Exercises for Dundas' lectures. II[1]

The following exercises can be performed over any field, but for simplicity I suggest you stick to the field $\mathbb{F}_2 = \{0, 1\}$ of mod 2 arithmetic ($1 + 1 = 0$ - all the rest as usual - one of the great features is that it is impossible to make sign errors). You might want to keep in mind that later on we will repeat some of it over the polynomial ring $\mathbb{F}_2[t]$.

1. How many simplicial complexes are there with set of vertices $\{v_0, v_1, v_2\}$.

2. Consider the point cloud in $\mathbb{R}^2$ consisting of the points

$$(-1, 0), (0, 0), (2, 0), (0, 2), (2, 2).$$

   For each $r \geq 0$, what is the Čech complex $\check{C}(r)$?

3. Repeat, this time with V being the sixth roots of unity. If you know about the Vietoris-Rips complex, do it for that as well.

4. Consider two linear maps $\mathbb{F}_2^{n_2} \xrightarrow{\partial_2} \mathbb{F}_2^{n_1} \xrightarrow{\partial_1} \mathbb{F}_2^{n_0}$ with the property that the composite $\partial_1\partial_2 = 0$.

   (a) Show that $\mathrm{im}\partial_2 \subseteq \ker \partial_1$.

   (b) Device an algorithm that given the matrix $[\partial_1]$ associated with ∂_1 produces a basis for $\ker \partial_1 = \mathrm{Null}[\partial_1]$.

   (c) Device an algorithm that expresses the image of ∂_2 (column space of $[\partial_2]$) in terms of the basis for $\ker \partial_1$ obtained above. A second output should be the dimension of the quotient space $\ker \partial_1/\mathrm{im}\partial_2$.

   (d) Incorporate the two algorithms into an algorithm that takes as input a simplicial complex with output the dimensions of the homology groups.

   (e) Implement the algorithm and run tests on simple examples (like the ones done in class).

5. Repeat the above exercise to give an algorithm for calculating bar codes for persistent homology. As a warm-up, do H_0 first. For further hints, you may for instance look at *Computing Persistent Homology* by Afra Zomorodian and Gunnar Carlsson (provide link). Implement the algorithm and run tests on simple examples (like the ones done in class).

6. When we get that far: find the Čech and Vietoris-Rips complexes and calculate the associated persistent homologies/write down the bar codes for the point clouds in exercise 2 and 3.

[1] If you read this on Tuesday: Since I did not have time for that many examples in class – except for question 1 - 4 which are accessible with what we did on Tuesday – you probably want to have a peek at the slides (avaliable at the home page of the conference) before answering them

7. Check the calculations of homology/persistent homology done in class.

8. *If* I had time to present both the Čech and the Vietoris-Rips complexes, prove that they are *intertwined* in the sense that

$$\check{C}(r) \subseteq VR(r) \subseteq \check{C}(\sqrt{2}r).$$

This is often used as a justification for the relevance of persistent homology based on the Vietoris-Rips complex.

9. For a finite subset $V \subseteq \mathbb{R}^d$, let $B(r)$ be the union of all the closed balls of radius r with centers in the points in V. For the examples (with $d = 2$ done in class or which you tested your other exercises on, check that the homology of $\check{C}(r)$ describes the shape of $B(r)$ fathfully. (Actually, a theorem says that the homology of $\check{C}(r)$ is the same as that of $B(r)$ (as discussed in the later lectures), lending credibility to $\check{C}(r)$.)