⁴⁾ and to give a method of determining the rotation number by counting the algebraic number of times that the curve cuts itself (if the curve has only simple singularities, — see Lemma 2).

This paper may be considered as a continuation of a paper of H. Hopf ²⁾; we assume a knowledge of the first part of his paper.

1. *Regular closed curves.*

Ordinarily, a curve in the plane is defined as a point set with certain properties; but when we allow singularities, this mode of definition cannot be used. (See footnote ³⁾.) Our first purpose is therefore to define a regular closed curve.

Let E be the Euclidean plane. Let E' be the vector plane (which we might let coincide with E), with origin O . Let I be the closed interval $(0, 1)$. Any differentiable function $f(t)$ with values in E has, as its derivative, a function $f'(t) = \frac{df(t)}{dt}$ with values in E' . By a *parametrized regular closed curve*, or *parametrized curve* for short, we shall mean a differentiable function $f(t)$ defined in I and with values in E , such that

$$(1) \quad f(1) = f(0), \quad f'(1) = f'(0), \quad f'(t) \neq 0 \text{ for } t \text{ in } I.$$

¹⁾ Presented to the American Mathematical Society, Sept., 1936.

²⁾ HEINZ HOPF, Über die Drehung der Tangenten und Sehnen ebener Kurven [Compositio Math. 2 (1935), 50—62].

The first two conditions are the conditions for the curve to be closed; the last condition makes t a „regular parameter”. To any such f there corresponds a unique differentiable function \bar{f} defined in $(-\infty, \infty)$, such that

$$(2) \quad \bar{f}(t) = f(t) \text{ in } I, \bar{f}(t+1) = \bar{f}(t), \bar{f}'(t) \neq 0,$$

and conversely.

It is natural to call two parametrized curves equivalent if one can be obtained from the other by a change of parameter (preserving orientation). The exact definition is: f and g are *equivalent* ($f \sim g$) if there exists a function $\eta(t)$ in $(-\infty, \infty)$ whose first derivative is continuous and positive, and is such that

$$(3) \quad \eta(t+1) = \eta(t) + 1, \bar{g}(t) = \bar{f}(\eta(t)).$$

Obviously $f \sim f$, $f \sim g$ implies $g \sim f$, and $f \sim g$ and $g \sim h$ imply $f \sim h$. Hence the parametrized curves fall into classes; we call each of these a *regular closed curve*, or curve for short. With any curve C is associated many (equivalent) *parametrizations* f . Let \bar{C} be the corresponding set of points in the plane E (all points $f(t)$). C is by no means determined by \bar{C} ³.

Given any C , a parametrization g may be chosen so that $|g'(t)|$ is constant, that is, so that the parameter is a constant times the arc length.

To prove this, set

$$(4) \quad L(t) = \int_0^t |\bar{f}'(s)| ds, \quad L = L(1).$$

$L = L(C)$ is the length of C . As $\bar{f}'(t) \neq 0$, $L(t)$ is a differentiable increasing function; hence we may solve $L \cdot s = L(t)$ for t , giving $t = \eta(s)$. The derivative $\eta'(s)$ is continuous and positive. As \bar{f} is periodic,

$$L(t+1) - L(t) = \int_t^{t+1} |\bar{f}'(s)| ds = \int_0^1 |\bar{f}'(s)| ds = L;$$

hence $\eta(s+1) = \eta(s) + 1$. Therefore

$$\bar{g}(t) = \bar{f}(\eta(t))$$

³) Let \bar{C} be the unit circle in E ; then for each integer $n \neq 0$ there is a corresponding curve C_n with $\bar{C}_n = \bar{C}$, determined by letting $f(t)$ traverse \bar{C} in the positive sense n times while t runs over I . Again, if we take an ellipse and pull the ends of the minor axis together till they are tangent, then there are four corresponding curves, in each of which the corresponding $f(t)$ traverses each point but one of the ellipse only once.

is a parametrization of C . Moreover,

$$(5) \quad \bar{g}'(t) = \bar{f}'(\eta(t)) \frac{L}{L'(\eta(t))}, \quad |\bar{g}'(t)| = L.$$

If h is any parametrization with $|h'(t)| = k$, then $k = L$ and $\bar{h}(t) = \bar{g}(t+a)$ for some constant a .

First, as $h \sim g$, there is an η such that $\bar{h}(t) = \bar{g}(\eta(t))$. As

$$h'(t) = g'(\eta)\eta'(t), \text{ hence } k = L\eta'(t),$$

we have

$$1 = \eta(1) - \eta(0) = \int_0^1 \eta'(t) dt = \int_0^1 \frac{k}{L} dt = \frac{k}{L},$$

and $k = L$. Hence $\eta'(t) = 1$, and $\eta(t) = t + a$.

Let f_0 and f_1 be parametrized curves. We say one may be deformed into the other if $f_u(t)$ may be defined for $0 < u < 1$ such that it is continuous in both variables for $0 \leq t \leq 1$, $0 \leq u \leq 1$, and each f_u is a parametrized curve.

If f_0 and f_1 are parametrizations of C , then one may be deformed into the other within C , that is, we can make each f_u a parametrization of C .

To prove this, say $f_1(t) = f_0(\eta(t))$. Set

$$(6) \quad \eta_u(t) = u\eta(t) + (1-u)t, \quad f_u(t) = \bar{f}_0(\eta_u(t)),$$

for $0 \leq u \leq 1$. Then $\eta_0(t) = t$, $\eta_1(t) = \eta(t)$, so that \bar{f}_0 and \bar{f}_1 bear the proper relation to f_0 and f_1 . As

$$\eta_u(t+1) = u[\eta(t)+1] + (1-u)(t+1) = \eta_u(t) + 1,$$

$$\frac{d\eta_u(t)}{dt} = u \frac{d\eta(t)}{dt} + (1-u) > 0 \text{ for } 0 \leq u \leq 1,$$

each f_u is a parametrized curve equivalent to f_0 .

We say C may be deformed into C' if some parametrization of C may be deformed into one of C' . By the above statement, this is independent of the parametrizations chosen.

2. The deformation theorem.

The following lemma is fundamental in this section.

LEMMA 1. Let $f'(t)$ be a continuous vector function in I , such that $f'(t) \neq 0$. If p is a point of E , then

$$(7) \quad f(t) = p + \int_0^t f'(s) ds$$

is a parametrized curve if and only if

$$(8) \quad f'(1) = f'(0), \quad \int_0^1 f'(s) ds = 0.$$

This is obvious. The last relation may be stated as follows: The average value of $f'(s)$ is O .

Given any parametrized curve f , we define its rotation number $\gamma(f)$ as the total angle through which $f'(t)$ turns as t traverses I . The function $f^*(t) = \frac{f'(t)}{|f'(t)|}$ is a map of I into the unit circle; $\gamma(f)$ is 2π times the degree of this map. (See Hopf. loc. cit., 1c, and our equation (11).)

If f may be deformed into g , then $\gamma(f) = \gamma(g)$.

For $\gamma(f_u)$ is continuous in u , and is an integral multiple of 2π ; hence it is constant. Hence, by 1, we may define $\gamma(C)$ for a curve C as $\gamma(f)$ for any parametrization f of C .

THEOREM 1⁴). *The curves C_0 and C_1 may be deformed into each other if and only if $\gamma(C_0) = \gamma(C_1)$.*

One half of the theorem was proved above. Suppose now that $\gamma(C_0) = \gamma(C_1) = \gamma$. Let g_0 and f_1 be parametrizations of C_0 and C_1 such that

$$|g'_0| \equiv L(C_0) = L_0, \quad |f'_1| \equiv L(C_1) = L_1.$$

Set

$$g_u(t) = g_0(0) + \left[u \frac{L_1}{L_0} + (1-u) \right] [g_0(t) - g_0(0)];$$

this deforms the parametrized curve g_0 into one g_1 . Set $f_0 = g_1$; then $|f'_0| \equiv |g'_1| \equiv L_1$. We must deform f_0 into f_1 .

The proof runs as follows. We consider the maps f'_0 and f'_1 of I into the circle K of radius L_1 . They are both of degree $\frac{\gamma}{2\pi}$; hence one map may be deformed into the other, say by the maps h_u . We alter each h_u by a translation to obtain a map f'_u whose average lies at O ; these functions then define the required deformation, at least if $\gamma \neq 0$.

We begin by defining the vector function

$$(9) \quad \theta(t) = (L_1 \cos t, L_1 \sin t);$$

this gives an angular coordinate t in K . Suppose first that $\gamma \neq 0$. By rotations in the plane E we may alter f_0 and f_1 so that

⁴) This theorem, together with a straightforward proof, was suggested to me by W. C. GRAUSTEIN.

$f'_0(0) = f'_1(0) = \theta(0)$. As $f'_i(t)$ lies on K , we may give it an angular measure $F_i(t)$:

$$(10) \quad f'_i(t) = \theta(F_i(t)), \text{ with } F_i(0) = 0 \quad (i = 0, 1).$$

(See Hopf., loc. cit., 1a.) Then, by definition of γ ,

$$(11) \quad F_i(1) = \gamma \quad (i = 0, 1).$$

Set

$$(12) \quad \begin{aligned} F_u(t) &= uF_1(t) + (1-u)F_0(t), \\ h_u(t) &= \theta(F_u(t)) \end{aligned} \quad (0 \leq t \leq 1),$$

$$(13) \quad \begin{aligned} f'_u(t) &= h_u(t) - \int_0^1 h_u(s) ds, \\ f_u(t) &= f_0(0) + u[f_1(0) - f_0(0)] + \int_0^t f'_u(s) ds. \end{aligned}$$

It is clear that $\int_0^1 f'_u(t) dt = 0$. As $F_u(0) = 0$, $F_u(1) = \gamma$, and γ is an integral multiple of 2π ,

$$f'_u(1) - f'_u(0) = \theta(F_u(1)) - \theta(F_u(0)) = \theta(\gamma) - \theta(0) = 0.$$

Finally, as $\gamma \neq 0$ and hence $h_u(t)$ passes over all of K , its average value lies interior to K ; therefore for no t does $h_u(t)$ equal the average, and $f'(t) \neq 0$. This proves that each f_u is a regular closed curve. As $f_u(t)$ is continuous in both variables, and it reduces to f_0 and f_1 for $u = 0$ and $u = 1$, it is a deformation of f_0 into f_1 , as required.

Suppose now that $\gamma = 0$. If we alter $F_u(t)$ so that it is constant for no u , then again $f'(t) \neq 0$, and the above proof will hold. Choose a t_0 for which $F_1(t_0) \neq 0$, and deform $F_0(t)$ in a small neighborhood of t_0 into $F_1(t)$ in this neighborhood; now deform the new F_0 into F_1 by the process given above. Then (as $F_u(0) = 0$) no F_u is constant.

3. Crossing points of curves.

Let $f(t)$ be a parametrized curve. Let p be a point of the plane. If there are exactly two numbers t_1, t_2 , such that

$$0 \leq t_1 < t_2 < 1, \quad f(t_1) = f(t_2) = p,$$

and if $f'(t_1)$ and $f'(t_2)$ are independent vectors, we call p a (*simple*) *crossing point* of the curve. This is evidently independent of the

parametrization. If the curve has no singularities other than a finite number of simple crossing points, we say the curve is *normal*.

LEMMA 2. *Any curve may be made normal by an arbitrarily small deformation.*

Given $\varepsilon > 0$, cut I into intervals I_1, \dots, I_p so small that each corresponding arc $A_i = f(I_i)$ is of diameter $< \varepsilon$, and the tangents at different points of A_i differ by at most ε . By a small deformation we may clearly obtain arcs A'_i such that neither end of any A'_i touches other points of the curve. Now for any i and j , it is easy to replace A'_j by an arc A''_j arbitrarily near it and with the same ends so that A''_j cuts A'_i in simple crossing points only⁵). Alter thus A'_2 in relation to A'_1 ; then A'_3 in relation to A'_1 ; then A'_3 in relation to A'_2 , altering it so slightly that its relation to A'_1 is not impaired, etc.

Let f be a parametrized curve, and let \bar{C} be the corresponding set of points $f(t)$ in the plane. We say f has an *outside starting point* if there is a line of support to \bar{C} ⁶) containing $f(0)$.

Let $f(t_1) = f(t_2)$, $t_1 < t_2$, be a crossing point. If the vectors $f'(t_1)$ and $f'(t_2)$ are oriented relative to each other in the *opposite* manner to the (fixed) x - and y -axes, we say the crossing point is *positive*; otherwise, *negative*⁷). If we set $\bar{g}(t) = \bar{f}(t + \tau)$ with $t_1 < \tau \leq t_2$, then the above crossing point changes its type. Corresponding to any normal parametrized curve are the numbers

$$(14) \quad \left. \begin{array}{l} N^+ \\ N^- \end{array} \right\} \text{ of crossing points of } \left\{ \begin{array}{l} \text{positive} \\ \text{negative} \end{array} \right\} \text{ type.}$$

These may be found by following the curve from its starting point, and watching the intersections with the part of the curve already traversed.

THEOREM 2. *If f is a normal parametrized curve with an outside starting point, then*

$$(15) \quad \gamma(f) = 2\pi[\mu + (N^+ - N^-)], \quad \mu = \pm 1.$$

If the axes are moved so that the x -axis is the line of support at

⁵) The proof is simplified by first replacing $f(t)$ by a function $g(t)$ with continuous second derivatives. The lemma is contained in Theorem 2 of H. WURTEY, Differentiable manifolds [Annals of Math. 37 (1936)]. (We replace I by the unit circle M and use (b) of the theorem.)

⁶) That is, a straight line touching \bar{C} and having each point of \bar{C} on it or on a single side of it.

⁷) An example of a positive crossing point is given in Fig. 2.

$f(0)$ and the curve is on the same side of this line as the positive y -axis, then $\mu = +1$ or -1 according as $f'(0)$ is in the positive or negative x -direction.

In particular, if the curve has no singularities, then $\gamma = \pm 2\pi$, which is the „Umlaufsatz“.

Let T be the triangle of all pairs of numbers

$$(t_1, t_2), \quad 0 \leq t_1 \leq t_2 \leq 1.$$

Let $l(t_1, t_2)$ be the smaller of $t_2 - t_1$ and $(1+t_1) - t_2$. Set

$$(16) \quad \psi(t_1, t_2) = \frac{f(t_2) - f(t_1)}{l(t_1, t_2)} \text{ if } l(t_1, t_2) \neq 0,$$

$$\psi(t, t) = f'(t), \quad \psi(0, 1) = -f'(0).$$

ψ is continuous in T , and is 0 at (t_1, t_2) if and only if $t_1 < t_2$ (but not $t_1 = 0, t_2 = 1$), and $f(t_1) = f(t_2)$, i.e. if and only if $f(t_1)$ is a crossing point⁸).

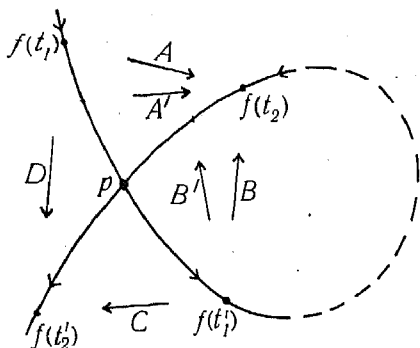


Fig. 2.

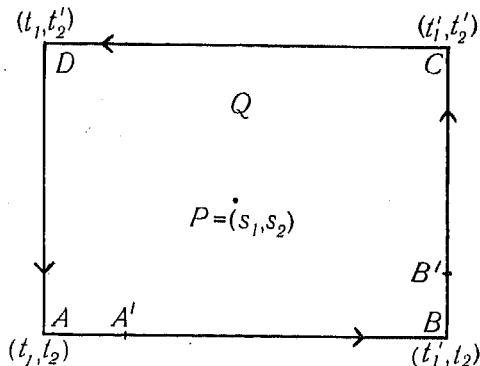


Fig. 3.

Take any crossing point $p = f(s_1) = f(s_2)$; suppose it is positive. As $s_1 < s_2$, $P = (s_1, s_2)$ is not on the hypotenuse of T . As $f(0)$ is an outside point, it is obviously not a crossing point; hence $f(0) \neq f(t)$ for $0 < t < 1$, and P is on neither side of T . As $P \neq (0, 1)$, it follows that P is interior to T . Choose numbers t_1, t'_1 very close to s_1 , and t_2, t'_2 very close to s_2 , so that

⁸ This function replaces the function $f(s_1, s_2)$ of Hopf (p. 54). It will be seen that $N = N^+ - N^-$ is the algebraic number of times that T covers 0 under ψ (see footnote¹⁰).

$$t_1 < s_1 < t'_1, \quad t_2 < s_2 < t'_2,$$

and

$f(t_1), f(t'_1), f(t_2), f(t'_2)$ are equidistant from p .

Let Q be the rectangle in T containing P , with coordinates (t_1, t_2) , etc. It is easily seen that if we run around Q once in the positive sense, the corresponding ψ runs around O once in the positive sense. For running around each side of Q turns the vector ψ through an angle of approximately $\frac{\pi}{2}$ (see the diagram); hence it turns, in all, approximately 2π ; but it turns an integral multiple of 2π , and hence exactly 2π ⁹⁾. If the crossing point is negative, the result is obviously -2π .

Let P_1, \dots, P_m be the points of T corresponding to crossing points, and let Q_1, \dots, Q_m be corresponding rectangles enclosing them, no two of which have common points. Cut the rest of T into triangles Q_{m+i}, \dots, Q_r . If we run around the boundary of any Q_{m+i} , ψ runs around O zero times¹⁰⁾. To show this, consider the vector $\psi^* = \frac{\psi}{|\psi|}$. This is defined throughout Q_{m+i} , and its values are on the unit circle. Hence an angular coordinate may be defined, giving the position of ψ^* throughout Q_{m+i} (see Hopf, loc. cit., 1b). If we run around the boundary of Q_{m+i} , the angular coordinate comes back to its original value, and hence ψ has turned around zero times.

Let $\alpha_1, \dots, \alpha_l$ be all sides of triangles or rectangles in T . Let $\alpha_1, \dots, \alpha_k$ be those lying on the boundary B of T , oriented the same as T ; the remaining α_i are oriented arbitrarily. With each α_i we associate a number $\varphi(\alpha_i)$, the angle through which ψ turns when α_i is traversed in the positive direction. Let $\varphi(Q_i)$ be the angle through which ψ turns when the boundary of Q_i is traversed in the positive direction; similarly for $\varphi(T)$. Now

$$(17) \quad \sum_{i=1}^r \varphi(Q_i) = \varphi(T).$$

For each $\varphi(Q_i)$ may be expressed as a sum $\sum^{(i)} \pm \varphi(\alpha_j)$, summing over the boundary lines of Q_i ; when these sums are added, the two terms corresponding to each α_j interior to T cancel, and we are left with the sum over the α_j on the boundary of T .

⁹⁾ By choosing the proper degree of approximation, it is easy to make this reasoning rigorous.

¹⁰⁾ Hence, in all cases, $\varphi(Q_i)$ (see below) is the algebraic number of times that the map ψ of Q_i covers O .

We have seen above that

$$(18) \quad \sum_{i=1}^r \varphi(Q_i) = \sum_{i=1}^m \varphi(Q_i) = 2\pi(N^+ - N^-) = 2\pi N.$$

Suppose $\mu = 1$. If S_1 , S_2 and H are the positively oriented sides and hypotenuse of T (see Fig. 1), it is easily seen that

$$(19) \quad \varphi(S_1) = \varphi(S_2) = -\pi.$$

(See Hopf, pp. 54–55. The change in sign is caused by the difference in orientation of S_1 and S_2 from that used by Hopf.) Hence, using (17) and (18),

$$2\pi N = \varphi(T) = \varphi(H) + \varphi(S_1) + \varphi(S_2) = \gamma - 2\pi,$$

which gives (15). If $\mu = -1$, the only change is that $\varphi(S_1) = \varphi(S_2) = \pi$, and (15) again follows.

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