Lisbon school July 2017: eversion of the sphere

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1 Lectures

1.1 exercise

1. Examples and non examples of immersions in the plane For each of the following functions $f\colon \mathbf{R} \to \mathbf{R}^2$ determine whether it is a $\mathcal{C}^1$ -immersion. Draw the corrseponding curve in the plane.

   - f(t)=(t,t^2)
   - f(t)=(cos(2π t),sin(2π t))
   - f(t)=(t^{3,t2})
   - f(t)=(t, $\sqrt[3]{\{t^2\}}$)
   - f(t)=(sin(2π t),sin(4π t))
   - f(t)=cos(6π t)$\cdot$ (cos(2π t),sin(2π t))

1.2 lecture 3 LECTURE3

1. The path lifting property We say that a map p: E$\to$ B has the **path lifting property** (or $\mathbf{HPL_0}$) if given

   - a path $\omega\colon [0,1] \to B$ starting at some point $b_0 = \omega(0)$, and
   - a point $e_0 \in E$ such that $p(e_0) = b_0$,

   Then there exists a path $\tilde{\omega}\colon [0,1] \to E$ such that

   (a) $p\tilde{\omega} = \omega$, and

   (b) $\tilde{\omega}(0) = e_0$.

$$\begin{tikzcd} & E \arrow[d, "p"] \\ {[0,1]} \arrow[ur, dashed, "\exists\tilde{\omega}"] \arrow[r, "\omega"'] & B \end{tikzcd} \qquad \begin{tikzcd} \ni e_0 \arrow[d, mapsto] \\ \ni \omega(0). \end{tikzcd} \qquad \mathit{or} \qquad \begin{tikzcd} \{0\} \arrow[r, "e_0"] \arrow[d, hook] & E \arrow[d, "p"] \\ {[0,1]} \arrow[ur, dashed, "\exists\tilde{\omega}"] \arrow[r, "\omega"'] & B. \end{tikzcd}$$

2. Proof of the path lifting property of p: step 1. Step 1: Prove the existence of a lifting under the extra assumption that the path $g\colon [0,1] \to S^1$ is missing the rightmost point (1,0) of the circle.
   <u>Hint</u>: Set $A := S^1 \setminus \{1\}$ the circle minus that rightmost point. Thus g([0,1])⊂ A. We have a homeomorphism $p_1 :]0,1[\xrightarrow{\cong} A$.
   Then $p^{-1}(A) = \mathbf{R} \setminus \mathbf{Z}$. Contemplate 10 minutes the following:

$$\begin{tikzcd}
 & e_0 \in \mathbf{R} \arrow[d, "p"] & p^{-1}(A) = \cup_{n\in\mathbf{Z}}]n, n+1[\arrow[l, hook', "incl"'] \arrow[d] &]0,1[\times \mathbf{Z} \ni (t_0, n_0) \arrow[l, "\cong", "\psi"'] \arrow[d, "proj_1"] \\
{[0,1]} \arrow[r, "g"'] \arrow[ur, dashed, "\exists ? \tilde{g}"] \arrow[rr, bend right, "g"'] & S^1 & A \arrow[l, hook', "inclusion"'] &]0,1[\arrow[l, "\cong", "p_1"'] \arrow[u, dotted, bend left, "\sigma"]
\end{tikzcd}$$

   where ψ(t,n):=t+n, (t$_0$,n$_0$):=ψ^{-1} (e$_0$) and σ(t):=(t,n$_0$).
   Show that $\tilde{g}(t) := \psi(\sigma(p_1^{-1}(g(t))))$ is a lift of g along p: $p\tilde{g} = g$.

3. Proof of the path lifting property: remaining steps Step 1bis: Prove the existence of a lifting under the extra assumption that the path $g\colon [0,1] \to S^1$ is missing the leftmost point (-1,0) of the circle. (Completely analoguous to step (1))

   Step 2: show that we can decompose the interval $[0,1]$ in small subintervals [t$_i$, t$_{i+1}$] with 0=t$_0$ <t$_1$ < . . . < t$_{N-1}$< t$_N$ =1 such that each restricted paths g| [t$_i$, t$_{i+1}$] misses either (-1,0) or (1,0)

   Step 3: Use the previous steps to construct inductively the lifting $\tilde{g}$ and finish the proof

   Exercise: complete the details of the proof.

4. The homotopy of paths lifting property We say that a map p: E→ B has the <u>**homotopy**</u> **of path lifting property** (or $\mathbf{HPL_1}$) if given

   - a homotopy $\Omega\colon [0,1] \times [0,1] \to B, (t,u) \mapsto \Omega(t,u)$, and
   - a path $\epsilon^0 \in E$ such that $p(\epsilon^0(u)) = \Omega(0,u)$,

   then there exists a homotopy $\tilde{\Omega}\colon [0,1] \times [0,1] \to E$ such that

   (a) $p\tilde{\Omega} = \Omega$, and
   (b) $\tilde{\omega}(0,u) = \epsilon^0(u)$ for all $u \in [0,1]$.

$$\begin{tikzcd}
{[0,1] \times \{0\}} \arrow[r, "\epsilon^0"] \arrow[d, hook] & E \arrow[d, "p"] \\
{[0,1] \times [0,1]} \arrow[r, "\Omega"'] \arrow[ur, dashed, "\exists \tilde{\Omega}"] & B.
\end{tikzcd}$$

Exercise: prove the above homotopy lifting theorem. Hint Very similar the proof of the path lifting property. Look at Hacher "Algebraic topology", proof of property (c) on page 29 (freely available on the web). Dont be scared by the apparent complexity of Hatcher: have faith and struggle.

2 exercises

2.1 The bundle $S^3 \to$ SO(3) whose fibre is a pair of points.

The goal of this exercise is to prove that there exists a bundle

$$p \colon S^3 \longrightarrow SO(3)$$

whose fibre is $\mathbf{Z}/2\mathbf{Z}$ (i.e. a space with two points) and where S^3 is the 3-dimensional sphere [1]

$$S^3 = \left\{(a,b,c,d) \in \mathbf{R}^4 : a^2 + b^2 + c^2 + d^2 = 1\right\}.$$

The proof is mostly algebra related to the quaternionic numbers.

We first review quickly the notion of quaternion numbers H. They are defined as the 4-dimensional real vectorspace [2]

$$\mathbf{H} := \{a + bi + cj + dk \ : \ a,b,c,d \in\}$$

and with a multiplication $\mu_{\mathbf{H}} \colon \mathbf{H} \times \mathbf{H} \to \mathbf{H}$ charaterized as the unique map such that

- $\mu_{\mathbf{H}}$ is R-linear in each variable
- 1 is a unit for the multiplication (on the left and on the right)
- \$$\mu_{\mathbf{H}}$(i,i)=-1,$\mu_{\mathbf{H}}$(j,j)=-1,$\mu_{\mathbf{H}}$(k,k)=-1\
- $\mu_{\mathbf{H}}$(i,j)=k\,,\,$\mu_{\mathbf{H}}$(j,k)=i\,,\,$\mu_{\mathbf{H}}$(k,i)=j\,,\,\$

It turns out that this unique map is an associative (but not commutative) multiplication. For $w, w' \in \mathbf{H}$ we just write $w \cdot w' := \mu_{\mathbf{H}}(w, w')$.

We define the conjugate of a quaternionic number w=a+bi+cj+dk by $\overline{w} := a - bi - cj - dk$ and its modulus by $|w| = \sqrt{w \cdot \overline{w}} = \sqrt{a^2 + b^2 + c^2 + d^2}$. The inverse, for $w \neq 0$, is defined by $w^{-1} = \overline{w}/|w|^2$.

The 3-sphere can be seen as the space of quaternionic numbers of modulus 1

$$S^3 = \{w \in \mathbf{H} : |w| = 1\}$$

Define the pure quaternionic numbers as the subset

$$\mathbf{H}_{\text{pure}} = \{bi + cj + dk : b,c,d \in \mathbf{R}\}$$

[1] Actually S^3 is itself a group, like SO(3), which is sometimes called SU(2) and p is a continuous homomorphism of topological groups with many important applications, in particular in quantum mecanics: the fact that p is a 2 sheet cover is related to the fact that some particles have spin $\pm 1/2$!!!

[2] in analgous way as the complex numer can be seen as a 2-dimensional real vector space with a multiplication

which is a 3-dimensional vector space that we will identify to R^3.

Fix z$\in$ S^3 a quaternionic number of modulus 1 and define

$$R_z\colon \mathbf{H} \to \mathbf{H}\,,\; w \mapsto z^{-1}\cdot w \cdot z.$$

Prove the following facts:

- R_z preserves $\mathbf{H}_{\text{pure}}$ i.e. $R_z(\mathbf{H}_{\text{pure}}) \subset \mathbf{H}_{\text{pure}}$ (hint: check the action on i, j and k independently)
- R_z preserves the modulus: $|R_z(w)|=|w|$
- if we identify in the obvious way $\mathbf{H}_{\text{pure}}$ with R^3 then R_z acts on R^3 as an element of SO(3)
- this defines a continuous map $p\colon S^3 \to SO(3), z \mapsto R_z$
- p(z)=p(-z)
- if p(z)=p(z') then z=$\pm$ z' (to prove this consider the action of R_z on i,j and k.)
- p is surjective (to prove this show that for z=(cos(θ)+sin(θ)i) then R_z is a rotation of angle 2θ about the i-axis in $\mathbf{H}_{\text{pure}}$, and similarly replacing i by j or k. Note then that any rotation in R^3 is a composition of rotations along each of the three axes.)
- for each rotation $A \in SO(3)$, $p^{-1}(A)$ is a pair $\{z, -z\}$
- p is a bundle with fibre Z/2Z

2.2 A criterion to prove the simple connectivity of a space.

We say that a space X is simply-connected if it is path-connected (that is, any two points in X can be connected by a path) and if moreover its fundamental group is trivial, $\pi_1(X, x_0)$ is a singleton (for some $x_{0\in}$ X), in other words if every based loop $\omega\colon [0,1] \to X$ is based homotopic to the constant loop $[c_{x_0}\colon [0,1] \to X, t \mapsto x_0]$.

We will give a criterion to check the simple-connectivity from "pieces" of the space.

1. On the connectivity of a space covered by connected supspaces. Let X be a space and let A,B$\subset$ X be open subsets such that A$\cup$ B=X and such that A and B are path connected and the intersection A$\cap$ B is non empty. Prove that X is path connected.
   Deduce from this that any sphere $S^n := \{x \in \mathbf{R}^{n+1} : \|x\| = 1\}$ is path connected for n$\geq$1.

2. Baby van Kampen Let X be a space and let A,B$\subset$ X be open subsets such that A$\cup$ B=X. Assume that A and B are simply-connected and that A$\cap$ B is non empty and path connected. Prove that X is simply-connected. (Hint: given a loop ω based at x_0, prove that it can be decompose in finitely many pieces each belonging to either A or B. Extend this pieces by a path connecting there extremities in both ways to x_0 in A$\cap$ B. Show that each of this extended pieces are contractible loops and conclude). Look at Hatcher Lemma 1.15 for a picture and help.
Deduce from this that any sphere $S^n := \{x \in \mathbf{R}^{n+1} : \|x\| = 1\}$ is simply-connected[3] for n$\geq$2.
Where hy does this argument fails to prove that S^1 is simply-connected ?

2.3 Euler characteristic of bundles with finite fibres

Let E$\to$ B be a bundle whose fibre is a space with exactly k points, for some integer k$\geq$1.

- Prove that the euler characterstics of E and B are related by the formulas χ(E)=k$\cdot\chi$(B) (hint: consider a triangulation of the base B with small enough simplices and lift it to a triangulation of E).
- Check that this formula is correct for the bundle $S^{2\to}$ P where P is the projective space
- Compute by hand the Euler characteristic of S^3 and deduce that of SO(3). Is the result compatible with the Betti numbers of SO(3) computed by Bjorn ?
- Use this to give restrictions to the possible bundles E$\to$ B where both E and B are compact surfaces.
- Challenge: for each case where there is no restriction in the previous item try to build such a bundle between surfaces. For example build a bundle E$\to$ B where E is the torus with 3 holes and B is the torus with 2 holes.

2.4 The unit tangent bundle of the sphere is not trivial

1. Fundamental group of a product. Prove that if (X,x_0) and (Y,y_0) are two based spaces then

$$\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$$

2. Deduce that $TS^{2\cong}$ SO(2) is not homeorphic to S^2 S^1

[3]the special case n=3 claims that S^3 is 3-connected. The famous Poincaré conjecture proved by Perlman in 2002 was the fact that S^3 is the only compact 3-manifold without boundary which is simply-connected.

2.5 Conjugacy classes in the fundamental group

Consider the set [$S^{1,X}$] of **unbased** homotopy classes of unbased loops in X. Comsider the obvious map

$$\Phi\colon \pi_1(X, x_0) \to [S^1, X]$$

sendic a based homotopy class to its unbased homotopy class. Prove that if X is path connected then Φ is surjective and two elements [α], [β]$\in \pi_1$(X,x_0) are sent to the same unbased homotopy class if and only if [α] and [β] are conjugate in π_1(X,x_0) in other words if and only if there exits [ω] such that [α]=[ω]$^{-1}$[β][ω]. Hint exrecise 6 of Hatcher p.38

Deduce from this that [$S^{1,SO}$(3)]=Z/2.