Exercises for Day #3 (Dugger)

1. For every compact 2-manifold (T_g or N_r) find geometric representatives for the homology classes and compute all products of basis elements. For T_g do this with real (or integer) coefficients, and for N_r do it for $\mathbb{F}_2$-coefficients.

2. Let X be the 3-manifold obtained from the cube $[-1,1]^3$ by doing the following gluing:

   - $(1,y,z) \sim (-1,-y,z)$
   - $(x,1,z) \sim (x,-1,z)$
   - $(x,y,1) \sim (x,y,-1)$.

   Compute $H_*(X;F)$ for all fields F that you know.

3. (a) Let $\rho\colon S^2 \to S^2$ be reflection in the equator. Define a 3-manifold X by taking $S^2 \times I$ and gluing $(x,0)$ to $(\rho(x),1)$ for all $x \in S^2$. Compute the homology groups of X.

   (b) Now let $f\colon T \to T$ be an automorphism of the torus that you choose. Define a similar 3-manifold Y be taking $T \times I$ and gluing $(x,0)$ to $(f(x),1)$ for all $x \in T$. Compute the homology groups of Y.

4. Here is another interesting 3-manifold. Fix a prime p and let $\zeta = e^{\frac{2\pi i}{p}}$. Consider the 3-ball

$$D^3 = \{(x,y,z) \in \mathbb{R}^3 \,|\, x^2+y^2+z^2 \leq 1\} = \{(z,t) \in \mathbb{C} \times \mathbb{R} \,\big|\, |z|^2 + t^2 \leq 1\}.$$

   For (z,t) such that $|z|^2 + t^2 = 1$ and $t \leq 0$, identify (z,t) with $(\zeta z, -t)$. This gives a space called $L(p,1)$.

   (a) Convince yourself that $L(p,1)$ is a 3-manifold.

   (b) Identify the equator of D^3 with the unit complex numbers and mark off the pth roots of unity $a_0, a_1, \ldots, a_{p-1}$. Then mark the north and south poles, and draw in the geodesic circles from N to S that pass through each a_i. Convince yourself that this picture gives a cell decomposition for $L(p,1)$ that consists of two 0-cells, $p+1$ 1-cells, p 2-cells, and one 3-cell. Compute the homology groups (or Betti numbers) of $L(p,1)$. For this calculation I suggest doing it both with $\mathbb{R}$-coefficients and with $\mathbb{F}_p$-coefficients.

5. (Challenge) Compute the intersection product on the homology groups of the space Q defined in lecture.

6. (Challenge) Compute the homology groups of the Poincaré homology sphere (defined in lecture).