EXERCISES: Examples and non examples of immersions in the plane

For each of the following functions $f : \mathbf{R} \to \mathbf{R}^2$ determine whether it is a $\mathcal{C}^1$ -immersion. Draw the corrseponding curve in the plane.

- f(t)=(t,t^2)
- f(t)=(cos(2π t),sin(2π t))
- f(t)=($t^{3,t2}$)
- f(t)=(t, $\sqrt[3]{\{t^2\}}$)
- f(t)=(sin(2π t),sin(4π t))
- f(t)=cos(6π t)$\cdot$ (cos(2π t),sin(2π t))

Prove that there exists no immersion of a closed curve in the real line.

EXERCISES: on the degree of a self map of the circle

- Prove that if a self map of the circle, g:$S^1 \to S^1$ is not surjective then it is homotopic to a constant map.
- Find and prove a formula that de degree of the composition of two self-maps of the circle in terms of the degrees of each of the map.
- Prove that the degree of a map is well defined and does not depend on the choice of the lifting.
- Complete the details of the proof of the existence of a lifting along p of path in the circle. After that do the same for the general homotopy lifting theorem for $p\colon R\to S^1$.
- Prove that if two self maps of the circle have the same degree then they are homotopic. Hint Assume first that the two maps have the same value at (0,1) and show that they liftings can be taken with same origins and same endpoint. Show that there is

Challenges

- Spot a mistake in Whitney's paper.
  Hint: with his definition of deformation on p. 278 (not same as ours), the proof of the first half of his theorem 1 is wrong. Find an explicit counterexample to his theorem 1 (with his definition). Fix his definition to make theorem 1 correct.
- Consider closed regular curves f: [0,1]$\to$ S^2 in the 2-sphere $S^2 = \{(x, y, z) \in \mathbf{R}^2 \,:\, x^2 + y^2 + z^2 = 1\}$ instead of the plane.
  1. Show that the immersion $f_1 \colon t \mapsto (\cos(2\pi t), \sin(2\pi t), 0)$ traveling the equator in one direction is regularly homotopic to the one traveling the equator in the other direction, $f_{-1} \colon t \mapsto (\cos(2\pi t), -\sin(2\pi t), 0)$
  2. Show that f_1 is regularly homotopic to the immersion $f_3 \colon t \mapsto (\cos(6\pi t), \sin(6\pi t), 0)$ traveling the equator 3 times
  3. Is f_1 regularly homotopic to the immersion $f_2 \colon t \mapsto (\cos(4\pi t), \sin(4\pi t), 0)$ traveling the equator 2 times ?

Further challenges

- Prove the magic formula of Whitney in the special case when they are no self-intersections. This is the "Umlaufsatz of Hopf". Hint define $\psi(t_1, t_2) = (f(t_2) - f(t_1))/dist(t_1, t_2)$ on the half-open solid triangle $0 \leq t_1 < t_2 \leq 1$ and extend continuously on the boundary $t_1 = t_2$ by the derivative $f'(t_1)$. Look at `http://www.mathematik.com/Hopf/` for a pictorial proof.