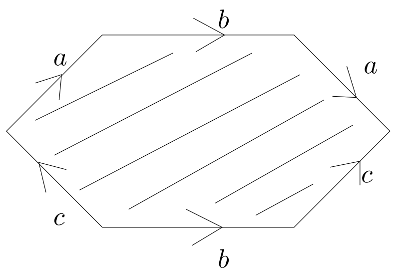
Exercises for Day #2 (Dugger)

1. Go back and do all the exercises for day #1 (you should be able to do them now!)
2. Compute the Betti numbers for every space you can think of, using both the fields $\mathbb{R}$, $\mathbb{Z}/2$, and $\mathbb{Z}/3$ for coefficients. At least do all the spaces X and $X \times S^1$ where X is a compact 2-manifold. Also do the space Y obtained from a torus T by collapsing a circle, to give a "bent banana".

The following exercises provide one approach to the classification of all compact 2-manifolds. The idea is to replace the topological problem with a combinatorial one, by encoding the cut-and-paste rules into algebraic formulas.

Suppose one has a polygon with labelled edges, representing a quotient space. Assume every edge is labelled, and every label occurs exactly twice. Such a quotient space is a 2-dimensional manifold.

If we have a diagram such as



we will represent this by the string "$abac^{-1}b^{-1}c$". Notice that we could also represent it by the string "$b^{-1}cabac^{-1}$", as well as others. When talking about these strings I will use letters x, y, z to stand for one symbol (like a or a^{-1}) and letters A, B, C to stand for a block of symbols (like aba, or $c^{-1}b^{-1}$).

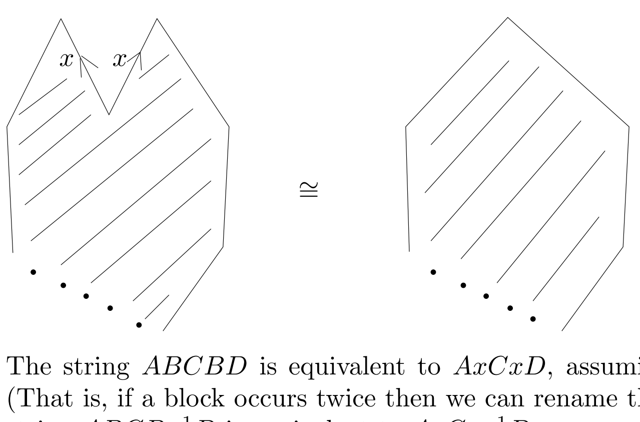
If B is a block $x_1x_2\cdots x_k$, let B^{-1} denote the block $x_k^{-1}x_{k-1}^{-1}\cdots x_1^{-1}$. Our convention is that $(x^{-1})^{-1} = x$, so that for instance if B is the block abc^{-1} then B^{-1} is $cb^{-1}a^{-1}$.

We'll say that two strings are 'equivalent' if the corresponding quotient spaces are homeomorphic. Here are some basic cases of equivalent strings:

(i) The strings $x_1x_2\dots x_n$ and $x_nx_1x_2\dots x_{n-1}$ are equivalent.

(ii) If you insert "xx^{-1}" or "$x^{-1}x$" into any string which doesn't have an x, you get an equivalent string. [Taking this and working backwards, if you remove either "xx^{-1}" or "$x^{-1}x$" from a given string then you get an equivalent one].

Here is a picture explaining where (ii) comes from:

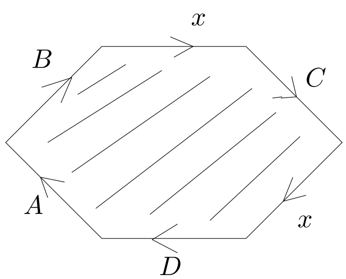


(iii) The string $ABCBD$ is equivalent to $AxCxD$, assuming the letter x doesn't appear in A, C, or D. (That is, if a block occurs twice then we can rename the whole block to just one letter). Likewise, the string $ABCB^{-1}D$ is equivalent to $AyCy^{-1}D$.

Based on all this, do the following exercises:

3. Using (i)–(iii), explain why a string AC is equivalent to both $ABB^{-1}C$ and $AB^{-1}BC$, assuming that none of the letters in B appears in AC.

4. Use a topological cut-and-paste argument to prove that the string $ABxCxD$ is equivalent to $AyCB^{-1}yD$. Start with the diagram below:



$\cong$?????

5. Again using topological cut-and-paste arguments, prove that $AxBCxD$ is equivalent to $AyCyB^{-1}D$. Also prove that $AxBCx^{-1}D$ is equivalent to $AyCBy^{-1}D$, and $ABxCx^{-1}D$ is equivalent to $AyCy^{-1}BD$.

We can summarize the rules you've established in questions 3–5 as follows:

(1) $AxBCxD \sim AxCxB^{-1}D$

(2) $ABxCxD \sim AxCB^{-1}xD$

(3) $AxBCx^{-1}D \sim AxCBx^{-1}D$

(4) $ABxCx^{-1}D \sim AxCx^{-1}BD$.

[Note that I am changing y's to x's in the strings on the right-hand-side. This is okay by (iii)—I can always rename a letter as something else].

The rules can be paraphrased as follows:

- If x occurs twice in a string (without any inverses on it), then a block next to the first x can be moved to a block on the same side of the second x, *but the new block has to be inverted*. [This is Rules (1) and (2).]
- If x and x^{-1} occur in a string, then a block next to the first one can be moved to a block *on the other side* of the second one, and the block doesn't get inverted. [Rules (iii) and (iv)].

As an example of using these rules, we can write

$$abacb^{-1}ddc \sim abacdbdc \sim cdbda^{-1}b^{-1}a^{-1}c \sim cdabda^{-1}b^{-1}c.$$

For the first one we have used Rule (2) with $A = abac$, $B = b^{-1}$, $C = []$ (the empty block), $D = c$, and $x = d$. For the second equivalence we have used Rule (2) again, this time with $A = []$, $B = aba$, $C = dbd$, $D = []$, and $x = c$. Finally, in the third equivalence we have used Rule (4) (in reverse) with $A = cd$, $B = a^{-1}$, $x = b$, $C = da^{-1}$, $D = c$.

6. Using the rules (i)–(iii) and (1)–(4) established above, prove the following equivalences:
   (a) $abab \sim xx$.
   (b) $abab^{-1} \sim xxyy$ (this shows $K \cong \mathbb{R}P^2 \# \mathbb{R}P^2$).

(c) $aabcb^{-1}c^{-1} \sim xxyyzz$ (this shows $\mathbb{R}P^2 \# T \cong \mathbb{R}P^2 \# \mathbb{R}P^2 \# \mathbb{R}P^2$).

[Part (c) can be tricky. Here is one possible approach. First, start with $xxyy$ and write down some other four element strings which are equivalent to this one. A list of four or five of these will be useful to you. Second, go back to $aabcb^{-1}c^{-1}$ and start by changing the $aabc$ block to $ac^{-1}b^{-1}a$. Now try to move the two c^{-1}'s together.]

7. In this problem you will develop the techniques to prove that every 2-dimensional manifold is homeomorphic to one in the following list:

$$(*) \qquad S^2, \quad T, \quad T\#T, \quad T\#T\#T, \quad \ldots, \quad \mathbb{R}P^2, \quad \mathbb{R}P^2\#\mathbb{R}P^2, \quad \mathbb{R}P^2\#\mathbb{R}P^2\#\mathbb{R}P^2, \quad \ldots$$

Note that S^2 is represented by the string xx^{-1} (think about why), which is equivalent to the empty string.

Let S be a string of letters and their 'inverses', in which each letter appears twice (meaning either twice as itself or once as x and once as x^{-1}). Use (i)–(iii) and (1)-(4) to do the following.

(a) Prove that if a letter x appears twice as itself, then S is equivalent to a string in which xx appears as the beginning. [That is, prove that $AxBxC \sim xxD$, for some block D.]

(b) Prove that a string of the form $AsCtDs^{-1}Et^{-1}$ is equivalent to one which starts out as $sts^{-1}t^{-1}$. [Hint: First move the s out front. Then move the s^{-1} and t^{-1} together, then move so that you have $ts^{-1}t^{-1}$, then move so that you have $sts^{-1}t^{-1}$.]

(c) Let M be the 2-dimensional manifold corresponding to the string

$$abca^{-1}befe^{-1}c^{-1}f.$$

Determine which of the manifolds in (*) is homeomorphic to M. [Start by doing what you did in (a), and one by one move all the double letters to the beginning.]

(d) Repeat part (c) for the manifold corresponding to the string

$$dbced^{-1}c^{-1}e^{-1}b^{-1}.$$

(e) Explain why any string of the form

$$a_1a_1a_2a_2\ldots a_ka_ks_1t_1s_1^{-1}t_1^{-1}\ldots s_rt_rs_r^{-1}t_r^{-1}$$

in which $k \geq 1$ is equivalent to one of the form

$$b_1b_1b_2b_2\ldots b_nb_n.$$

(f) Think about how you would prove that any string is equivalent to either one of the form $a_1a_1a_2a_2\ldots a_ka_k$ or $s_1t_1s_1^{-1}t_1^{-1}\ldots s_rt_rs_r^{-1}t_r^{-1}$.