Exercises for Dundas' lectures. III

1. Calculate the homology of the Klein bottle (c.f. the projective plane calculation).

2. a sequence of two composable homomorphisms of abelian groups

$$M' \xrightarrow{f} M \xrightarrow{g} M''$$

   is *exact* if $\ker g = \mathrm{im} f$, i.e., if for $m \in M$

$$g(m) = 0 \Leftrightarrow m = f(m') \text{ for an } m' \in M'.$$

   A sequence $\cdots \to M_{n+1} \to M_n \to M_{n-1} \to \dots$ is *exact* if each consecutive pair of morphisms are exact. Show that

   (a) $f\colon M' \to M$ is injective iff $0 \to M' \xrightarrow{f} M$ is exact

   (b) $g\colon M \to M''$ is injective iff $M \xrightarrow{g} M'' \to 0$ is exact

   (c) assume $\cdots \to M_{n+1} \to M_n \to M_{n-1} \to \dots$ is exact. Show that $M_n \to M_{n-1}$ is an isomorphism iff both $M_{n+1} \to M_n$ and $M_{n-1} \to M_{n-2}$ are zero

3. Run through the argument that $\tilde{h}_* D^n = 0$ from the lecture. Likewise the calculation of $\tilde{h}_* S^n$.

4. Naturality of the boundary map $\tilde{h}_{n+1} X/A \to \tilde{h}_n A$ means the following. If $f\colon (X, A) \to (Y, B)$ is a map of CW-pairs. Then f induces a map $X/A \to Y/B$ and the resulting diagram

$$\begin{array}{ccc} \tilde{h}_{n+1} X/A & \xrightarrow{\partial} & \tilde{h}_n A \\ \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} \\ \tilde{h}_{n+1} Y/B & \xrightarrow{\partial} & \tilde{h}_n B \end{array} \quad .$$

   is commutative. Show that if all the vertical maps in the (sequence of) diagrams above are isomorphisms, then $\{\tilde{h}_n X \to \tilde{h}_n Y\}_n$ also consists of isomorphisms (this follows from the so-called "five lemma": check it up).

5. The uniqueness of homology theories mentioned in the lecture comes from the inductive definition of CW-complexes. See if you can prove it, using the previous exercise.