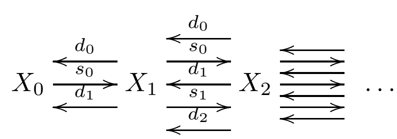
Exercises for Dundas' lectures. IV

1. Show that the integers $\mathbb{Z}$ can be defined as the set of equivalence classes in $\mathbb{N}\times\mathbb{N}$ under the equivalence relation generated by setting $(m+k, n+k) \sim (m, n)$. Get the picture that was on the blackboard Friday morning. How do you add and multiply?

   Toy with the problems you can imagine would appear if you tried to do this with (finite sets and bijections) instead of $\mathbb{N}$ (essentially by adding artificial maps $(M \coprod K, N \coprod K) \to (M, N)$) as sketched in the lecture.

2. Define the rationals similarly from $\mathbb{Z}\times\mathbb{Z}$ via a relation of the type $(mk, nk) \sim (m, n)$.

A *simplicial set* is a sequence of sets $X_0, X_1, \dots$ together with *face maps* $d_i = X(d^i)\colon X_n \to X_{n-1}$ and *degeneracy maps* $s_i = X(s^i)\colon X_n \to X_{n+1}$ for $0 \le i \le n$



satisfying the *simplicial identities*

$$d_i d_j = d_{j-1} d_i \quad \text{for } i < j$$

$$s_i s_j = s_j s_{i-1} \quad \text{for } i > j$$

and

$$d_i s_j = \begin{cases} s_{j-1} d_i & \text{for } i < j \\ \mathrm{i}d & \text{for } i = j, j+1 \\ s_j d_{i-1} & \text{for } i > j+1 \end{cases}.$$

1. Let G be a group. Define the *classifying space* BG as follows.

   The set of k-simplices is the set $B_k G = G \times \cdots \times G$ (k factors),

   $d_0(g_1, \dots, g_k) = (g_2, \dots, g_k)$,

   $d_k(g_1, \dots, g_k) = (g_1, \dots, g_{k-1})$, and for $0 < j < k$ we let

   $d_j(g_1, \dots, g_k) = (g_1, \dots, g_i \cdot g_{j+1}, \dots g_k)$ (multiply two entries and leave the rest), and

   $s_j(g_1, \dots, g_k) = (g_1, \dots, g_{j-1}, 1, g_j, \dots, g_k)$ (with the understanding that $s_0(g_1, \dots, g_k) = (1, g_1, \dots, g_k)$). Check that this is a simplicial set (just do enough to get the drift).

2. If you know what a (small) category $\mathcal{C}$ is: define the classifying space $B\mathcal{C}$ by letting $B_k\mathcal{C}$ be the set of composable strings $c_0 \leftarrow c_1 \leftarrow \cdots \leftarrow c_k$ with face maps defined by composition and degeneracies defined by inserting identity maps. Write out.

3. Let k be a natural number and consider the sequence $\{\Delta^k_j\}$ of sets, where Δ^k, j is the sets of order preserving functions from $[j] = \{0, 1, \ldots, j\}$ to $[k] = \{0, \ldots, k\}$. How can I think of Δ^k as a k-simplex: how should you define face and degeneracy maps? (Hint: start with low k - composition of ordered functions is the key). If you know category theory: can you recognize Δ^k as the classifying space of something?